Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Quadratic Variance Swap Models
Theory and Evidence

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Swiss Finance Institute
Ecole Polytechnique Fédérale de Lausanne

(joint with Elise Gourier and Loriano Mancini)

6th World Congress of the Bachelier Finance Society
Toronto, 26 June 2010

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

SWISS FINANCE INSTITUTE AT EPFL

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

SWISSQUOTE CONFERENCE

English   EPFL > CDM > Swiss Finance Ins... > EVENTS > Swissquote Confer...

Swissquote Conference on Interest Rate and Credit Risk

Date: Thursday 28th and Friday 29th October 2010

Venue: Ecole Polytechnique Fédérale de Lausanne (EPFL), Switzerland

Scope: Interest rate and credit risk constitute the major risk sources for banks, insurance companies, and other financial institutions. This conference brings together different perspectives and tools for the valuation and managing of these risks. It addresses academics and practitioners, and shall foster the interaction between individuals and across institutions.

Speakers include:

- Tomas Björk (Stockholm School of Economics)
- Giovanni Cesari (UBS)
- Pierre Collin-Dufresne (Columbia University)
- Rama Cont (University Paris 6)
- Jean-Pierre Danthine (Swiss National Bank)
- Mark Davis (Imperial College)
- David Lando (Copenhagen Business School)
- Alex Lipton (Bank of America Merrill Lynch and Imperial College)
- Dilip Madan (University of Maryland)
- Fabio Mercurio (Bloomberg)
- Antoon Pelsser (Maastricht University)
- Kenneth Singleton (Stanford University)

Scientific Committee: Tomas Björk (Stockholm School of Economics), Mark Davis (Imperial College), Damir Filipovic (EPFL), David Lando (Copenhagen Business School), Wolfgang Runggaldier (University of Padova), Kenneth Singleton (Stanford University)

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Outline

1 Variance Swaps

2 Quadratic Term Structures

3 Quadratic (Pearson) Diffusions

4 Model Estimation

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Realized Variance

- Filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$
- Price process (e.g. S&P 500 index): semimartingale S
- Annualized realized variance on $t = t_0 < \cdots < t_k = T$:

$$\mathrm{RV}_{t,T}^2 = \frac{k}{n} \sum_{i=1}^{k} \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2$$

  where $n = $ number of trading days per year

- Approximate by quadratic variation, for $k \to \infty$:

$$\sum_{i=1}^{k} \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 \to [\log S]_T - [\log S]_t$$

- Justified for daily sampling

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Variance Swaps

- A variance swap initiated at t with maturity T pays

$$\mathrm{RV}_{t,T}^2 - \mathrm{VS}_{t,T}^2$$

- $\mathrm{VS}_{t,T}^2 =$ variance swap rate fixed at t
- Assume deterministic risk-free rate r:

$$\mathrm{VS}_{t,T}^2 = \frac{1}{T-t} \mathbb{E}_{\mathbb{Q}}\left[[\log S]_T - [\log S]_t \mid \mathcal{F}_t\right]$$

- Provides hedging instrument against volatility increases, which often coincide with drops of stock prices

S-Characteristics

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

- Assume

$$\frac{dS_t}{S_{t-}} = r_t \, dt + \sigma_t \, dW_t + \int_{\mathbb{R}} (\mathrm{e}^x - 1) \left(\mu(dt, dx) - \nu_t(dx)dt \right)$$

- In particular,

$$\Delta \log S_t = \int_{\mathbb{R}} x \, \mu(dt, dx)$$

- Hence

$$[\log S]_T - [\log S]_t = \int_t^T \sigma_s^2 \, ds + \int_t^T \int_{\mathbb{R}} x^2 \, \mu(ds, dx)$$

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Model-Free Replication

- Britten–Jones and Neuberger [2], Jiang and Tian [11], Carr and Wu [4], a.o. showed:

$$
\begin{aligned}
&[\log S]_T - [\log S]_t \\
&= \int_0^{F_t} \frac{2}{K^2}(K - S_T)^+ \, dK + \int_{F_t}^\infty \frac{2}{K^2}(S_T - K)^+ \, dK \\
&\quad + 2 \int_t^T \left(\frac{1}{F_{s-}} - \frac{1}{F_t} \right) dF_s \\
&\quad - 2 \int_t^T \int_{\mathbb{R}} \left(e^x - 1 - x - \frac{x^2}{2} \right) \mu(ds, dx)
\end{aligned}
$$

