A Hybrid Commodity and Interest Rate Market Model

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Literature

- LIBOR Market Model (LMM): Miltersen/Sandmann/Sondermann (1997), Brace/Gatarek/Musiela (1997), Jamshidian (1997), Musiela/Rutkowski (1997)
- Multicurrency LIBOR Market Model: Schlögl (2002)
- LMM calibration: Pedersen (1998)
- Integrating commodity risk, interest rate risk, and stochastic convenience yields: Gibson/Schwartz (1990), Miltersen/Schwartz (1998), Miltersen (2003)

A model of forward LIBOR

The forward LIBOR $L(t,T)$ is defined in terms of zero coupon bond prices by

$$L(t,T) := \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right)$$

Note that irrespective of the model we choose, $L(t,T)$ is a martingale under $\mathbf{P}_{T+\delta}$.

Therefore, assuming deterministic volatility for $L(t,T)$ means that it is lognormally distributed under $\mathbf{P}_{T+\delta}$.

Setup

Discrete–tenor lognormal forward LIBOR model
(as in Musiela/Rutkowski (1997))

Horizon date T_N for some $N \in \mathbb{N}$, finite number of maturities

$$T_i = T_N - (N - i)\delta, \qquad i \in \{0, \dots, N\}$$

Dynamics of (domestic) forward LIBORs

$$dL(t, T_i) = L(t, T_i)\lambda(t, T_i)dW_{T_{i+1}}(t)$$

where

- $\lambda(\cdot, \cdot)$ is a deterministic function of its arguments
- $W_{T_{i+1}}(\cdot)$ is a Brownian motion under the time T_{i+1} forward measure

Note that lognormality in this model is a measure–dependent property.

Links between domestic forward measures

By Ito's lemma

$$d\left(\frac{B(t,T)}{B(t,T+\delta)}\right)=\frac{B(t,T)}{B(t,T+\delta)}\frac{\delta L(t,T)}{1+\delta L(t,T)}\lambda(t,T)dW_{T+\delta}(t)$$

Setting

$$\gamma(t,T,T+\delta)=\frac{\delta L(t,T)}{1+\delta L(t,T)}\lambda(t,T)$$

we can write

$$\left.\frac{d\mathbf{P}_{T_i}}{d\mathbf{P}_{T_{i+1}}}\right|_{\mathcal{F}_t}=\frac{B(t,T_i)}{B(t,T_{i+1})}\frac{B(0,T_{i+1})}{B(0,T_i)}=\mathcal{E}_t\left(\int_0^\cdot \gamma(u,T_i,T_{i+1})\cdot dW_{T_{i+1}}(u)\right)$$

Thus

$$dW_{T_i}(t)=dW_{T_{i+1}}(t)-\gamma(t,T_i,T_{i+1})dt$$

Adding a foreign economy, case 1

Assume lognormal forward LIBOR dynamics in the foreign economy as well

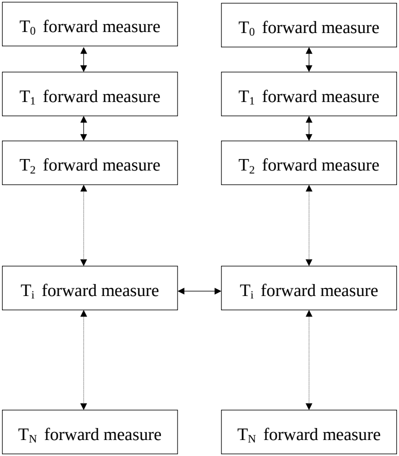
$$d\tilde{L}(t, T_i) = \tilde{L}(t, T_i)\tilde{\lambda}(t, T_i)d\tilde{W}_{T_{i+1}}(t)$$

Then the foreign forward measures are linked in a manner analogous to the domestic forward measures.

This leaves us with the freedom of specifying **one** further link (only) between a domestic and a foreign forward measure.

