Optimal Stock Selling Based on the Global Maximum

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(joint work with Dr. M. Dai and Z. Yang)

If I were an Innocent Investor...

- I just bought a stock and must sell it in one year
- Need to decide when to sell?
- Obviously, sell it at the maximum price of the whole year. However, this is an impossible mission.
- So, what about selling at the price "closest" to the maximum?

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- So, what about selling at the price "closest" to the maximum?
- This talk is using square error to measure "closeness" and studying the optimal selling strategy under this criterion.

The Model

- A Black-Scholes market with one stock and one saving account
- The *discounted* stock price follows, on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$,

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\mu \in (-\infty, \infty)$ and $\sigma > 0$ are constants

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- Let $M_s = \max_{0\leq t\leq s} S_t, 0 \leq s \leq T$ be the running maximum of stock price
- Consider the following optimal stopping problem

$$\inf_{0\leq \nu \leq T} \mathbb{E}[(S_\nu - M_T)^2],$$

  where $\mathbb{E}$ stands for the expectation, ν is an $\mathcal{F}_t$-stopping time.

Related (Probabilistic) Literature

- Graversen, Peskir and Shiryaev (2000), Theory Prob Appl, studied

$$\inf_{0 \le \nu \le T} \mathbb{E}[(S_\nu^0 - M_T^0)^2],$$

where $S_t^0 = W_t$, $M_T^0 = \max_{0 \le t \le T} W_t$ and obtained explicit optimal solution

$$\nu * = \inf\{t : M_t^0 - S_t^0 \ge z^* \sqrt{T - t}\}, z^* = 1.12 \dots$$

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- du Toit and Peskir (2007), Ann Prob, considered

$$\inf_{0\le\nu\le T} \mathbb{E}[(S^\mu_\nu - M^\mu_T)^2],$$

where $\mu \neq 0$.

Related (Financial) Literature

- Shiryaev, Xu and Zhou (2008), Quant Fin, studied the relative error between the selling price and global maximum,

$$\inf_{0\leq\nu\leq T} \mathbb{E}\left[\frac{S_\nu}{M_T}\right]$$

- "Bang-bang" strategy:
  - Sell at time T : $\mu > \frac{\sigma^2}{2}$
  - Sell at time 0 : $\mu \leq \frac{\sigma^2}{2}$

PDE Formulation

- The problem is
$$\inf_{0\le\nu\le T}\mathbb{E}[(S_\nu - M_T)^2]$$
- Not a standard optimal stopping problem, since M_T is not $\mathcal{F}_t$-adapted
- One more step:

$$\begin{aligned}\inf_{0\le\nu\le T}\mathbb{E}[(S_\nu - M_T)^2] &= \inf_{0\le\nu\le T}\mathbb{E}\Big\{\mathbb{E}[(S_\nu - M_T)^2 \mid \mathcal{F}_\nu]\Big\}\\ &= \inf_{0\le\nu\le T}\mathbb{E}\Big\{\phi(\nu, S_\nu, M_\nu)\Big\},\end{aligned}$$

where $\phi(t, S_t, M_t) = \mathbb{E}[(S_t - M_T)^2 \mid \mathcal{F}_t]$

PDE Formulation (Con't)

- Denote the value function

$$\psi(t, S_t, M_t) = \inf_{t \le \nu \le T} \mathbb{E}\Big\{ \phi(\nu, S_\nu, M_\nu) \mid \mathcal{F}_t \Big\}$$

- Dynamic programming equation (Variational Inequalities)

$$\begin{cases} \max\{-\partial_t \psi - \mathcal{L}^0 \psi, \psi - \phi\} = 0, & (t, S, M) \in D, \\ \partial_M \psi(t, M, M) = 0, \quad \psi(T, S, M) = (S - M)^2, \end{cases}$$

where $\mathcal{L}^0 = \frac{\sigma^2}{2} \partial_{SS} + \mu \partial_S$ and
$D = \{(t, S, M) : 0 < S < M,\ 0 \le t < T\}$.

