# Correcting the Optimal Stopping Bias in Monte Carlo Evaluation of Early Exercise Options

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SHARCNet/UWO

Bachelier Finance Society, Sat June 26 2010

# Optimal Stopping

Let $k$ take on the values $1, \ldots, N$ for a given $N$. Assume there is

1. a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,
2. a non-decreasing sequence of sub-sigma algebras $\mathcal{F}_k \subset \mathcal{F}$, and
3. a sequence of random variables (RVs) $P_k$ such that each $P_k$ is $\mathcal{F}_k$-measurabl

The discrete, finite-time-horizon, optimal-stopping problem is to determine

$$\sup_{\tau \in \mathcal{T}} \mathrm{E}\left[P_\tau\right] \tag{1}$$

where $\mathcal{T}$ is the class of all RVs $\Omega \to [1, \ldots, N]$ such that the event $[\tau = k] \in \mathcal{F}_k$
i.e., the class of all stopping times $\Omega \to [1, \ldots, N]$).

# Equivalently

$$H_k = \mathrm{E}\big[B_{k+1} \| \mathcal{F}_k\big], \tag{2}$$

$$B_k = \max(H_k, P_k), \tag{3}$$

where $B_N = P_N$ and $\mathrm{E}[B_1] = \sup_{\tau \in \mathcal{T}} \mathrm{E}[P_\tau]$.

# Tree notation

- Depth N, branching factor M
- Specify tree by choice at each node:
- $\mathbf{i} = (i_1, i_2, i_3, i_4)$
- Can specify node location by **i** and depth k

# Tree estimates (high biased)

$$H_k^{\mathbf{i}} = \mathrm{E}\left[B_{k+1}^{\mathbf{i}} \| \mathcal{F}_k\right],$$

$$B_N^{\mathbf{i}} = P_N^{\mathbf{i}}$$

$$B_k^{\mathbf{i}} = \max(H_k^{\mathbf{i}}, P_k^{\mathbf{i}}),$$

$$\tilde{H}_{k,M}^{\mathbf{i}+} = \frac{1}{M} \sum_{i_{k+1}=1}^{M} \tilde{B}_{k+1,M}^{\mathbf{i}+};$$

$$\tilde{B}_{k,M}^{\mathbf{i}+} = \max(\tilde{H}_{k,M}^{\mathbf{i}+}, P_k^{\mathbf{i}}),$$

where $\tilde{B}_{N,M}^{\mathbf{i}+} = P_N^{\mathbf{i}}.$

# Bias Correction

Let $\bar{H}^{\mathbf{i}\pm}_{k,M} = \mathrm{E}[\tilde{H}^{\mathbf{i}\pm}_{k,M} \| \mathcal{F}_k]$. A node of the high-biased stochastic tree at the $k^{\text{t}}$ stopping opportunity has a bias of $\bar{H}^{\mathbf{i}+}_{k,M} - H^{\mathbf{i}}_k = \mathrm{E}[\tilde{B}^{\mathbf{i}+}_{k+1} - B^{\mathbf{i}}_{k+1} \| \mathcal{F}_k]$. Expandin the inner terms gives

$$\mathrm{E}\left[\max(\tilde{H}^{\mathbf{i}+}_{k+1,M}, P^{\mathbf{i}}_{k+1}) - \max(H^{\mathbf{i}}_{k+1}, P^{\mathbf{i}}_{k+1}) \| \mathcal{F}_k\right].$$

Adding and subtracting $\mathrm{E}[\max(\bar{H}^{\mathbf{i}+}_{k+1,M}, P^{\mathbf{i}}_{k+1}) \| \mathcal{F}_k]$ splits this expression into local (Equation 11) and a global (Equation 12) component:

$$\mathrm{E}\left[\max(\tilde{H}^{\mathbf{i}+}_{k+1,M}, P^{\mathbf{i}}_{k+1}) - \max(\bar{H}^{\mathbf{i}+}_{k+1,M}, P^{\mathbf{i}}_{k+1}) \| \mathcal{F}_k\right] \quad \text{local} \quad (11$$

$$+ \mathrm{E}\left[\max(\bar{H}^{\mathbf{i}+}_{k+1,M}, P^{\mathbf{i}}_{k+1}) - \max(H^{\mathbf{i}}_{k+1}, P^{\mathbf{i}}_{k+1}) \| \mathcal{F}_k\right]. \quad \text{global} \quad (12$$

# Y = H - P

| | Held: $\tilde{Y}^{\mathbf{i+}}_{k+1,M} > 0$ | Exercised: $\tilde{Y}^{\mathbf{i+}}_{k+1,M} \leq 0$ |
|---|---|---|
| Should Hold: $\bar{Y}^{\mathbf{i+}}_{k+1,M} > 0$ | $0$ | $-\tilde{Y}^{\mathbf{i+}}_{k+1,M}$ |
| Should Exercise: $\bar{Y}^{\mathbf{i+}}_{k+1,M} \leq 0$ | $\tilde{Y}^{\mathbf{i+}}_{k+1,M}$ | $0$ |

