Groupoidification: A Categorification of Hecke Algebras

Alexander Hoffnung

University of California, Riverside   Email: alex@math.ucr.edu   Web: http://math.ucr.edu/~alex

Joint work with John Baez

Introduction

Groupoidification is a form of categorification:

Categorification:

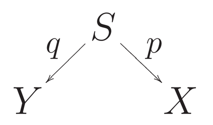
- $\circ$ Sets $\to$ Categories
- $\circ$ Functions $\to$ Functors

Groupoidification:

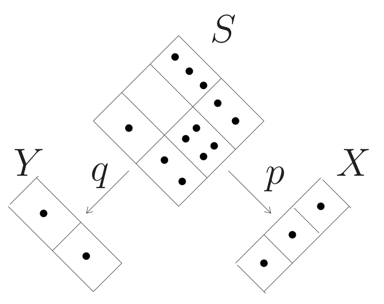
- $\circ$ Vector spaces $\to$ Groupoids
- $\circ$ Spans of vector spaces $\to$ Linear maps

Example: Groupoidified Hecke algebra

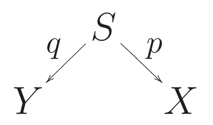
Spans of finite sets



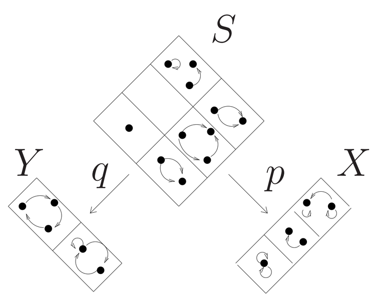
give matrices of natural numbers.



Spans of groupoids



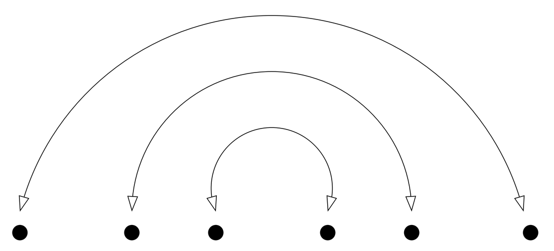
give matrices of non-negative real numbers.



For finite sets, the concept of cardinality is well-known.

What is the cardinality of a groupoid?

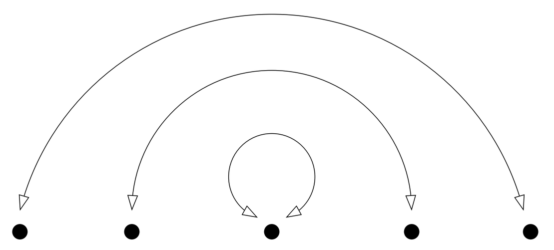
The concept of "categorified division" arises from thinking of group actions on sets. For example, we can ask:
Why is $6/2 = 3$?



$\mathbb{Z}/2$ acting on 6-element set

Since we are 'folding the 6-element set in half', we get $|S/G| = 3$. That is $|S/G| = |S|/|G|$.

Let's try the same trick starting with a 5-element set:



$\mathbb{Z}/2$ acting on 5-element set

We don't obtain a set with $2\frac{1}{2}$ elements! The reason is that the point in the middle gets mapped to itself, i.e., the action is not free.

To get the desired cardinality $2\frac{1}{2}$, we would need a way to count this point as 'folded in half'.

Action groupoid:

Let G be a group acting on a set S,

$$G \times S \to S$$

$$(g, s) \mapsto g \cdot s.$$

The **action groupoid** $S//G$ consists of

- $\circ$ objects: elements of S;
- $\circ$ morphisms: pairs $(g, s)\colon s \to g \cdot s$.

Groupoid cardinality:

$$|X| = \sum_{\text{isomorphism classes of objects } [x]} \frac{1}{|\mathrm{Aut}(x)|}\,.$$

Now,

$$|S//G| = |S|/|G|$$

whenever G is a finite group acting on a finite set S.

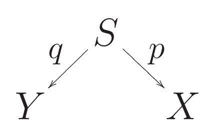
Degroupoidification:

We can think of degroupoidification as a functor from $Span(Groupoid)$ to $Vect$.

Given a groupoid X, we define the **vector space associated to** X by

$$\mathbb{R}^{\underline{X}} = \{\Psi\colon \underline{X} \to \mathbb{R}\}.$$

Given a 'nice' span:



there exists a unique linear operator

$$\tilde{S}\colon \mathbb{R}^{\underline{X}} \to \mathbb{R}^{\underline{Y}}.$$

The Hecke Bicategory

For every finite group G there is a bicategory $Hecke(G)$ enriched over the monoidal bicategory $Span(Groupoid_{Fib})$ consisting of:

- $\circ$ objects: finite G-sets
- $\circ$ morphisms: spans of G-sets (*categorified generators*)
- $\circ$ 2-morphisms: (equivalence classes of) maps of spans (*categorified relations*)