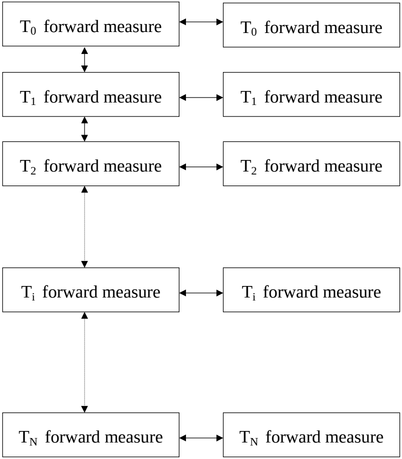
Measure Links 1

Domestic **Foreign**



Measure Links 2

Domestic **Foreign**



Linking domestic & foreign forward measures

$X(t)$: spot exchange rate in units of domestic currency per unit of foreign currency

Time T_i forward exchange rate:

$$X(t,T_i)=\frac{\tilde{B}(t,T_i)X(t)}{B(t,T_i)}$$

This is a martingale under $\mathbf{P}_{T_i}$. Conversely

$$\frac{1}{X(t,T_i)}=\frac{B(t,T_i)\frac{1}{X(t)}}{\tilde{B}(t,T_i)}$$

is a martingale under $\tilde{\mathbf{P}}_{T_i}$.

So we can write

$$dX(t,T_N)=X(t,T_N)\sigma_X(t,T_N)\cdot dW_{T_N}(t)$$

Domestic vs. foreign forward measures

$$\frac{d\tilde{\mathbf{P}}_{T_N}}{d\mathbf{P}_{T_N}} = \frac{X(T_N)\tilde{B}(T_N,T_N)B(0,T_N)}{X(0)\tilde{B}(0,T_N)B(T_N,T_N)} = \frac{X(T_N,T_N)}{X(0,T_N)}$$

resp. restricting $\mathbf{P}_{T_N}$, $\tilde{\mathbf{P}}_{T_N}$ to the information given at time t:

$$\left.\frac{d\tilde{\mathbf{P}}_{T_N}}{d\mathbf{P}_{T_N}}\right|_{\mathcal{F}_t} = \frac{X(t,T_N)}{X(0,T_N)}$$

By the dynamics assumed for$X(t,T_N)$,

$$\frac{d\tilde{\mathbf{P}}_{T_N}}{d\mathbf{P}_{T_N}} = \mathcal{E}_t\left(\int_0^{\cdot}\sigma_X(u,T_N)dW_{T_N}(u)\right) \qquad \mathbf{P}_{T_N}\text{–a.s.}$$

Thus

$$d\tilde{W}_{T_N}(t) = dW_{T_N}(t) - \sigma_X(t,T_N)dt$$

Forward exchange rate volatilities

Note that all measure relationships and therefore all volatilities are now fixed.

To determine the remaining forward exchange rate volatilities, inductively make use of the relationship

$$\frac{X(t,T_i)}{X(t,T_{i+1})} = \frac{B(t,T_{i+1})}{B(t,T_i)} \frac{\tilde{B}(t,T_i)}{\tilde{B}(t,T_{i+1})}$$

For ease of notation, consider just the first step of the induction. Writing all processes under the domestic time T_{N-1} forward measure and applying Ito's lemma then yields

$$dX(t,T_{N-1}) = X(t,T_{N-1}) \left((\tilde{\gamma}(t,T_{N-1},T_N) - \gamma(t,T_{N-1},T_N) + \sigma_X(t,T_N)) \cdot dW_{T_{N-1}}(t)\right)$$

Thus we must set

$$\sigma_X(t,T_{N-1}) = \tilde{\gamma}(t,T_{N-1},T_N) - \gamma(t,T_{N-1},T_N) + \sigma_X(t,T_N)$$

i.e. we can choose only one $\sigma_X(t,T_i)$ to be a deterministic function of its arguments.

So for FX options we can have a Black/Scholes–type formula for only one maturity, as all other forward exchange rates are not lognormal.

Adding a foreign economy, case 2

Assume lognormal forward LIBOR dynamics in the domestic economy only; assume lognormal forward exchange rates

$$dX(t, T_i) = X(t, T_i)\sigma_X(t, T_i)dW_{T_i}(t)$$

for σ_X a deterministic function of its arguments.

Thus for all maturities T_i

$$d\tilde{W}_{T_i}(t) = dW_{T_i}(t) - \sigma_X(t, T_i)dt$$

Since the derivation of the links between forward exchange rate volatilities did not depend on the lognormality assumptions, it is valid in the present context as well and therefore

$$\tilde{\gamma}(t, T_{i-1}, T_i) = \sigma_X(t, T_{i-1}) - \sigma_X(t, T_i) + \gamma(t, T_{i-1}, T_i)$$

A commodity as “foreign currency”

- The commodity market can naturally be considered as a “foreign interest rate market.”
- The currency is the physical commodity itself.
- The “zero coupon bond prices” $C(t, T)$ quote (as seen at time t) the amount of the commodity that has to be invested at time t to physically receive one unit of the commodity at time T.
- Thus the yield of $C(t, T)$ is the convenience yield (adjusted for storage costs, if applicable).
- Since convenience yields are implicit rather than explicitly quoted in the market, “Case 2” of the multicurrency model is applicable.