# Expectation of Table

$$\mathrm{E}\Big[\mathbb{1}_{\bar{Y}^{\mathrm{i}+}_{k+1,M}>0}\mathbb{1}_{\tilde{Y}^{\mathrm{i}+*}_{k+1,M}\le 0}(-\tilde{Y}^{\mathrm{i}+*}_{k+1,M}) \;+\; \mathbb{1}_{\bar{Y}^{\mathrm{i}+}_{k+1,M}\le 0}\mathbb{1}_{\tilde{Y}^{\mathrm{i}+*}_{k+1,M}>0}(\tilde{Y}^{\mathrm{i}+*}_{k+1,M})\Big\|\mathcal{F}_k\Big], \quad (17)$$

$$\int_0^\infty \int \int_D |\tilde{y}^*| \frac{1}{\sqrt{\bar{v}/M}}\phi\left(\frac{\tilde{y}^* - \bar{y}}{\sqrt{\bar{v}/M}}\right) f_{\bar{Y}^{\mathrm{i}+}_{k+1,M},\bar{V}^{\mathrm{i}+}_{k+1,M}\|\mathcal{F}_k}(\bar{y},\bar{v})\, \mathrm{d}\tilde{y}^*\mathrm{d}\bar{y}\,\mathrm{d}\bar{v},$$

where $D = (0,\infty)\times(-\infty,0]\cup(-\infty,0]\times(0,\infty)$ and $\phi$ is the normal density.

# Substituting z = y√M

$$\frac{1}{M}\int_0^\infty \int \int_D |\tilde{z}^*| \, \frac{1}{\sqrt{\bar{v}}}\phi\left(\frac{\tilde{z}^* - \bar{z}}{\sqrt{\bar{v}}}\right) f_{\bar{Y}^{\mathrm{i}+}_{k+1,M},\bar{V}^{\mathrm{i}+}_{k+1,M}\|\mathcal{F}_k}\left(\frac{\bar{z}}{\sqrt{M}},\bar{v}\right) \mathrm{d}\tilde{z}^* \mathrm{d}\bar{z}\, \mathrm{d}\bar{v}.$$

# Replace z with z* for one integration

$$\int_0^\infty \int_{-\infty}^\infty |\tilde{y}^*| \, \Phi\left(\frac{-|\tilde{y}^*|}{\sqrt{\bar{v}/M}}\right) f_{\tilde{Y}^{\mathrm{i}+*}_{k+1,M}, \bar{V}^{\mathrm{i}+}_{k+1,M} \| \mathcal{F}_k}(\tilde{y}^*, \bar{v}) \, \mathrm{d}\tilde{y}^* \, \mathrm{d}\bar{v},$$

which, in expectation form, is

$$\mathrm{E}\left[ |\tilde{Y}^{\mathrm{i}+*}_{k+1,M}| \, \Phi\left(\frac{-|\tilde{Y}^{\mathrm{i}+*}_{k+1,M}|}{\sqrt{\bar{V}^{\mathrm{i}+}_{k+1,M}/M}}\right) \Big\| \mathcal{F}_k \right]. \tag{19}$$

# Bias Correction

$$\mathrm{E}\Big[\max(\tilde{H}^{\mathbf{i}+}_{k+1,M}, P^{\mathbf{i}}_{k+1}) - \max(\bar{H}^{\mathbf{i}+}_{k+1,M}, P^{\mathbf{i}}_{k+1})\Big\|\mathcal{F}_k\Big] \quad \text{local} \tag{23}$$

$$+\,\mathrm{E}\Big[\max(\bar{H}^{\mathbf{i}+}_{k+1,M}, P^{\mathbf{i}}_{k+1}) - \max(H^{\mathbf{i}}_{k+1}, P^{\mathbf{i}}_{k+1})\Big\|\mathcal{F}_k\Big] \quad \text{global} \tag{24}$$

$$-\,\mathrm{E}\left[\left|\tilde{H}^{\mathbf{i}+}_{k+1,M} - P^{\mathbf{i}}_{k+1,M}\right| \Phi\left(\frac{-|\tilde{H}^{\mathbf{i}+}_{k+1,M} - P^{\mathbf{i}}_{k}|}{\sqrt{\tilde{V}^{\mathbf{i}+}_{k+1,M}/M}}\right)\right\|\mathcal{F}_k\Bigg] \quad \text{correctn} \tag{25}$$