- $F_t = S_t / P(t, T) = T$-futures price of S

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Model-Free Valuation

- Taking $\mathbb{Q}$-expectation:

$$VS_{t,T}^2 = \frac{2}{T-t} \int_0^\infty \frac{\Theta_t(K,T)}{P(t,T)K^2} \, dK + \epsilon$$

with error term

$$\epsilon = -\frac{2}{T-t} \mathbb{E}_{\mathbb{Q}} \left[\int_t^T \int_{\mathbb{R}} \left(e^x - 1 - x - \frac{x^2}{2} \right) \nu_s(dx) ds \mid \mathcal{F}_t \right]$$

- $\Theta_t(K,T) = $ out-of-the-money option price

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

VIX

- Chicago Board Options Exchange Volatility Index (VIX)

$$\mathrm{VIX}_t = \sqrt{VS^2_{t,t+30\ \mathrm{days}}} \times 100\,[\%]$$

  calculated as weighted blend of options on S&P 500 index
- Introduced in 1993, revised in 2003
- Industry benchmark for market volatility

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Term-Structure Models

- OTC variance swaps at many different maturities available
  ⇒ design and estimate term-structure models of variance
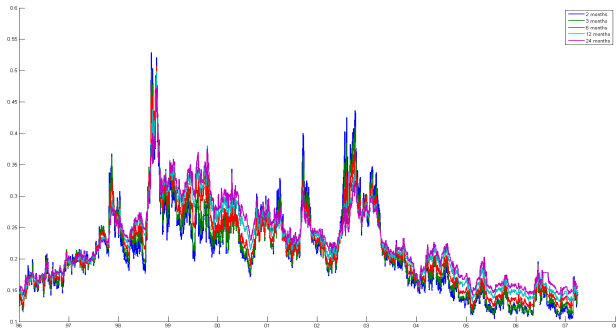 swap rates and risk premiums!

Figure: Variance swap rates $\sqrt{VS^2_{t,t+\tau}}$ on the S&P 500 index from
Jan 4, 1996 to Apr 2, 2007. Source: Bloomberg

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Forward Variance

- Define the forward variance

$$f(t, T) = \mathbb{E}_{\mathbb{Q}}\left[\sigma_T^2 + \int_{\mathbb{R}} x^2\, \nu_T(dx) \mid \mathcal{F}_t\right]$$

- Then the variance swap rates equal

$$\mathrm{VS}_{t,T}^2 = \frac{1}{T-t}\mathbb{E}_{\mathbb{Q}}\left[\int_t^T \sigma_s^2\, ds + \int_t^T \int_{\mathbb{R}} x^2\, \mu(ds, dx) \mid \mathcal{F}_t\right]$$

$$= \frac{1}{T-t}\int_t^T f(t, s)\, ds$$

- The spot variance is

$$v_t = \lim_{T \downarrow t} \mathrm{VS}_{t,T}^2 = f(t, t)$$

- Note the analogy to yields vs. forward rates

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Term-Structure Models: Program

- Exogenous (factor) model of $f(t, T)$, or v_t, under $\mathbb{P}$ and $\mathbb{Q}$

- Define index $\mathbb{Q}$-dynamics ...

$$\frac{dS_t}{S_{t-}} = r_t\,dt + \sigma_t\,dW_t + \int \left(e^{\delta_t(\xi)} - 1\right)(\mu(dt, d\xi) - \nu_t(d\xi)dt)$$

- ... such that spot variance satisfies

$$v_t = \sigma_t^2 + \int \delta_t(\xi)^2\,\nu_t(d\xi)$$

and

$$d[v, \log S]_t \leq 0 \quad (\text{``leverage effect''})$$

- E.g. Buehler [3], Egloff et al. [9], Cont and Kokholm [7], a.o.