Pedersen (1998) Calibration

- Calibration to market prices of caps (or caplets) and swaptions.
- Calibration of a non-parametric volatility function $\lambda(\cdot,\cdot)$, piecewise constant on a discretisation of both time to maturity and calendar time.
- Unconstrained non-linear optimisation of weighted sum of quality–of–fit and smoothness criteria.
- Correlation is exogenous to the calibration procedure: Assumed to be constant in time and estimated from historical data.
- Reduction of dimension via principal components analysis.

The nonparametric approach

Suppose we have n_{fac} factors (the dimension of the driving Brownian motion) and discretise process time into n_{cal} segments, and forward time (maturities) into n_{fwd} segments.

The *i*-th component of $(1 \leq i \leq n_{\text{fac}})$ of the volatility function $\lambda(t, T)$ will be given by

$$\lambda_i(t, x) = \lambda_{ijk} \ , \ t \in [t_{j-1}, t_j) \ , \ x \in [x_{k-1}, x_k)$$

where $x = T - t$ is the **forward tenor**, t_j, $j > 0$, and x_k, $k > 0$, are the chosen process and forward times, respectively.

For convenience set $t_0 = x_0 = 0$.

Objective function

$$w_{\text{caps}} \text{QOF}_{\text{caps}} + w_{\text{swaptions}} \text{QOF}_{\text{swaptions}} + \text{smooth}$$

Quality of fit $$\text{QOF} = \frac{1}{N} \sum_{i=1}^{N} \left(\frac{\overline{\text{PV}}_i}{\text{PV}_i} - 1 \right)^2$$

$$\text{smooth} = \text{scale}_{\text{fwd}} \cdot \text{smooth}_{\text{fwd}} + \text{scale}_{\text{cal}} \cdot \text{smooth}_{\text{cal}}$$

$$\text{smooth}_{\text{fwd}} = \sum_{j=1}^{n_{\text{cal}}} \sum_{j=2}^{n_{\text{fwd}}} \left(\frac{\text{vol}_{i,j}}{\text{vol}_{i,j-1}} - 1 \right)^2$$

$$\text{smooth}_{\text{cal}} = \sum_{j=2}^{n_{\text{cal}}} \sum_{j=1}^{n_{\text{fwd}}} \left(\frac{\text{vol}_{i,j}}{\text{vol}_{i-1,j}} - 1 \right)^2$$

Reducing the dimesionality of the problem

Original dimensionality: $n_{\text{fac}} \times n_{\text{cal}} \times n_{\text{fwd}}$

Separate volatility levels and correlation:

Volatility levels given by *volatility grid*

$$\text{vol}_{i,j} \;,\; 1 \leq i \leq n_{\text{cal}} \;,\; 1 \leq j \leq n_{\text{fwd}}$$

where $\text{vol}_{i,j}$ is the volatility as seen at time t_{i-1} (assumed constant until t_i) of the basic period rate $L(\cdot, t_{i-1} + x_j)$ for the forward period beginning at time $t_{i-1} + x_j$.

This is the object which will be calibrated.

Covariance and correlation — Principal components representation

Let vol be the vector of basic period forward rate volatilities as seen on time t_{j-1}. Let Corr be the corresponding correlation matrix. The covariance matrix is then computed as

$$\text{Cov} = \text{vol}^T \ \text{Corr vol}$$

Let Γ be the diagonal matrix containing the eigenvalues of Cov and V be the corresponding matrix of eigenvectors, i.e. we have the eigenvalue/eigenvector decomposition of Cov

$$\text{Cov} = V^T \Gamma V$$

As Cov is positive semidefinite, all entries γ_k on the diagonal of Γ will be non-negative and we have

$$\mathrm{Cov} = W^T W$$

where

$$w_{ik} = \sqrt{\gamma_k} v_{ik}$$

We can then extract the stepwise constant volatility function for forward LIBORs as

$$\lambda_{ijk} = w_{ik}$$

W will provide values for as many factors as the rank of the covariance matrix. For a given n_{fac}, we only use the rows of W corresponding to the n_{fac} largest eigenvalues.