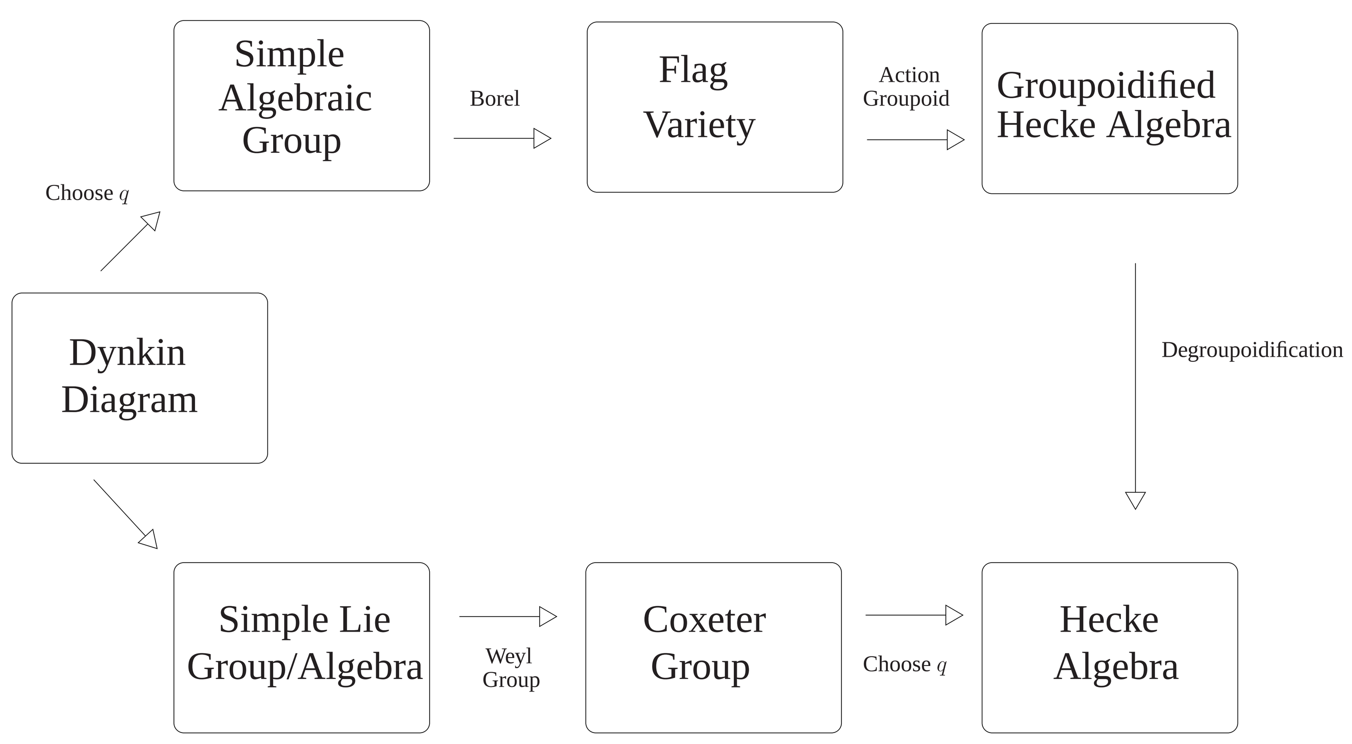
This bicategory categorifies the category $Perm(G)$ consisting of:

- $\circ$ objects: permutation representations of G, i.e., representations arising from actions of G on finite sets via 'free vector space' functor.
- $\circ$ morphisms: intertwining operators, i.e., G-equivariant linear operators.

The Hecke algebra

- $\circ$ Hecke algebras are q-deformed versions of the group algebras of symmetric groups. These naturally sit inside the category of permutation representations of $GL(n, \mathbb{F}_q)$.
- $\circ$ An algebra is a one-object category enriched over $Vect$. So the categorification of the Hecke algebra should be a one-object bicategory enriched over $Span(Groupoid_{Fib})$.

A hands-on view of the groupoidified Hecke algebra



Example: The A_2 Dynkin Diagram

- $\circ$ Fix a power of a prime q.
- $\circ$ $G = SL(3, \mathbb{F}_q)$ and take B to be the upper triangular matrices.
- $\circ$ $X = G/B$ is the set of complete flags in $\mathbb{F}_q^3$, i.e.,

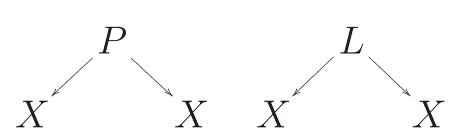
$$\{V_1 \subset V_2\} \quad \text{and} \quad \dim V_i = i.$$

- $\circ$ A_2 Dynkin diagram $\leftrightarrow$ Projective plane geometry
- $\circ$ Vertices represent "figures" and the edges represent "incidence relations"

- $\circ$ In $\mathbb{F}_q\mathrm{P}^2$, a flag is just a chosen point lying on a chosen line.

The groupoid

- $\circ$ The A_2 Hecke algebra has 2 generators P and L.
- $\circ$ There should be one span of G-sets in our groupoid for each generator - these correspond to 'changing a point' and 'changing a line'.



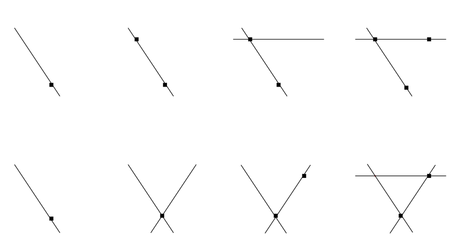
where

$$P = \{((p, l), (p', l)) \mid p \neq p'\} \text{ and } L = \{((p, l), (p, l')) \mid l \neq l'\}$$

- $\circ$ The multiplication in the Hecke algebra satisfies the following relations:

$$P^2 = (q-1)P + q \qquad L^2 = (q-1)L + q$$

- $\circ$ We can see this at the groupoidified level by counting.
- $\circ$ The Hecke algebra also satisfies the Yang-Baxter equation $PLP = LPL$.



In the projective plane:

- $\circ$ Any two distinct points determine a unique line
- $\circ$ Any two distinct lines determine a unique point

References

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