The Obstacle Function $\phi(t, S, M)$

- Recall

$$\begin{aligned} \phi(t, S_t, M_t) &= \mathbb{E}[(S_t - M_T)^2 \mid \mathcal{F}_t] \\ &= S_t^2 - 2S_t\mathbb{E}[M_T \mid \mathcal{F}_t] + \mathbb{E}[M_T^2 \mid \mathcal{F}_t] \\ &=: S_t^2 - 2S_t\phi_1(t, S_t, M_t) + \phi_2(t, S_t, M_t), \end{aligned}$$

where $\phi_i(t, S_t, M_t) = \mathbb{E}[M_T^i \mid \mathcal{F}_t]$.

- Then, $\phi_i(t, S, M)$ satisfies

$$\begin{cases} -\partial_t\phi_i - \mathcal{L}^0\phi_i = 0, & (t, S, M) \in D, \\ \partial_M\phi_i(t, M, M) = 0, & \phi_i(T, S, M) = M^i. \end{cases}$$

Change of Variables

- Denote $\tau = T - t, \quad x = \ln \frac{M}{S}, \quad u_i(\tau, x) = \frac{\phi_i(t,S,M)}{S^i}$, $u(\tau, x) = \frac{\phi(t,S,M)}{S^2}$.
- Then, u_1 and u_2 satisfy

$$\begin{cases} \partial_\tau u_1 - \mathcal{L}_x^1 u_1 = 0 \ \text{ in } \Omega, \\ \partial_x u_1(\tau, 0) = 0, \ u_1(0, x) = e^x, \end{cases} \quad \begin{cases} \partial_\tau u_2 - \mathcal{L}_x^2 u_2 = 0 \ \text{ in } \Omega, \\ \partial_x u_2(\tau, 0) = 0, \ u_2(0, x) = e^{2x}, \end{cases}$$

where $\mathcal{L}_x^1 = \frac{\sigma^2}{2} \partial_{xx} - \left(\mu + \frac{\sigma^2}{2} \right) \partial_x + \mu$,
$\mathcal{L}_x^2 = \frac{\sigma^2}{2} \partial_{xx} - \left(\mu + \frac{3\sigma^2}{2} \right) \partial_x + (2\mu + \sigma^2)$,
$\Omega = (0, T] \times (0, \infty)$.

Change of Variables (con't)

- Denote $v(\tau, x) = \frac{\psi(t,S,M)-\phi(t,S,M)}{S^2}$
- Then, v satisfies

$$\begin{cases} \max\left\{\partial_\tau v - \mathcal{L}_x^2 v - H, v\right\} = 0 \ \text{ in } \Omega, \\ \partial_x v(\tau, 0) = 0, \ v(0, x) = 0, \end{cases}$$

where $H = \mathcal{L}_x^2 u - \partial_\tau u = 2\mu + \sigma^2 + 2\Big(\sigma^2 \partial_x u_1 - (\mu + \sigma^2) u_1\Big)$,

$\mathcal{L}_x^2 = \frac{\sigma^2}{2}\partial_{xx} - \left(\mu + \frac{3\sigma^2}{2}\right)\partial_x + (2\mu + \sigma^2)$.

- Define the selling region (the stopping region) as follows:

$$SR = \{(\tau, x) \in [0, \infty) \times (0, T] : v(\tau, x) = 0\}.$$

The Optimal Selling Strategy: Good Stock($\mu > 0$)

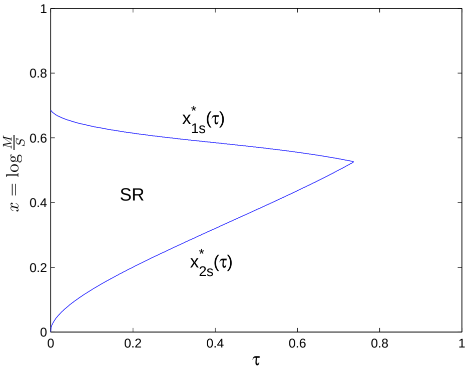


Figure: Two optimal selling boundaries. Parameter values used: $\mu = 0.045,\ \sigma = 0.3,\ T = 1.$

The Optimal Selling Strategy: Bad Stock $(-\sigma^2 \leq \mu \leq 0)$

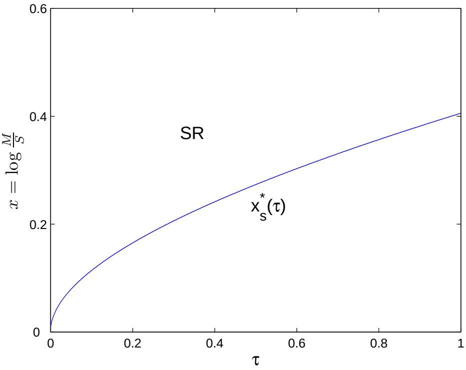


Figure: The monotonically increasing optimal selling boundary. Parameter values used: $\mu = -0.010, \ \sigma = 0.3, \ T = 1.$