Spot measure dynamics

Brownian motion under the rolling spot LIBOR measure **Q** is related to BM under the T_i forward measure by

$$dW_{T_i}(t) = \phi(t, T_i)dt + dW_Q(t)$$

with $\phi(\cdot, T_i)$ defined recursively as

$$\phi(t, T_i) - \phi(t, T_{i-1}) = \gamma(t, T_{i-1}, T_i) = \frac{\delta L(t, T_{i-1})}{1 + \delta L(t, T_{i-1})}\lambda(t, T_{i-1})$$

Under an appropriate extension of the discrete–tenor LMM to continuous tenor, **Q** coincides with the spot risk–neutral measure and the futures price corresponds to the expected future spot price under this measure.

Futures vs. forward

Thus for the futures price $G(t,T)$ observed at time t for maturity T, we have

$$\begin{aligned} G(t,T) &= E_{\mathbf{Q}}[X(T,T)|\mathcal{F}_t] \\ &= X(t,T)E_{\mathbf{Q}}\left[\exp\left\{\int_t^T \sigma_X(u,T)dW_Q(u)\right.\right. \\ &\quad \left.\left. -\frac{1}{2}\int_t^T \sigma_X^2(u,T)du + \int_t^T \sigma_X(u,T)\phi(u,T)du\right\}\middle|\mathcal{F}_t\right] \\ &\approx X(t,T)\exp\left\{\int_t^T \sigma_X(u,T)\overline{\phi}(u,T)du\right\} \end{aligned}$$

where $\overline{\phi}$ is the “frozen coefficient” approximation for ϕ.

Merging Interest Rate & Commodity Calibrations

Step 1:

Calibrate LMM for interest rates using Pedersen approach. An output of this is the matrix $W^{(I)}$.

Step 2:

Calibrate the volatility of forward commodity prices to the market using an appropriately modified Pedersen approach. An output of this is the matrix $W^{(C)}$.

Step 3:

Suppose we have an exogenously given covariance matrix Σ_{CI} of all forward LIBORSs and commodity prices.

In order to achieve an approximate fit to this covariance matrix, we exploit the property that multivariate normally distributed random variables are invariant under orthonormal rotations.

We seek a square matrix Q, which minimises

$$\|\Sigma_{CI} - W^{(C)}Q(W^{(I)})^{\top}\|$$

and

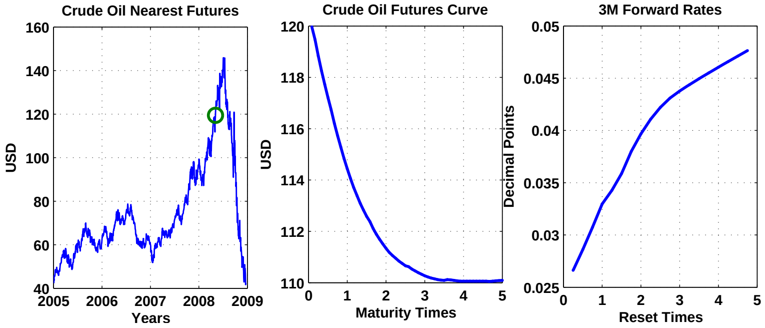
$$\|QQ^{\top} - \mathbb{I}\|$$

We then replace $W^{(C)}$ by $W^{(C)}Q$ when determining the volatility functions for forward commodity prices.

Notes

- The dimension of Q is the total number of factors, which may be greater than or equal to the greater of the number of factors in $W^{(I)}$ and $W^{(C)}$.
- $W^{(I)}$ and $W^{(C)}$ are padded with zeroes where needed.
- Due to the dependence of the convexity adjustment on interest rate volatilities, steps 2 and 3 need to be repeated iteratively.

The commodity and interest rate market on the calibration date 5 May 2008

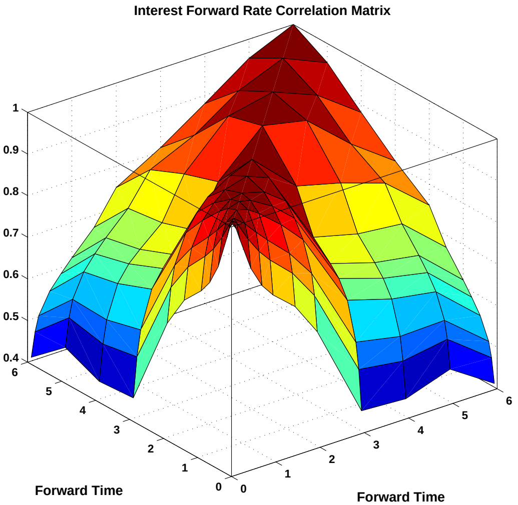


Left: The WTI Crude Oil nearest futures between 2005 and end of 2008. The circle indicates the calibration date.