# Fixing Local  Also Fixes Global

**Global Stage k error bounded**

**Formula:**

- It is bounded by stage k+1 bias
- Jensen +
- |max(x,y) – max(u,v)|

  ≤ |x-u| + |y-v|
- Final stage bias = 0
- So can ignore stage k
- bias

$$
\begin{aligned}
&\left|\mathrm{E}\left[\max(\bar{H}^{\mathbf{i}\pm}_{k+1,M}, P^{\mathbf{i}}_{k+1}) - \max(H^{\mathbf{i}}_{k+1}, P^{\mathbf{i}}_{k+1}) \,\|\, \mathcal{F}_k\right]\right| \\
&\quad < \mathrm{E}\left[|\bar{H}^{\mathbf{i}\pm}_{k+1,M} - H^{\mathbf{i}}_{k+1}| \,\|\, \mathcal{F}_k\right] = \mathrm{E}\left[|\mathrm{E}[\tilde{H}^{\mathbf{i}\pm}_{k+1,M} \,\|\, \mathcal{F}_{k+1}] - H^{\mathbf{i}}_{k+1}| \,\|\, \mathcal{F}_k\right],
\end{aligned}
$$

# Moments and Bounds

**The idea**

- $W^p$ is $p^{th}$ absolute moment of B
- $W^p$ is bounded above by $U^p$
- Makes sense for $U^1$, pre-visible option value clearly higher than $B^1$.
- Can also prove for $p > 1$

**The bound**

$$\tilde{U}^{\mathbf{i},p}_{k,M} = \frac{1}{M}\sum_{i_{k+1}=1}^{M} \cdots \frac{1}{M}\sum_{i_N=1}^{M} \max_{\tau\in[k,\ldots,N]} |P^{\mathbf{i}}_{\tau}|^p,$$

$$U^{\mathbf{i},p}_{k} = \mathrm{E}\Big[\max_{\tau\in[k,\ldots,N]} |P^{\mathbf{i}}_{\tau}|^p \Big| \mathcal{F}_k\Big],$$

# Moment Bounding Lemma

**Lemma 2.1** (Bounds). *For all* $\mathbf{i}$, $1 \leq p$, *and* $k$,

1. $|P_k^{\mathbf{i}}|^p \leq_1 U_k^{\mathbf{i},p}$
2. $|H_k^{\mathbf{i}}|^p \leq_1 U_k^{\mathbf{i},p}$ *and* $|\tilde{H}_{k,M}^{\mathbf{i}+}|^p \leq \tilde{U}_{k,M}^{\mathbf{i},p}$,
3. $|B_k^{\mathbf{i}}|^p \leq_1 U_k^{\mathbf{i},p}$ *and* $|\tilde{B}_{k,M}^{\mathbf{i}+}|^p \leq \tilde{U}_{k,M}^{\mathbf{i},p}$, *and*
4. $|V_k^{\mathbf{i}}|^p \leq_1 U_k^{\mathbf{i},2p}$ *and* $|\tilde{V}_{k,M}^{\mathbf{i}+}|^p \leq (M/(M-1))^p \tilde{U}_{k,M}^{\mathbf{i},2p}$.

# Bounds Consistency Lemma

**Lemma 2.2** (Bounds consistency). *For all* $\mathbf{i}$, $1 \leq q \leq p$, $k$, *and* $\mathcal{G} \subset \mathcal{F}_k$, *if* $U^p < \infty$, *then*

1. $\tilde{U}^{\mathbf{i},q}_{k,M}$ *and* $U^{\mathbf{i},q}_k$ *are integrable,*
2. $\tilde{U}^{\mathbf{i},q}_{k,M} \rightarrow_1 U^{\mathbf{i},q}_k$ *and* $1/M \sum_{i_k=1}^{M} \tilde{U}^{\mathbf{i},q}_{k,M} \rightarrow_1 \mathrm{E}[U^{\mathbf{i},q}_k \| \mathcal{F}_{k-1}]$, *and*
3. $\mathrm{E}[\tilde{U}^{\mathbf{i},q}_{k,M} \| \mathcal{G}] =_1 \mathrm{E}[U^{\mathbf{i},q}_k \| \mathcal{G}]$.

# Estimator Consistency Lemma

**Theorem 2.3** (Estimator consistency). *For all* $\mathbf{i}$, $2 \leq p$, $k$, *and* $\mathcal{G} \subset \mathcal{F}_k$, *if* $U^p <$ $\infty$, *then*[6]

1. $P_k^{\mathbf{i}}$, $H_k^{\mathbf{i}}$, $B_k^{\mathbf{i}}$, $V_k^{\mathbf{i}}$, $\tilde{H}_{k,M}^{\mathbf{i}+}$, $\tilde{B}_{k,M}^{\mathbf{i}+}$, *and* $\tilde{V}_{k,M}^{\mathbf{i}+}$ *are integrable,*
2. $\tilde{H}_{k,M}^{\mathbf{i}+} \to_1 H_k^{\mathbf{i}}$, $\tilde{B}_{k,M}^{\mathbf{i}+} \to_1 B_k^{\mathbf{i}}$, *and* $\tilde{V}_{k,M}^{\mathbf{i}+} \to_1 V_k^{\mathbf{i}}$, *and*
3. $\mathrm{E}[\tilde{H}_{k,M}^{\mathbf{i}+} \| \mathcal{G}] \to_1 \mathrm{E}[H_k^{\mathbf{i}} \| \mathcal{G}]$, $\mathrm{E}[\tilde{B}_{k,M}^{\mathbf{i}+} \| \mathcal{G}] \to_1 \mathrm{E}[B_k^{\mathbf{i}} \| \mathcal{G}]$, *and* $\mathrm{E}[\tilde{V}_{k,M}^{\mathbf{i}+} \| \mathcal{G}] \to_1$ $\mathrm{E}[V_k^{\mathbf{i}} \| \mathcal{G}]$.