The Optimal Selling Strategy: Very Bad Stock ($\mu < -\sigma^2$)

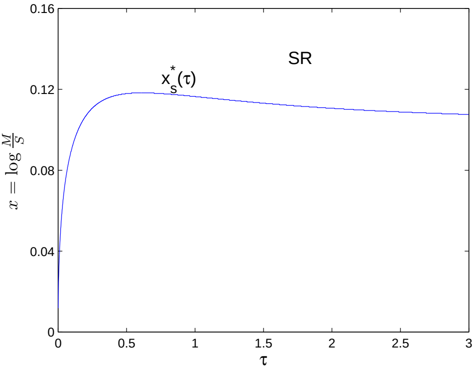


Figure: The nonmonotone optimal selling boundary. Parameter values used: $\mu = -0.032, \ \sigma = 0.4, \ T = 3.$

The Proof

- Recall

$$\begin{cases} \max\left\{\partial_\tau v - \mathcal{L}_x^2 v - H, v\right\} = 0 \ \text{ in } \Omega, \\ \partial_x v(\tau, 0) = 0, \ v(0, x) = 0, \end{cases}$$

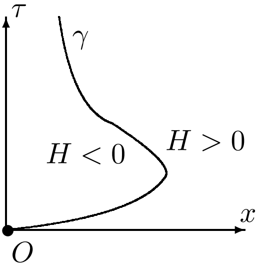
- So,

$$\begin{aligned} SR &= \{(\tau, x) : v = 0\} \\ &\subseteq \{(\tau, x) : \partial_\tau 0 - \mathcal{L}_x^2 0 - H \leq 0\} \\ &= \{(\tau, x) : H \geq 0\} \end{aligned}$$

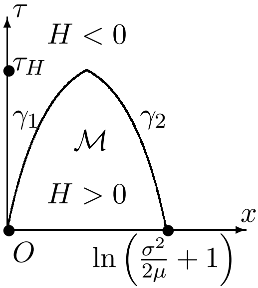
The Set $\{(\tau, x) : H \geq 0\}$

Lemma: Recall $H(\tau, x) = 2\mu + \sigma^2 + 2\Big(\sigma^2 \partial_x u_1 - (\mu + \sigma^2) u_1\Big)$.

- If $\mu \leq 0$, $\partial_x H > 0$;
- If $\mu \geq -\sigma^2$, $\partial_\tau H < 0$;
- If $\mu > 0$, $\partial_x H(\tau, x) = 0$ has at most one solution for any give $\tau > 0$;



case $\mu \leq 0$



case $\mu > 0$

The Main Results: $\mu \leq 0$

With the help of previous lemma, we have

- $\partial_x v \geq 0$ if $\mu \leq 0$;
- $\partial_\tau v \leq 0$ if $\mu \geq -\sigma^2$;
- These are due to

$$\partial_\tau v - \mathcal{L}_x^2 v = H, \text{in } \{(\tau, x) : v < 0\}.$$

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- Define $x_s^*(\tau) = \inf\{x \in (0, +\infty) : v(\tau, x) = 0, \forall \tau \in (0, T]\}$.

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- Define $x_s^*(\tau) = \inf\{x \in (0, +\infty) : v(\tau, x) = 0, \forall \tau \in (0, T]\}$.
- Thanks to $\partial_x v \geq 0$, we can show

$$\begin{aligned} SR &= \{(\tau, x) : v(\tau, x) = 0\} \\ &= \{(\tau, x) : x \geq x_s^*(\tau), 0 < \tau \leq T\}. \end{aligned}$$

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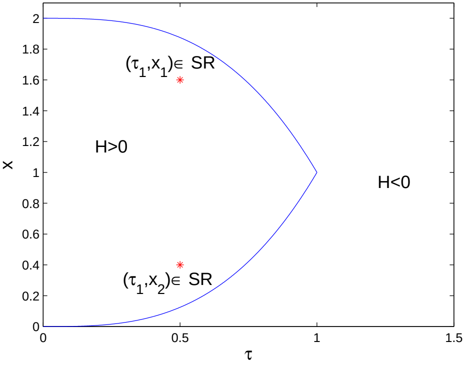
- $\partial_\tau v \leq 0$ gives the monotonicity of the free boundary.

