Large Traders and Illiquid Options: Hedging vs. Manipulation

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Motivation

- Classical theory of option pricing assumes that hedging of derivatives has no impact on the price process of the underlying.
- In practice, we observe particularly large trading activities when derivatives mature ("**witches' sabbaths**").
- Another example for a price impact: the battle for control of **Volkswagen**

  Financial Times vom 29 Oct 2008:

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Research questions

- Given this empirical evidence, what are the **optimal manipulation strategies** of large traders with price impact that hold/issued illiquid derivatives ?
- What is the large trader's **indifference price** (reservation price) of an illiquid derivative ?
- Extensive literature on price impact models:

  Back (1992), Bank, Baum (2004), Çetin, Jarrow, Protter (2004), Çetin, Rogers (2007), Cvitanić, Ma (1996), DeMarzo, Urošević (2006), Frey, Stremme (1997), Glosten, Milgrom (1985), Horst, Naujokat (2008), Jarrow (1994), Kyle (1985)

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Model considered in Kraft and K. (2010)

- Investment opportunities of large trader
  (1) money market account with zero interest
  (2) risky small cap stock S, whose drift rate is affected by the €-amount $(\theta_t)_{t\in[0,T]}$ the large trader holds in stocks.
- stock dynamics: $dS_t = S_t\left[(\mu_0 + \mu_1\theta_t)\,dt + \sigma\,dW_t\right]$
  typically: $\mu_1 < 0$, "squeezing" ($\mu_1 > 0$, "herding")
- Justified as equilibrium stock price process by DeMarzo and Urošević (2006)
- This leads to the gain process X given by $X_0 = 0$ and

$$dX_t = \frac{\theta_t}{S_t}\,dS_t = \theta_t(\mu_0 + \mu_1\theta_t)\,dt + \theta_t\sigma\,dW_t$$

- Moreover, large trader issues an **illiquid derivative** on the stock with time T payoff $h(S_T)$ ("over the counter")
- total wealth at time $T = p^h - h(S_T) + X_T$
- To switch from seller's to buyer's viewpoint replace h by $-h$.

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Model

Recall the stock dynamics: $dS_t = S_t\,[(\mu_0 + \mu_1\theta_t)\,dt + \sigma\,dW_t]$

- **Immediate observation**: despite of the price impact $\mu_1 \neq 0$ the large trader can perfectly replicate the claim $h(S_T)$ at the same costs as in the corresponding standard Black-Scholes model with $\mu_1 = 0$.
- *One* explanation: distribution of price process under "martingale measure" does not depend on $(\theta_t)_{t\in[0,T]}$.
  Replication costs $=$ expected payoff under martingale measure
- $\rightsquigarrow$ we have the reference Black-Scholes hedge θ^{BS} and price p^{BS}
- But due to the price impact there appears a **trade-off**:
  - **Hedging** (removing risk by offset transactions)
  - **Manipulation** (systematic influence on the non-hedged derivative position to the own advantage)

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Utility-based hedging and indifference pricing

- Exponential utility: $u(Y) = E[-\exp(-\alpha Y)]$, $\alpha >$ **0 risk aversion**
- p^h is the **seller's indifference price** for the derivative payoff $h(S_T)$ iff

$$\sup_\theta E\left[-\exp(-\alpha(p^h - h(S_T(\theta)) + X_T(\theta)))\right]$$
$$= \sup_\theta E[-\exp(-\alpha(X_T(\theta)))]$$

Utility with derivative deal $\stackrel{!}{=}$ Utility without derivative deal

- **New**: $h(S_T(\theta))$ depends on θ.
  $X_T(\theta)$ is no longer linear in the strategy θ
  $\Longrightarrow$ in general $p^h \neq p^{\mathrm{BS}}$

Hedging manipulation strategy $:= \widehat{\theta}$(with claim) $- \widehat{\theta}$(without claim)

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Hamilton-Jacobi-Bellman equation