# Moment Consistency Lemma

**Lemma 2.4** (Moment consistency). *For all* $\mathbf{i}$, $1 \leq q \leq p$, $k$, *and* $\mathcal{G} \subset \mathcal{F}_k$, *if* $U^p < \infty$, *then*

1. $\bar{W}^{\mathbf{i}+,q}_{k,M}$ *is integrable,*

2. $\bar{W}^{\mathbf{i}+,q}_{k,M} \longrightarrow_1 W^{\mathbf{i},q}_k$, *and*

3. $\mathrm{E}[\bar{W}^{\mathbf{i}+,q}_{k,M} \| \mathcal{G}] \longrightarrow_1 \mathrm{E}[W^{\mathbf{i},q}_k \| \mathcal{G}]$.

# Formal Justification

In this section, we prove the high-bias corrected estimators have a reduced order of bias if the fourth moment of the rewards is finite and there is some region about the stopping boundary for which the sampling density is continuous and the variance is bound.

Bias

$$\mathrm{E}\left[\max(\tilde{H}^{\mathbf{i}+}_{k+1,M}, P^{\mathbf{i}}_{k+1}) - \max(\bar{H}^{\mathbf{i}+}_{k+1,M}, P^{\mathbf{i}}_{k+1}) \| \mathcal{F}_k\right], \text{ local} \quad (29)$$

$$\mathrm{E}\left[\max(\bar{H}^{\mathbf{i}+}_{k+1,M}, P^{\mathbf{i}}_{k+1}) - \max(H^{\mathbf{i}}_{k+1}, P^{\mathbf{i}}_{k+1}) \| \mathcal{F}_k\right], \text{ global} \quad (30)$$

$$\mathrm{E}\left[-|\tilde{H}^{\mathbf{i}+}_{k+1,M} - P^{\mathbf{i}}_{k+1,M}|\, \Phi\left(\frac{-|\tilde{H}^{\mathbf{i}+}_{k+1,M} - P^{\mathbf{i}}_{k+1}|}{\sqrt{\tilde{V}^{\mathbf{i}+}_{k+1,M}/M}}\right) \Big\| \mathcal{F}_k\right]. \text{ correction} \quad (31)$$

# We will manipulate the local error

**The Goal**

- to show that the local error term – the bias correction term is o(1/M)
- f(M) is o(1/M) means:
  $\lim_{M \to \infty} Mf(M) = 0$

**Equivalent Local Bias**

$$\mathrm{E}\big[\mathbb{1}_{\bar{Y}^{\mathbf{i}+}_{k+1,M}>0}\mathbb{1}_{\tilde{Y}^{\mathbf{i}+}_{k+1,M}\leq 0}(-\tilde{Y}^{\mathbf{i}+}_{k+1,M}) +$$

$$\mathbb{1}_{\bar{Y}^{\mathbf{i}+}_{k+1,M}\leq 0}\mathbb{1}_{\tilde{Y}^{\mathbf{i}+}_{k+1,M}>0}(\tilde{Y}^{\mathbf{i}+}_{k+1,M})\|\mathcal{F}_k\big],$$

# Want to exchange for normal (*)

**Theorem 3.3** (Non-uniform Berry-Esseen). *If $X_1, X_2, \ldots, X_M$ are iid RVs with*

1. $E[X_i] = 0$,
2. $\mathrm{E}[|X_i|^2] = V > 0$, *and*
3. $\mathrm{E}[|X_i|^3] = W < \infty$,

*then, for some universal constant $C$,*

$$\left| \mathrm{P}\left[ \frac{1}{\sqrt{VM}} \sum_{i=1}^{M} X_i \leq x \right] - \Phi(x) \right| \leq C \frac{W}{V\sqrt{VM}(1+|x|)^3}.$$

# Y* is 'normal' version of Y

**Theorem 3.4** (Exchange of $\tilde{Y}^{\mathbf{i}+}_{k+1,M}$ and $\tilde{Y}^{\mathbf{i}+*}_{k+1,M}$ in bias). *For all* $\mathbf{i}$, *k, and* $\mathcal{G} \subset \mathcal{F}_k$, *if*