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Factor Model

- Forward variance factor model

$$f(t, T) = g(T - t, X_t)$$

- State space $\mathcal{X} \subseteq \mathbb{R}$ open
- Jump-diffusion state process

$$dX_t = b(X_t)dt + \Sigma(X_t)dW_t + \int_{\mathbb{R}} \xi \left(\mu(dt, d\xi) - \nu(X_{t-}, d\xi)dt \right)$$

- b, Σ, γ, g nice enough ... in particular linear growth

$$b(x)^2 + \Sigma(x)^2 + \int_{\mathbb{R}} \xi^2 \, \nu(x, d\xi) \le K(1 + x^2)$$

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Quadratic Term Structure

Theorem 2.1.
The forward variance model admits a quadratic term structure:

$$g(t, x) = \phi(t) + \psi(t)x + \pi(t)x^2$$

if (and essentially only if) the state process X is quadratic:

$$b(x) = b + \beta x$$
$$\Sigma^2(x) = a + \alpha x + A x^2$$
$$\nu(x, d\xi) = n(d\xi) + \nu(d\xi)x + N(d\xi)x^2.$$

Moreover, . . .

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Quadratic Term Structure

Theorem 2.1 (cont'd).

. . . the functions ϕ, ψ, π satisfy the linear ODE

$$\frac{d}{dt} \begin{pmatrix} \phi \\ \psi \\ \pi \end{pmatrix} = \begin{pmatrix} 0 & b & a + \int_{\mathbb{R}} \xi^2 \, n(d\xi) \\ 0 & \beta & 2b + \alpha + \int_{\mathbb{R}} \xi^2 \, \nu(d\xi) \\ 0 & 0 & 2\beta + A + \int_{\mathbb{R}} \xi^2 \, N(d\xi) \end{pmatrix} \begin{pmatrix} \phi \\ \psi \\ \pi \end{pmatrix}$$

with initials ϕ_0, ψ_0, π_0 determined by the spot variance function

$$g_0(x) \equiv g(0, x) = \phi_0 + \psi_0 x + \pi_0 x^2.$$

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Proof

- Kolmogorov backward equation:

$$\frac{\partial g(t,x)}{\partial t} = b(x)\frac{\partial g(t,x)}{\partial x} + \frac{1}{2}\Sigma^2(x)\frac{\partial^2 g(t,x)}{\partial x^2}$$
$$+ \int_{\mathbb{R}} \left(g(t, x+\xi) - g(t,x) - \frac{\partial g(t,x)}{\partial x}\xi \right) \nu(x, d\xi)$$

- Reads here:

$$\phi'(t) + \psi'(t)x + \pi'(t)x^2 = b(x)\left(\psi(t) + 2\pi(t)x + \Sigma^2(x)\pi(t)\right)$$
$$+ \int_{\mathbb{R}} \pi(t)\xi^2 \, \nu(x, d\xi)$$
$$= \psi(t)\mathbf{P_1(x)} + \pi(t)\mathbf{P_2(x)}$$

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Proof

- Hence

$$\mathbf{P_1(x)} = b(x)$$

$$\mathbf{P_2(x)} = 2b(x)x + \Sigma^2(x) + \int_{\mathbb{R}} \xi^2 \, \nu(x, d\xi)$$

are quadratic polynomials in x

- Necessity follows for diffusion case ($\nu(x, d\xi) = 0$)
- Separate terms in 1, x, x^2 yields the result

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Quadratic Processes

Theorem 2.2.
The process X is quadratic if and only if

$$\mathbb{E}\left[X_t^n \mid X_0 = x\right] = \sum_{k=0}^{n} M_{kn}(t) x^k$$

for all $n \geq 0$. Moreover, the $(n+1) \times (n+1)$-matrix M solves the ODE

$$\frac{d}{dt} M(t) = B M(t)$$
$$M(0) = Id$$

That is, $M(t) = \mathrm{e}^{Bt}$, where $\ldots$

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Quadratic Processes

Theorem 2.2 (cont'd).

. . . the matrix B is upper triangular, and reads for the diffusion case (for simplicity):

$$B = \begin{pmatrix} \mathbf{0} & \mathbf{b} & \mathbf{2\frac{a}{2}} & 0 & \cdots & 0 \\ \mathbf{0} & \beta & \mathbf{2\left(b+\frac{\alpha}{2}\right)} & 3\cdot 2\frac{a}{2} & 0 & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{2\left(\beta+\frac{A}{2}\right)} & 3\left(b+2\frac{\alpha}{2}\right) & \ddots & 0 \\ 0 & 0 & 0 & 3\left(\beta+2\frac{A}{2}\right) & \ddots & n(n-1)\frac{a}{2} \\ \vdots & & & 0 & \ddots & n\left(b+(n-1)\frac{\alpha}{2}\right) \\ 0 & & \cdots & & 0 & n\left(\beta+(n-1)\frac{A}{2}\right) \end{pmatrix}$$