The Main Results: $\mu > 0$

- With $\mu > 0$, we have $\partial_\tau v \leq 0$, which implies that $(\tau_2, x) \in SR$, if $(\tau_1, x) \in SR$ and $\tau_2 < \tau_1$.



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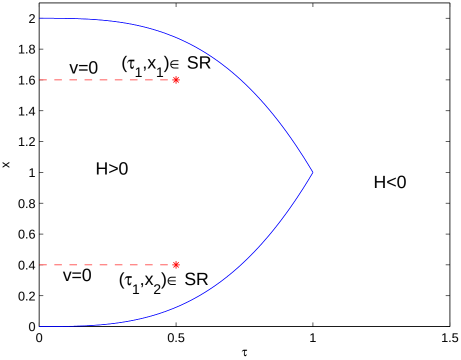

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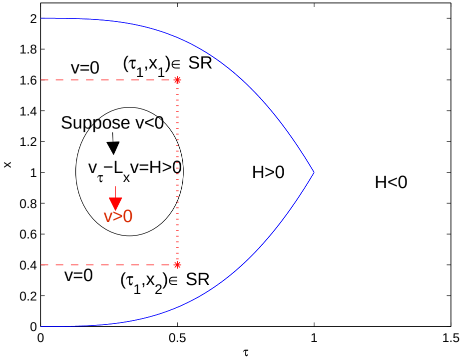

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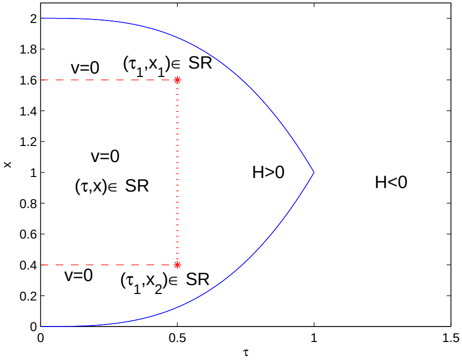

The Main Results: $\mu > 0$

- The sell region SR is connected;
- We can define

$$\begin{aligned} x_{1s}^*(\tau) &= \inf\{x \in [0, +\infty) : v(\tau, x) = 0\} \\ x_{2s}^*(\tau) &= \sup\{x \in [0, +\infty) : v(\tau, x) = 0\} \end{aligned}$$

- It is easy to show

$$SR = \{(\tau, x) : x_{1s}^*(\tau) \leq x \leq x_{2s}^*(\tau), 0 < \tau \leq \tau^*\}.$$

- The monotonicity of $x_{is}^*(\tau)$ follows by $\partial_\tau v \leq 0$.

Smoothness of the Free Boundary

- For $\mu \geq -\sigma^2$, we have $\partial_\tau v \leq 0$. So, one can easily establish the smoothness of $x_s^*(\tau)$ following Friedman (1975).
  - First, show $x_s^*(\tau) \in C^{3/4}((0,T])$
  - Then, show $x_s^*(\tau) \in C^1((0,T])$
  - By a bootstrap argument, show $x_s^*(\tau) \in C^\infty((0,T])$

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  - Then, show $x_s^*(\tau) \in C^1((0,T])$
  - By a bootstrap argument, show $x_s^*(\tau) \in C^\infty((0,T])$
- For $\mu < -\sigma^2$, we change of variables. Let $y = x - \mu/\sigma^2\tau$, and $V(\tau,y) = v(\tau,x)$.
  - Show $\partial_\tau V(\tau,y) \leq 0$ and $\partial_y V(\tau,y) \geq 0$
  - Apply Friedman (1975) to show smoothness of the corresponding $y_s^*(\tau)$, which gives the desired result

Conclusion

- We examine the optimal decision to sell a stock with the criteria of minimizing the square error between the selling price and the global maximum.
- For good stock, i.e. $\mu > 0$, the optimal selling boundary has two branches and only exists when time to maturity is not long enough.
- For bad stock, i.e. $\mu \leq 0$, the optimal selling boundary only has one branch and always exists.
- The smoothness of the free boundary is also established.

Thank you !