1. $U^3 < \infty$,
2. $\mathrm{E}[|\bar{Y}^{\mathbf{i}+}_{k+1,M} - Y^{\mathbf{i}}_{k+1}| \| \mathcal{G}] =_1 O(1/M)$, *and*
3. *there is some* $\mathcal{G}$*-measurable* $\epsilon > 0$ *such that* $f_{Y^{\mathbf{i}}_{k+1} \| \mathcal{F}_k}(y)$ *exists for and has a* $\mathcal{G}$*-integrable bound on* $[|y| \leq \epsilon]$ *a.e.,*

*then the* $\mathcal{G}$*-conditional-expected absolute difference between Equations 32 and 33 is* $o(1/M)$ *a.e.*

local error ✓ (33) up to $o(1/M)$

$$\mathrm{E}\left[\mathbb{1}_{\tilde{Y}^{\mathbf{i}+}_{k+1,M}>0}\mathbb{1}_{\tilde{Y}^{\mathbf{i}+*}_{k+1,M}<0}\left(-\tilde{Y}^{\mathbf{i}+*}_{k+1,M}\right) + \mathbb{1}_{\tilde{Y}^{\mathbf{i}+}_{k+1,M}<0}\mathbb{1}_{\tilde{Y}^{\mathbf{i}+*}_{k+1,M}>0}\left(\tilde{Y}^{\mathbf{i}+*}_{k+1,M}\right)\|\mathcal{F}_k\right] \quad (33)$$

# Local – Correction o(1/M)

- We need to show
- E[|Local – (34)|] is o(1/M)

$$\mathrm{E}\left[ |\tilde{Y}^{\mathbf{i}+*}_{k+1,M}| \, \Phi\left( \frac{-|\tilde{Y}^{\mathbf{i}+*}_{k+1,M}|}{\sqrt{\bar{V}^{\mathbf{i}+}_{k+1,M}/M}} \right) \middle\| \mathcal{F}_k \right] \tag{34}$$

# It is!

**Theorem 3.5** (Exchange of $\bar{Y}^{\mathbf{i}+}_{k+1,M}$ and $\bar{V}^{\mathbf{i}+}_{k+1,M}$ for $Y^{\mathbf{i}}_{k+1}$ and $V^{\mathbf{i}}_{k+1}$ in difference). *For all* $\mathbf{i}$, $k$, *and* $\mathcal{G} \subset \mathcal{F}_k$, *if*

1. $U^2 < \infty$,
2. $\mathrm{E}[|\bar{Y}^{\mathbf{i}+}_{k+1,M} - Y^{\mathbf{i}}_{k+1}| \| \mathcal{G}] =_1 o(1/M)$, *and*
3. *there is some* $\mathcal{G}$*-measurable* $\epsilon > 0$ *such that* $f_{Y^{\mathbf{i}}_{k+1} \| \mathcal{F}_k}(y)$ *exists for and has a* $\mathcal{G}$*-integrable bound on* $[|y| \leq \epsilon]$ *a.e.,*

*then the* $\mathcal{G}$*-conditional expected absolute difference in exchanging* $\bar{Y}^{\mathbf{i}+}_{k+1,M}$ *for* $Y^{\mathbf{i}}_{k+1}$ *and* $\bar{V}^{\mathbf{i}+}_{k+1,M}$ *for* $V^{\mathbf{i}}_{k+1}$ *in*

$$\mathrm{E}\left[\frac{\bar{V}^{\mathbf{i}+}_{k+1,M}/M}{\sqrt{\bar{V}^{\mathbf{i}+}_{k+1,M}/M}}\,\phi\left(\frac{\bar{Y}^{\mathbf{i}+}_{k+1,M}}{\sqrt{2\bar{V}^{\mathbf{i}+}_{k+1,M}/M}}\right) - |\bar{Y}^{\mathbf{i}+}_{k+1,M}|\,\Phi\left(\frac{-|\bar{Y}^{\mathbf{i}+}_{k+1,M}|}{\sqrt{2\bar{V}^{\mathbf{i}+}_{k+1,M}/M}}\right) \Bigg\| \mathcal{F}_k\right]$$

*is* $o(1/M)$ *a.e.*

# Separating Scales

**Theorem 3.6** (Separation of scales). *For all* $\mathbf{i}$, $k$, *and* $\mathcal{G} \subset \mathcal{F}_k$, *if*

1. $U^3 < \infty$,
2. $\mathrm{E}[|\bar{Y}^{\mathbf{i}+}_{k+1,M} - Y^{\mathbf{i}}_{k+1}||\mathcal{G}] =_1 o(1/M)$, *and*
3. *there is some* $\mathcal{G}$*-measurable* $\epsilon > 0$ *such that*
   (a) $V^{\mathbf{i}}_{k+1}$ *has a* $\mathcal{G}$*-measurable bound on* $[|Y^{\mathbf{i}}_{k+1}| \leq \epsilon]$ *a.e.*,
   (b) $f_{Y^{\mathbf{i}}_{k+1}||\mathcal{F}_k}(y)$ *exists for and has a* $\mathcal{G}$*-integrable bound on* $[|y| \leq \epsilon]$ *a.e. and is continuous in* $y$ *about zero, and*