- Zhou [13], Forman and Sørensen [10], Cuchiero et al. [8]

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Eigenpolynomials and Moments

- Spectral decomposition

$$B = S \operatorname{diag}\left(\lambda_0, \ldots, \lambda_n\right) S^{-1}$$

with eigenvalues $\lambda_k = B_{kk}$, $k = 0, \ldots, n$

- Gives eigenpolynomials $p_k(x) = \sum_{j=0}^{k} S_{jk} x^j$ s.t.

$$\mathbb{E}\left[p_k(X_t) \mid X_0 = x\right] = e^{\lambda_k t} p_k(x), \quad k = 0, \ldots, n$$

- Stationary moments given by first row of S^{-1}:

$$\mathbb{E}\left[X_t^k\right] = S_{0k}^{-1}, \quad k = 0, \ldots, n$$

- Very efficiently computable! (e.g. with Mathematica)

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Quadratic Interest Rate vs. Variance Swap Models

- An interest rate factor model $r = r(X)$ admits a quadratic term structure if

$$\mathbb{E}\left[e^{-\int_0^t r(X_s)ds} \mid X_0 = x\right] = e^{\Phi(t) + \Psi(t)x + \Pi(t)x^2}$$

- Ahn, Dittmar, and Gallant [1], Leippold and Wu [12], a.o.

- Chen, Filipović, and Poor [5]: the only (!) consistent jump-diffusion state process X is Gaussian

$$dX_t = (b + \beta X_t)\,dt + \Sigma\,dW_t$$

- Quadratic variance swap term-structure models are much more flexible!

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Model Identification

- The quadratic property of X is invariant w.r.t. affine transformations

$$X \mapsto c + \gamma X$$

- Can be offset with affine transformation of the quadratic forward variance function:

$$\phi + \psi x + \pi x^2 \rightsquigarrow \left(\phi + \psi c + \pi c^2\right) + \left(\psi \gamma + 2\pi \gamma\right) x + \pi \gamma^2 x^2$$

- Need canonical representation of X for econometric model identification!

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps
Quadratic
Term
Structures
Quadratic
(Pearson)
Diffusions
Model
Estimation

Canonical Representation

- Forman and Sørensen [10]

Theorem 3.1.
*Denote by $D = \alpha^2 - 4aA$ the discriminant of the diffusion
function of the quadratic diffusion process*

$$dX_t = (b + \beta X_t)\, dt + \sqrt{a + \alpha X_t + A X_t^2}\, dW_t.$$

*Suppose $A > 0$. Then X falls in one of the following three
equivalence classes . . .*

Class 1: $D < 0$

- State space $\mathcal{X} = \mathbb{R}$
- Canonical representative:

$$dX_t = (b + \beta X_t)dt + \sqrt{1 + AX_t^2}\, dW_t$$

  for $b \geq 0$ and $\beta \in \mathbb{R}$

- If $\beta < 0$: $\exists$ stationary density

$$\mu(x) \propto \left(1 + Ax^2\right)^{\beta/A - 1} \exp\left[\frac{2b}{\sqrt{A}} \arctan[\sqrt{A}x]\right]$$

  (Pearson's type IV, or skew t-distribution)

- For $A \to 0$: Gaussian limit with stationary density

$$\mu(x) \propto \exp\left[2bx + \beta x^2\right]$$

Class 2: $D = 0$

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

- State space $\mathcal{X} = (0, \infty)$
- Canonical representative:

$$dX_t = (b + \beta X_t)dt + \sqrt{AX_t^2}\, dW_t$$

  for $b \geq 0$ and $\beta \in \mathbb{R}$
- If $b > 0$ and $\beta < 0$: $\exists$ stationary density

$$\mu(x) \propto x^{2\beta/A-2} \exp\left[-\frac{2b}{Ax}\right]$$

  (inverse Gamma distribution)
- Also called GARCH diffusion

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Class 3: $D > 0$

- State space $\mathcal{X} = (0, \infty)$
- Canonical representative:

$$dX_t = (b + \beta X_t)dt + \sqrt{X_t + AX_t^2}\, dW_t$$

for $b \geq \frac{1}{2}$ and $\beta \in \mathbb{R}$

- If $\beta < 0$: $\exists$ stationary density

$$\mu(x) \propto x^{2b-1}(1 + Ax)^{2\beta/A - 2b - 1}$$

(scaled F-distribution)