Assume that $\mu_1 < \frac{1}{2}\alpha\sigma^2$. Large trader's **value function**:

$$G(t,x,s) = \sup_{\theta} E\left[-\exp(-\alpha(-h(S_T(\theta)) + X_T(\theta)))\right]$$

has to satisfy Hamilton-Jacobi-Bellman equation

$$\max_{\vartheta\in\mathbb{R}} \Big\{ G_t + \vartheta(\mu_0 + \mu_1\vartheta)G_x + (\mu_0 + \mu_1\vartheta)sG_s + \frac{1}{2}\sigma^2\vartheta^2 G_{xx} + \frac{1}{2}\sigma^2 s^2 G_{ss} + \sigma^2\vartheta s G_{xs} \Big\} = 0,$$

where $G(T,x,s) = -\exp(-\alpha(x - h(s)))$.

Ansatz for value function: $G(t,x,s) = -\exp(-\alpha x)F(t,z)$ with $z = \ln(s)$

HJB equation becomes

$$\max_{\vartheta\in\mathbb{R}} \left\{ -F_t + \left(\vartheta(\mu_0 + \mu_1\vartheta)\alpha - \frac{1}{2}\sigma^2\vartheta^2\alpha^2\right) F + \left(\sigma^2\vartheta\alpha + \frac{1}{2}\sigma^2 - \mu_0 - \mu_1\vartheta\right) F_z - \frac{1}{2}\sigma^2 F_{zz} \right\} = 0, \quad \text{where} \quad F(T,z) = \exp(\alpha h(\exp(z))).$$

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Optimal strategy

$$\widehat{\theta}_t = \underbrace{\frac{\mu_0}{\alpha\sigma^2 - 2\mu_1}}_{\text{maximizer without claim}} + \underbrace{\left(1 + \frac{\mu_1}{\alpha\sigma^2 - 2\mu_1}\right)}_{=:\ \text{hedge multiplier}} \underbrace{\frac{F_z(t, \ln(S_t))}{\alpha F(t, \ln(S_t))}}_{=\partial_z p^h(t, \ln(S_t))}.$$

$\mu_1 = 0$ (Black-Scholes) $\rightsquigarrow$ hedge multiplier $= 1$ (perfect hedging)
$\mu_1 < 0$ $\rightsquigarrow$ hedge multiplier < 1 (underhedging)
$\mu_1 > 0$ $\rightsquigarrow$ hedge multiplier > 1 (overhedging)

Interpretation for the case $\mu_1 < 0$: large trader replicates e.g. 80% of the claim. The hedging portfolio suffers a loss from the price impact of the hedging activity (as price impact is negative). But the **opposite** derivative position profits from it. Taken together the 20% unhedged position profits from the price impact of 80% hedging activity.

Plugging the optimal stock position in the HJB-equation yields

$$0 = -F_t + -(\mu_0 - \frac{1}{2}\sigma^2)F_z - \frac{1}{2}\sigma^2 F_{zz} + \frac{1}{2}\frac{(\alpha\mu_0 F - \mu_1 F_z + \sigma^2\alpha F_z)^2}{\alpha(\alpha\sigma^2 - 2\mu_1)F}$$

Non linear !

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Solution of the HJB-equation

To knock out the nonlinear term we use a trick applied in papers by Henderson, Hobson, and Zariphopoulou

$$\text{Ansatz:} \qquad F(t,z) = g(t,z)^\beta$$

and thus $g(T,z) = \exp\left(\frac{\alpha}{\beta} h(\exp(z))\right)$.