# Separating Scales II

*(c)* $f^{ac}_{Y^{\mathbf{i}}_{k+1},V^{\mathbf{i}}_{k+1}\|\mathcal{F}_k}(y,v)$, $f^{s_j}_{Y^{\mathbf{i}}_{k+1}\|\mathcal{F}_k}(y)$, *and* $v^j_{\mathcal{F}_k}(y)$ *exist such that*

$$\begin{aligned}
&\mathrm{E}\left[\mathbb{1}_{|Y^{\mathbf{i}}_{k+1}|\leq\epsilon}\, g(Y^{\mathbf{i}}_{k+1},V^{\mathbf{i}}_{k+1})\|\mathcal{F}_k\right] \\
&\quad =_1 \int\int_{-\epsilon}^{\epsilon} g(y,v)\; f^{ac}_{Y^{\mathbf{i}}_{k+1},V^{\mathbf{i}}_{k+1}\|\mathcal{F}_k}(y,v)\,\mathrm{d}y\,\mathrm{d}v \\
&\qquad + \sum_j \int_{-\epsilon}^{\epsilon} g(y, v^j_{\mathcal{F}_k}(y))\; f^{s_j}_{Y^{\mathbf{i}}_{k+1}\|\mathcal{F}_k}(y)\,\mathrm{d}y
\end{aligned}$$

*for all integrable* $g(Y^{\mathbf{i}}_{k+1},V^{\mathbf{i}}_{k+1})$ *and are continuous in* $y$ *about zero,*

*then the* $\mathcal{G}$*-conditional-expected absolute difference between Equations 33 and 34 is* $o(1/M)$ *a.e.*

# Y*, Vbar not observable

Equation 34 cannot be used directly as $\tilde{Y}^{\mathbf{i}+*}_{k+1,M}$ is not observable in the context of a simulation. Using $\tilde{Y}^{\mathbf{i}+}_{k+1,M}$ instead gives

$$\mathrm{E}\left[ |\tilde{Y}^{\mathbf{i}+}_{k+1,M}| \, \Phi\left( \frac{-|\tilde{Y}^{\mathbf{i}+}_{k+1,M}|}{\sqrt{\bar{V}^{\mathbf{i}+}_{k+1,M}/M}} \right) \Big\| \mathcal{F}_k \right], \tag{35}$$

which still contains the unobservable $\bar{V}^{\mathbf{i}+}_{k+1,M}$. Replacing it wtih $\tilde{V}^{\mathbf{i}+}_{k+1,M}$ gives

$$\mathrm{E}\left[ |\tilde{Y}^{\mathbf{i}+}_{k+1,M}| \, \Phi\left( \frac{-|\tilde{Y}^{\mathbf{i}+}_{k+1,M}|}{\sqrt{\tilde{V}^{\mathbf{i}+}_{k+1,M}/M}} \right) \Big\| \mathcal{F}_k \right]. \tag{36}$$

# It’s OK to swap Y* and Y

**Theorem 3.7** (Exchange of $\tilde{Y}^{\mathbf{i}+}_{k+1,M}$ and $\tilde{Y}^{\mathbf{i}+*}_{k+1,M}$ in correction). *For all* $\mathbf{i}$, $k$, *and* $\mathcal{G} \subset \mathcal{F}_k$, *if*

1. $U^3 < \infty$,
2. $\mathrm{E}[|\bar{Y}^{\mathbf{i}+}_{k+1,M} - Y^{\mathbf{i}}_{k+1}| \| \mathcal{G}] =_1 O(1/M)$, *and*
3. *there is some* $\mathcal{G}$*-measurable* $\epsilon > 0$ *such that* $f_{Y^{\mathbf{i}}_{k+1} \| \mathcal{F}_k}(y)$ *exists for and has a* $\mathcal{G}$*-integrable bound on* $[|y| \leq \epsilon]$ *a.e.*,

*then the* $\mathcal{G}$*-conditional-expected absolute difference between Equations 34 and 35 is* $o(1/M)$ *a.e.*

# It's OK to swap Vbar for Vtilde

**Theorem 3.8** (Exchange of $\bar{V}^{\mathbf{i}+}_{k+1,M}$ and $\tilde{V}^{\mathbf{i}+}_{k+1,M}$ in correction). *For all* $\mathbf{i}$, $k$, *and* $\mathcal{G} \subset \mathcal{F}_k$, *if*

1. $U^4 < \infty$,
2. $\mathrm{E}[|\bar{Y}^{\mathbf{i}+}_{k+1,M} - Y^{\mathbf{i}}_{k+1}| \| \mathcal{G}] =_1 O(1/M)$, *and*
3. *there is some* $\mathcal{G}$*-measurable* $\epsilon > 0$ *such that* $f_{Y^{\mathbf{i}}_{k+1} \| \mathcal{F}_k}(y)$ *exists for and has a* $\mathcal{G}$*-integrable bound on* $[|y| \leq \epsilon]$ *a.e.*,