- For $A \to 0$: affine limit case with stationary density

$$\mu(x) \propto x^{2b-1} \exp[2\beta x]$$

(Gamma distribution)

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Measure Change

- Aim: equivalent change of measure $\mathbb{Q} \sim \mathbb{P}$ preserving quadratic property of X:

$$dX_t = \left(b^{\mathbb{Q}} + \beta^{\mathbb{Q}} X_t \right) dt$$
$$+ \sqrt{a + \alpha X_t + A X_t^2} \underbrace{\left(dW_t + \frac{\ell + \lambda X_t}{\sqrt{a + \alpha X_t + A X_t^2}} \, dt \right)}_{=dW_t^{\mathbb{Q}}}$$

- With return variance risk premium parameters

$$\ell = b - b^{\mathbb{Q}}, \quad \lambda = \beta - \beta^{\mathbb{Q}}$$

- Problem: Novikov's condition fails in general
- But equivalent measure change works here, due to ...

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

. . . Measure Change Theorem

Theorem 3.2 (Cheridito, Filipović, and Yor [6]).

Let b, σ, λ be locally bounded functions on $\mathcal{X}$, and

$$X_t = X_0 + \int_0^t b(X_s)\, ds + \int_0^t \Sigma(X_s)\, dW_s.$$

Assume that the martingale problem for

$$\widetilde{\mathcal{A}}f(x) = (b(x) + \Sigma(x)\lambda(\mathbf{x}))\, f'(x) + \frac{1}{2}\Sigma(x)^2 f''(x)$$

is well posed in $\mathcal{X}$. Then stochastic exponential

$$\mathcal{E}_t(\lambda(\mathbf{X})^\top \bullet W) = \exp\left(\int_0^t \lambda(\mathbf{X_s})^\top dW_s - \frac{1}{2} \int_0^t \|\lambda(\mathbf{X_s})\|^2 \, ds \right)$$

is a martingale.

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Application: Martingality of S

- Recall quadratic state process

$$dX_t = (b + \beta X_t)\, dt + \sqrt{a + \alpha X_t + A X_t^2}\, dW_t^{1}$$

- Let $\rho : \mathcal{X} \to [-1, 1]$ be Lipschitz and s.t. for some $x^* > 0$:

$$\rho(x) \begin{cases} \leq 0, & x \geq x^*, \\ \geq 0, & x \leq -x^* \end{cases}$$

- Model the discounted S&P 500 index process as

$$\frac{dS_t}{S_t} = \sqrt{g_0(X_t)} \left(\rho(X_t)\, dW_t^1 + \sqrt{1 - \rho^2(X_t)}\, dW_t^2 \right)$$

- It satisfies (with high probability) the "leverage effect"

$$d[g_0(X), \log S]_t = g_0'(X_t)\sqrt{g_0(X_t)}\rho(X_t) \leq 0$$

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Application: Martingality of S

- Question: is S a true $\mathbb{Q}$-martingale? (vital for pricing!)
- Yes! Write S as stochastic exponential

$$S_t = S_0 \, \mathcal{E}_t \left(\lambda(\mathbf{X})^\top \bullet W \right)$$

with

$$\lambda(\mathbf{x}) = \sqrt{\mathbf{g_0(x)}} \left(\frac{\rho(x)}{\sqrt{1 - \rho^2(x)}} \right)$$

- ... and note that the martingale problem for

$$\widetilde{\mathcal{A}} f(x) = \left(b + \beta x + \sqrt{a + \alpha x + A x^2} \sqrt{\mathbf{g_0(x)}} \rho(\mathbf{x}) \right) f'(x)$$
$$+ \frac{1}{2} \left(a + \alpha x + A x^2 \right) f''(x)$$

is well posed in $\mathcal{X}$ (Yamada–Watanabe)

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Generalized Method of Moments

- Model parameters

$$\theta = (a(=0,1), \alpha(=0,1), A, b, \beta, \ell, \lambda, \phi_0, \psi_0, \pi_0)$$

- Observations: 5-vector of VS rates

$$Y_t = (\mathrm{VS}^2_{t,t+\tau_1}, \ldots, \mathrm{VS}^2_{t,t+\tau_5})^\top, \quad t = 1, \ldots, T$$