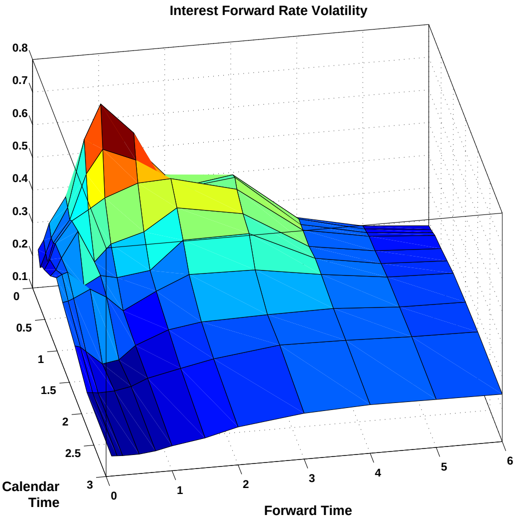
Middle: The futures curve as seen at calibration date with maturities up to five years. Right: The 3–month USD

forward rates for reset dates (expiries) between 3 months and 4 years and 9 months.

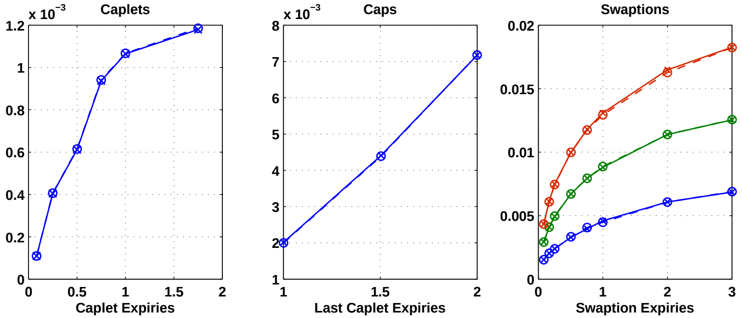
Historically estimated interest rate correlation matrix



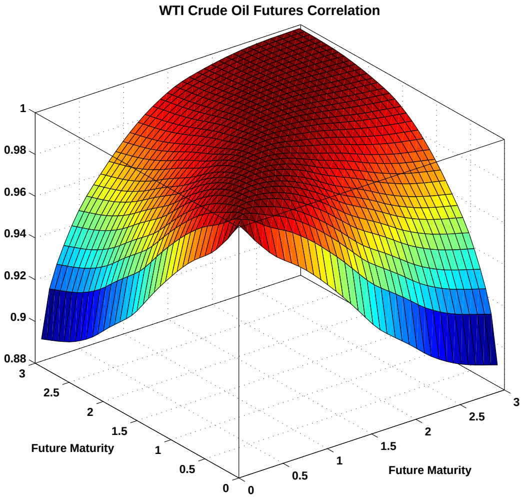
Calibrated interest rate volatility matrix



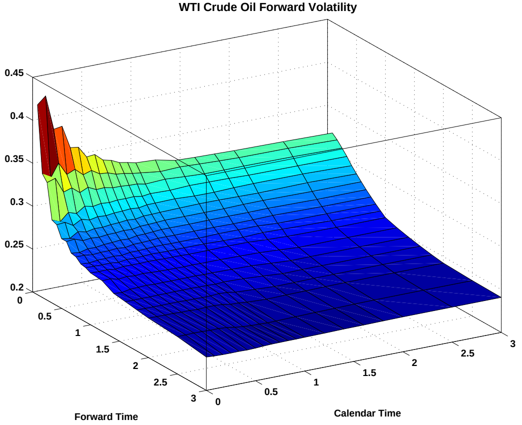
Market prices vs. model prices



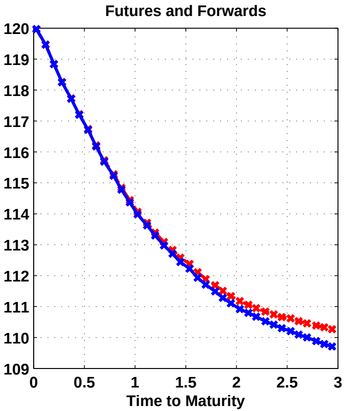
Historically estimated commodity correlation matrix