The HJB-equation becomes

$$0 = -\frac{\beta}{\alpha} g_t - \frac{\gamma}{\alpha}(\mu_0 - \frac{1}{2}\sigma^2) g_z - \frac{1}{2}\frac{\gamma}{\alpha}\sigma^2[(\beta-1)\frac{g_z^2}{g} + g_{zz}] + \frac{1}{2}\frac{\left(\mu_0 g - \frac{\gamma}{\alpha}\mu_1 g_z + \beta\sigma^2 g_z\right)^2}{(\alpha\sigma^2 - 2\mu_1) g}$$

To knock out the terms with $\frac{g_z^2}{g}$ we choose

$$\beta = \frac{1}{1 - \frac{(\sigma^2 - \mu_1/\alpha)^2}{\sigma^2(\sigma^2 - 2\mu_1/\alpha)}} < 0$$

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$$\begin{aligned} 0 &= -\frac{\beta}{\alpha} g_t - \frac{\gamma}{\alpha}(\mu_0 - \frac{1}{2}\sigma^2) g_z - \frac{1}{2}\frac{\gamma}{\alpha}\sigma^2[(\beta-1)\frac{g_z^2}{g} + g_{zz}] \\ &\quad + \frac{1}{2}\frac{\left(\mu_0 g - \frac{\gamma}{\alpha}\mu_1 g_z + \beta\sigma^2 g_z\right)^2}{(\alpha\sigma^2 - 2\mu_1) g} \end{aligned}$$

To knock out the terms with $\frac{g_z^2}{g}$ we choose

$$\beta = \frac{1}{1 - \frac{(\sigma^2 - \mu_1/\alpha)^2}{\sigma^2(\sigma^2 - 2\mu_1/\alpha)}} < 0$$

$$g_t - \underbrace{\frac{1}{2}\frac{\alpha}{\beta}\frac{\mu_0^2}{\alpha\sigma^2 - 2\mu_1}}_{=\widetilde{r}} g + \underbrace{\left(\mu_0 - \frac{1}{2}\sigma^2 - \frac{\mu_0(\alpha\sigma^2 - \mu_1)}{\alpha\sigma^2 - 2\mu_1}\right)}_{=\eta_Z} g_z + \frac{1}{2}\sigma^2 g_{zz} = 0.$$

This PDE is linear and thus it possesses a Feynman-Kac stochastic representation

$$g(t,z) = \exp(-\widetilde{r}(T-t))\widetilde{E}\left[\exp\left(\frac{\alpha}{\beta}h(\exp(Z_T))\right)\right], \quad \text{wobei}$$

Z_T is normally distributed with expectation $\eta_Z \cdot (T-t)$ & variance $\sigma^2 \cdot (T-t)$

For the **seller's indifference price** this yields

$$p^h = \frac{1}{\frac{\alpha}{\beta}} \ln\left(\widetilde{E}\left[\exp\left(\frac{\alpha}{\beta}h(\exp(Z_T))\right)\right]\right).$$

As $\beta < 0$ this would formally correspond to the exponential principles (under the artificial measure $\widetilde{P}$) with the artificial **negative risk aversion** $\frac{\alpha}{\beta}$.

Consequence: **many things turn around**

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- seller's indifference price is concave (and not convex as in (in)complete frictionless markets)
- Every claim $h \geq 0$ has a finite seller's indifference price (even if Black-Scholes replication costs and expectation w.r.t. P are infinite)
- Hedging manipulation strategy $\rightarrow \theta^{\text{Black-Scholes}}$
  if risk aversion $\alpha \rightarrow \infty$
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Extension: Two Large Traders

- θ^i is the €-amount that the i-th trader invests in stocks ($i = 1, 2$)
- Stock price dynamics:

$$dS_t = S_t \left((\mu_0 + \mu_1 \theta_t^1 + \mu_1 \theta_t^2)\, dt + \sigma\, dW_t \right)$$

- i-th player's liquid wealth reads

$$dX_t^i = \frac{\theta_t^i}{S_t}\, dS_t = \theta_t^i (\mu_0 + \mu_1 \theta_t^1 + \mu_1 \theta_t^2)\, dt + \theta_t^i \sigma\, dW_t, \quad i = 1, 2.$$

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Extension: Two Large Traders (continued)

- Consider the case that the first trader holds a short and the second a long position in the **same** illiquid derivative with payoff $h(S_T)$
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- **Result**: The game has the following **Nash equilibrium**:

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and $v(t,s)$ is the Black-Scholes price of the claim $h(S_T)$

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Many thanks for your attention !