- Define (martingale increments) vector-valued function

$$h_t = h(Y_t, Y_{t-1}, \theta)$$

such that

$$\frac{1}{T} \sum_{t=1}^{T} h_t \approx \mathbb{E}_{\theta_0} [h(G(X_t), G(X_{t-1}), \theta_0)] = 0$$

where $G(x) = (g(\tau_1, x), \ldots, g(\tau_5, x))^\top$

- Notice: eigenpolynomials of X_t in closed form: Forman and Sørensen [10] provide explicit optimal
  martingale estimating functions for X_t. Problem: X_t is not observed

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps
Quadratic
Term
Structures
Quadratic
(Pearson)
Diffusions
Model
Estimation

Extended Kalman Filter

- De facto standard estimation method
- Pros:
  - Faster and more stable than GMM (simulation study)
  - Uses all data
  - Filters latent factor X_t (useful for pricing)
- Cons: asymptotically inconsistent (but here finite sample!)
- Approximate state transition equation (QML):

$$X_{t+1} \sim \mathcal{N}\left(\mathbb{E}_\theta[X_{t+1} \mid X_t], \operatorname{var}_\theta[X_{t+1} \mid X_t]\right)$$

- Linearize observation equation:

$$Y_{t+1} = G(\hat{\mathbf{X}}_{t+1|t}, \theta) + G'(\hat{\mathbf{X}}_{t+1|t}, \theta)(X_{t+1} - \hat{\mathbf{X}}_{t+1|t}) + \epsilon_{t+1}$$

where $\hat{\mathbf{X}}_{t+1|t}$ = predicted state

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Data
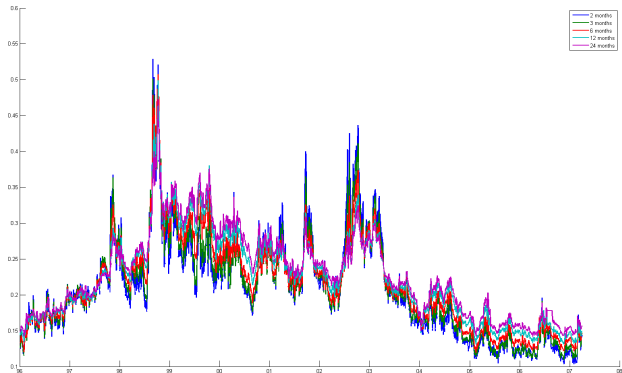

Figure: Variance swap rates $\sqrt{VS_{t,t+\tau}^2}$ on the S&P 500 index from Jan 4, 1996 to Apr 2, 2007. Source: Bloomberg

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Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Summary Statistics

Panel A: Variance swap rates

Maturity	Mean	Std	Skew	Kurt	Q_{22}	ADF
2	20.76	6.80	0.87	4.09	49,806.04	-3.84
3	20.90	6.54	0.78	3.87	52,308.54	-3.65
6	21.48	6.32	0.78	3.93	54,570.82	-3.45
12	22.25	6.06	0.62	3.19	56,549.61	-3.07
24	22.86	5.90	0.55	2.75	57,460.86	-2.86
Panel B: Calculated VIX						
2	20.68	6.14	0.71	3.38	51,405.71	-3.63
3	20.71	5.80	0.63	3.20	51,697.76	-3.40
6	20.78	5.23	0.47	2.79	55,149.03	-3.22

Table: Panel A: Summary statistics of the variance swap rates on the
S&P 500 index at different maturities (in months) from January 4,
1996 to April 2, 2007, for a total of 2832 observations.

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Summary Statistics cont'd

- Panel A cont'd: The table reports mean, standard deviation (Std), skewness (Skew), kurtosis (Kurt); the Ljung–Box portmanteau test for up to 22nd order autocorrelation, Q_{22}, 10% critical value is 30.81; the augmented Dickey–Fuller test for unit root involving 22 augmentation lags, a constant term and time trend, ADF, 10% critical value is -3.16.

- Panel B: summary statistics of the two-, three- and six-month VIX calculated using SPX options and applying the revised CBOE VIX methodology.