*then the* $\mathcal{G}$*-conditional-expected absolute difference between Equations 35 and 36 is* $o(1/M)$ *a.e.*

# The final theorem I

**Theorem 3.9** (Order after bias correction). *For all* $\mathbf{i}$, $k$, *and* $\mathcal{G} \subset \mathcal{F}_k$, *if*

1. $U^4 < \infty$,
2. $\mathrm{E}[|\bar{Y}^{\mathbf{i}+}_{k+1,M} - Y^{\mathbf{i}}_{k+1}||\mathcal{G}] =_1 o(1/M)$, *and*
3. *there is some* $\mathcal{G}$*-measurable* $\epsilon > 0$ *such that*

   *(a)* $V^{\mathbf{i}}_{k+1}$ *has a* $\mathcal{G}$*-measurable bound on* $[|Y^{\mathbf{i}}_{k+1}| \leq \epsilon]$ *a.e.*,

# The final theorem II

(b) $f_{Y^{\mathbf{i}}_{k+1}\|\mathcal{F}_k}(y)$ *exists for and has a* $\mathcal{G}$*-integrable bound on* $[|y| \leq \epsilon]$ *a.e. and is continuous in* $y$ *about zero, and*

(c) $f^{ac}_{Y^{\mathbf{i}}_{k+1},V^{\mathbf{i}}_{k+1}\|\mathcal{F}_k}(y,v)$, $f^{S_j}_{Y^{\mathbf{i}}_{k+1}\|\mathcal{F}_k}(y)$, *and* $v^j_{\mathcal{F}_k}(y)$ *exist such that*

$$\mathrm{E}\big[\mathbb{1}_{|Y^{\mathbf{i}}_{k+1}|\leq\epsilon}\, g(Y^{\mathbf{i}}_{k+1},V^{\mathbf{i}}_{k+1})\|\mathcal{F}_k\big]$$
$$=_1 \int\int_{-\epsilon}^{\epsilon} g(y,v)\, f^{ac}_{Y^{\mathbf{i}}_{k+1},V^{\mathbf{i}}_{k+1}\|\mathcal{F}_k}(y,v)\, \mathrm{d}y\, \mathrm{d}v$$
$$+\sum_j \int_{-\epsilon}^{\epsilon} g(y, v^j_{\mathcal{F}_k}(y))\, f^{S_j}_{Y^{\mathbf{i}}_{k+1}\|\mathcal{F}_k}(y)\, \mathrm{d}y$$

*for all integrable* $g(Y^{\mathbf{i}}_{k+1},V^{\mathbf{i}}_{k+1})$ *and are continuous in* $y$ *about zero,*

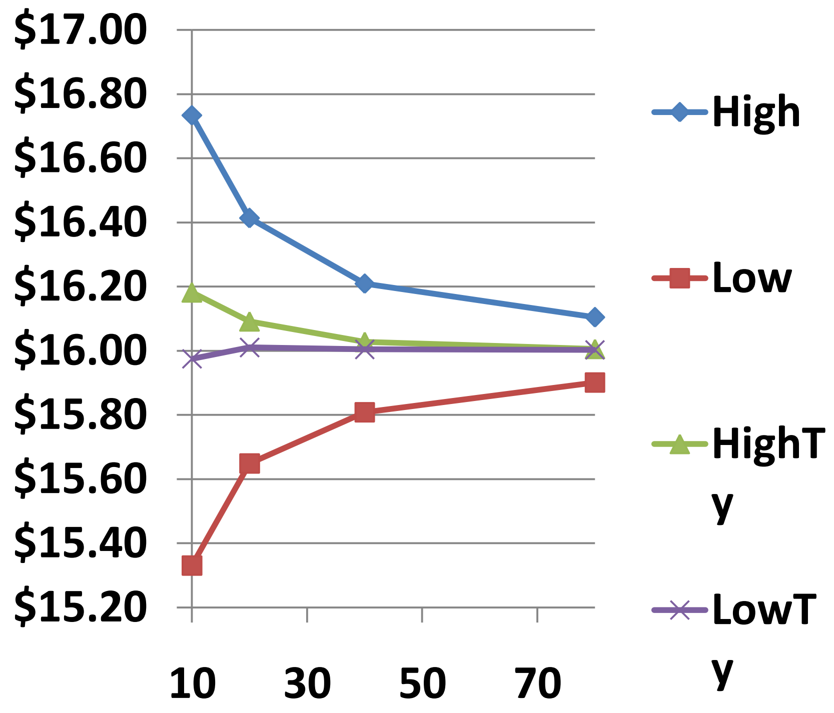
*then* $\mathrm{E}[|\bar{Y}^{\mathbf{i}+}_{k,M} - Y^{\mathbf{i}}_k|\|\mathcal{G}] =_1 o(1/M)$.