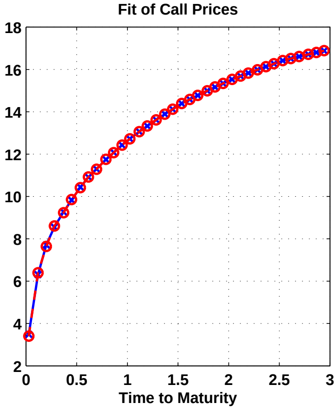


Calibrated commodity volatility matrix

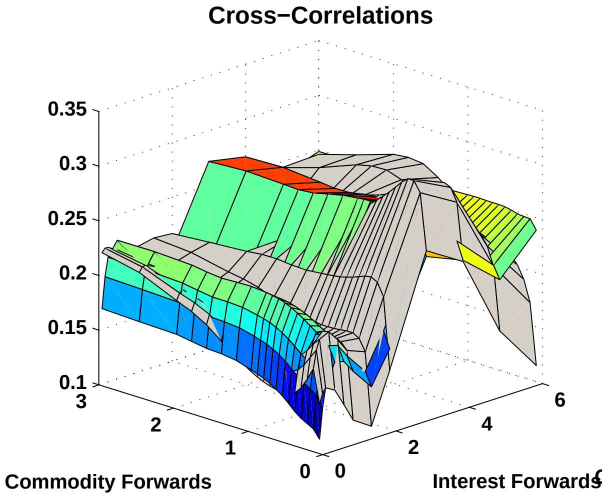


Commodity futures vs. forwards & call option prices

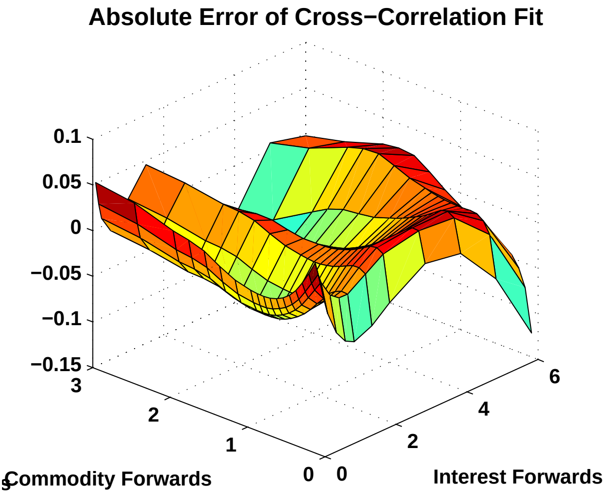




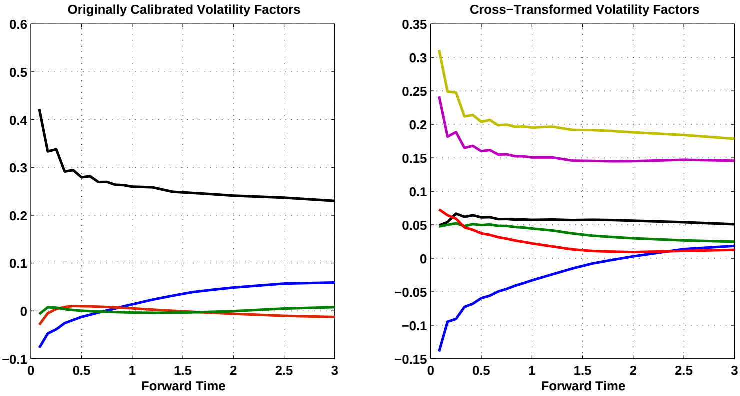
Target & model cross correlations



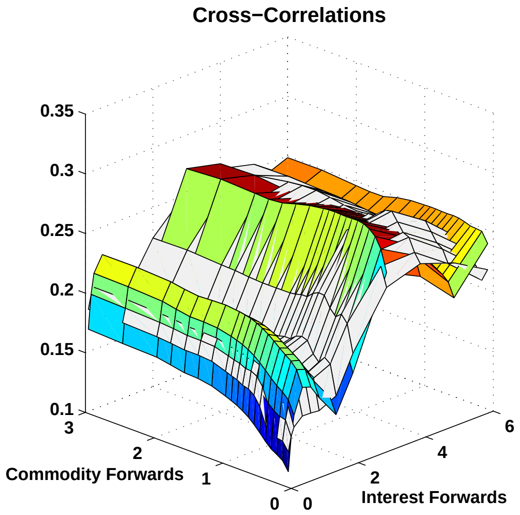
Six-factor correlation fitting errors



Factors before & after rotation



Target & model cross correlations



Twelve-factor correlation fitting errors