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Variance Swap
Models

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Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Principal Component Analysis

- PCA of variance swap curve $\tau \mapsto \sqrt{VS_{t,t+\tau}^2}$

- One major factor (level), explains 96% **of variance**
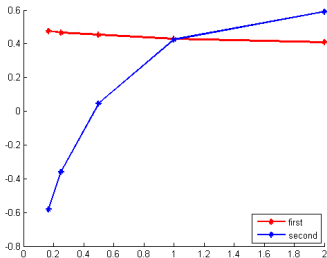
- Second factor (slope), explains 3% of variance

Figure: First two variance swap curve loadings

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Model Specifications

- Full specification: no restrictions

$$dX_t = (b + \beta X_t)\, dt + \sqrt{a + \alpha X_t + A X_t}\, dW_t$$

$$f(t, t) = \phi_0 + \psi_0 X_t + \pi_0 X_t^2$$

- Nested restricted specifications:
  - $A = 0$: affine X-dynamics
  - $\pi = 0$: linear spot variance function
  - $\psi_0^2 = 4\phi_0 \pi_0$: spot variance function has exactly one zero
  - $\phi_0 = \psi_0 = 0$: sv function has exactly one zero at $x = 0$

- Report likelihood ratio

$$\mathrm{LR} = 2\left(\log L_{\mathrm{Full}} - \log L_{\mathrm{REST}}\right) \sim \chi^2_{\#\ \mathrm{rest}}$$

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Estimation Results: Overview

Type	Full	$A = 0$	$\pi = 0$	$\psi_0^2 = 4\phi_0\pi_0$	$\phi_0 = \psi_0 = 0$
a	0	0	0	0	0
α	1	1	1	0	0
A	0.402	0	1.620	0.462	0.320
b	2.005	0.529	4.083	0.979	0.098
β	-0.742	-0.264	-0.839	-1.034	-0.849
ℓ	-0.023	-0.003	-1.273	0.300	0.032
λ	-0.243	0.020	-1.503	-0.590	-0.458
ϕ_0	0.016	0.000	0.693	0.002	0
ψ_0	-0.002	0.007	0.008	0.010	0
π_0	0.002	0.009	0.003	0.013	1.640
LR	0	**386**	**620**	264	286

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Estimation Results

- Full model of class 3 gives best fit

$$dX_t = (b + \beta X_t)\, dt + \sqrt{X_t + AX_t^2}\, dW_t$$

- All nested restricted specifications strongly rejected, in particular the affine ones ("$A = 0$", "$\pi = 0$")

- Class 3 combines affine behavior for small X_t and quadratic behavior for large X_t

- Quadratic terms allow for extreme movements and hump shaped VS term structure

- Drawback: spot variance function bounded away from zero ($\geq 0.1^2$)

Quadratic
Variance Swap
Models
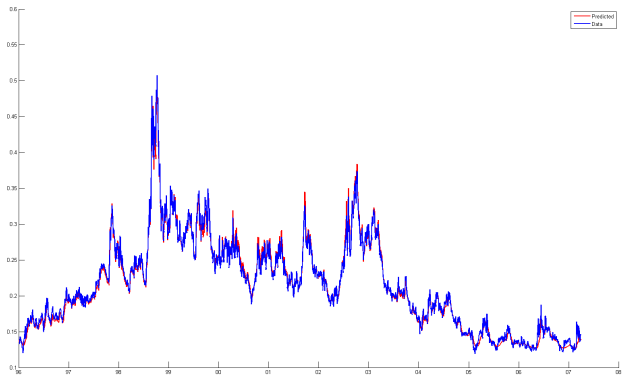
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Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

In-Sample Analysis: Predicted VS

Figure: Very good fit of predicted (filtered) variance swap rates vs. data for 6 months maturity

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Variance Swap
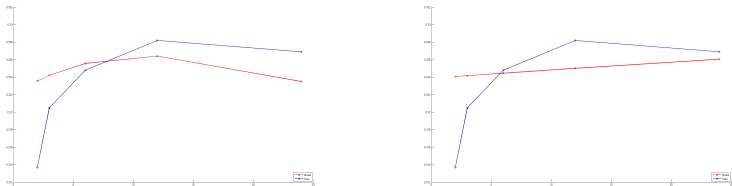Models

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Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

In-Sample Analysis: Humps

Figure: Hump shaped VS term structure on 15-Dec-1998. Quadratic model (left), linear model ($\pi = 0$) right.