# Some results

**The parameters**

- Max call on 5 uncorrelated stocks
- $C(S^1_T, S^2_T, S^3_T, S^4_T, S^5_T, T) = \max(S^1_T - K^1, S^2_T - K^2, S^3_T - K^3, S^4_T - K^4, S^5_T - K^5, 0)$
- For all stocks: $S_0 = 90$, $K = 100$, $\sigma = 20\%$, $q = 10\%$
- $N = 3$, $T = 3$, $r = 5\%$

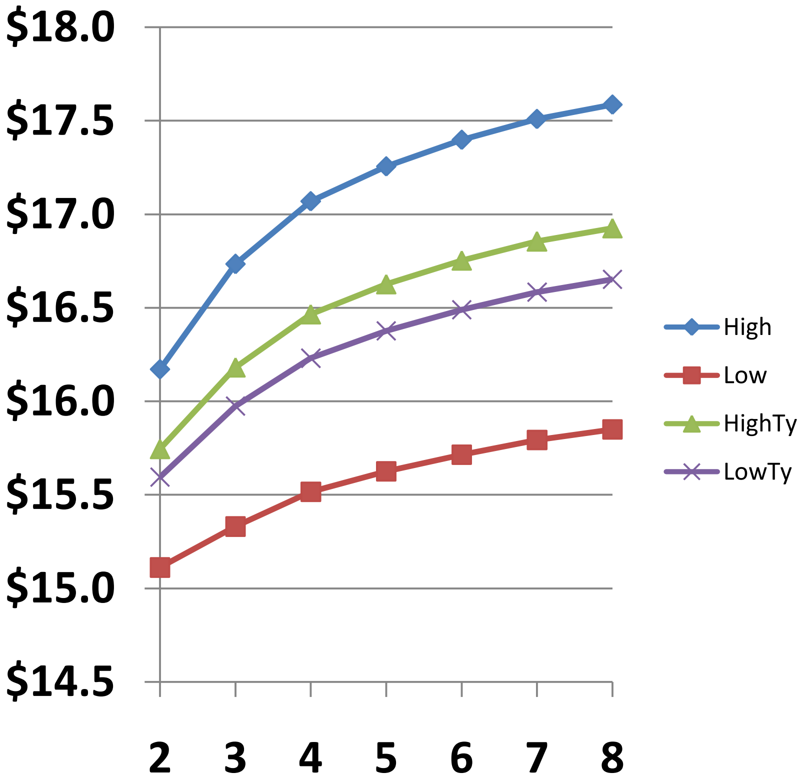
**Price vs. Branching Factor**



# Works with more Exercise opps!

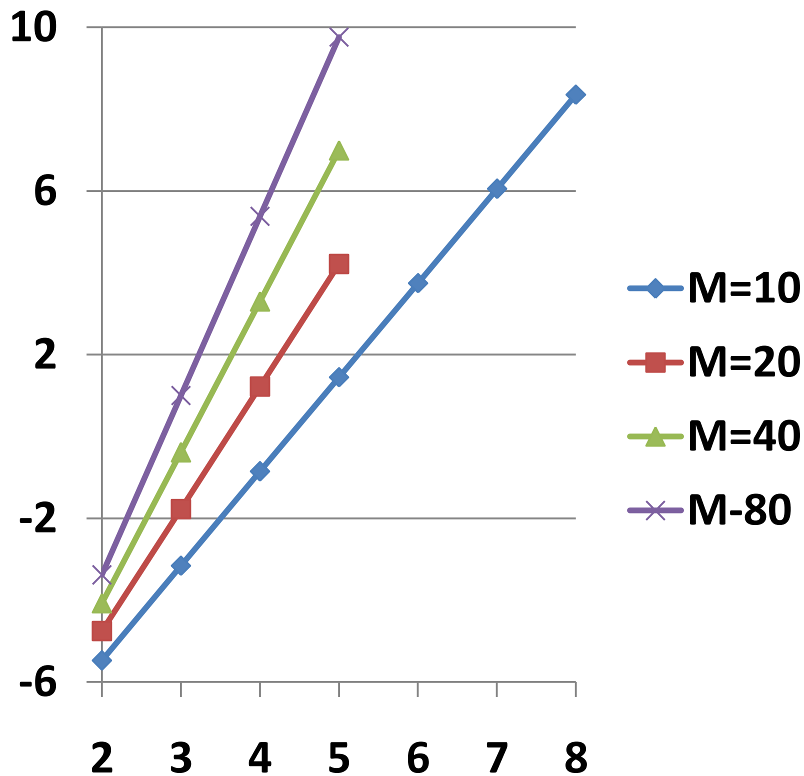
**Price vs Exercise Opps.**

**(M=10, same parameters)**




**Log (time in hours) vs. ExOpp**

**(M=10, same parameters)**



# Thanks for listening

- Any Questions?