Quadratic
Variance Swap
Models

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Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
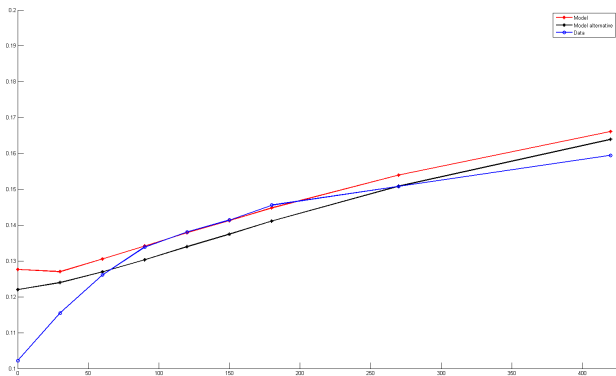Estimation

Out-of-Sample Analysis: VIX Futures Pricing

Figure: Challenging exercise for the model: VIX futures on Feb 15, 2007. Maturities 30, 60, 90, 120, 150, 180, 270, 420 days. Systematic pricing error for VIX due to jumps. Acceptable result.

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions
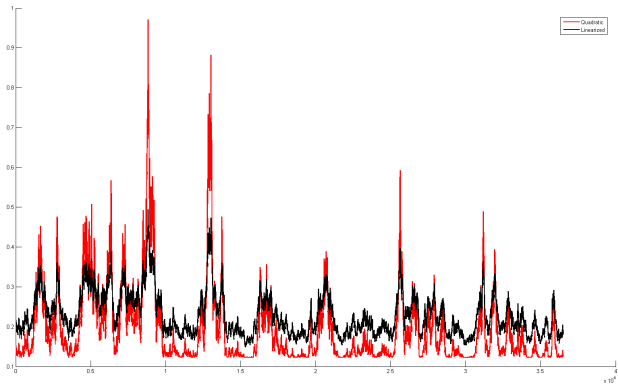
Model
Estimation

Nonlinear Phenomena

Figure: Simulation of a spot volatility trajectory over 100 years. In black the linearized model. We see volatility spikes and clustering. Do we need jumps after all?

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

Conclusion

- Need for variance swap term-structure models
- Quadratic term structure (closed form) led to quadratic factor process
- Quadratic models are much more flexible than linear-affine models
- Data imply strong statistical evidence for quadratic terms
- Quadratic models capture nonlinear phenomena: rare events, volatility clustering

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

References I

D. H. Ahn, R. F. Dittmar, and A. R. Gallant.
Quadratic term structure models: Theory and evidence.
The Review of Financial Studies, 15:243–288, 2002.

M. Britten-Jones and A. Neuberger.
Option prices, implied price processes and stochastic
volatility.
Journal of Finance, 55(2):839–866, 2000.

H. Buehler.
Consistent variance curve models.
Finance Stoch., 10(2):178–203, 2006.

P. Carr and L. Wu.
Variance risk premiums.
The Review of Financial Studies, 22(3):1311–1341, 2009.

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

References II

📄 L. Chen, D. Filipović, and H. V. Poor.
Quadratic term structure models for risk-free and
defaultable rates.
Math. Finance, 14(4):515–536, 2004.

📄 P. Cheridito, D. Filipović, and M. Yor.
Equivalent and absolutely continuous measure changes for
jump-diffusion processes.
Ann. Appl. Probab., 15(3):1713–1732, 2005.

📄 R. Cont and T. Kokholm.
A Consistent Pricing Model for Index Options and
Volatility Derivatives.
SSRN eLibrary, 2009.

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

References III

📄 C. Cuchiero, M. Keller-Ressel, and J. Teichmann.
Polynomial processes and their applications in
mathematical finance.
Working Paper, 2008.

📄 D. Egloff, M. Leippold, and L. Wu.
The term structure of varaince swap rates and optimal
variance swap investments.
Journal of Financial annd Quantitative Analysis,
forthcoming, 2010.

📄 J. L. Forman and M. Sørensen.
The Pearson diffusions: a class of statistically tractable
diffusion processes.
Scand. J. Statist., 35(3):438–465, 2008.

Quadratic
Variance Swap
Models

D. Filipović

Variance
Swaps

Quadratic
Term
Structures

Quadratic
(Pearson)
Diffusions

Model
Estimation

References IV

📄 G. J. Jiang and Y. S. Tian.
The model-free implied volatility and its information content.
The Review of Financial Studies, 18(4):1305–1342, 2005.

📄 M. Leippold and L. Wu.
Asset pricing under the quadratic class.
Journal of Financial and Quantitative Analysis, 37:271–295, 2002.

📄 H. Zhou.
Itô conditional moment generator and the estimation of short-rate processes.
J. Financial Econometrics, 1:250–271, 2003.