Convertible Subordinated Debt Financing and Optimal Investment Timing

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1. Introduction (1)

- Many companies issue convertible debt as a means of debt financing.

- Convertible debt is an attractive security
  – The seller may not incur the debt in conversion and can also enjoy interest tax shields relative to equity.
  – The buyer has an option to convert into equity when its value is higher.

- Firms issuing the convertible subordinated debt have been common.

1. Introduction (2)

- **Senior-sub structure for equity, straight debt and convertible debt**

Straight debt $\geq$ Convertible debt $\geq$ Equity

- Senior-sub structure
  - The junior security holders will not get paid at all until the senior security holders are completely paid off at default.

  (Black and Cox(1976), Sundaresen and Wang(2006))

1. Introduction (3)

- The firm’s investment and financing decisions
  – Real options framework

- The investment problems for the firm
  – All-equity financing

(McDonald and Siegel(1986), Dixit and Pindyck(1994))

  – Straight debt financing

(Mauer and Ott(2000), Mauer and Sarkar(2005),
Sundaresan and Wang(2006),
Lyandres and Zhdanov(2006a), Zhdanov(2007))

1. Introduction (4)

- The investment problems for the firm
  - Convertible debt financing

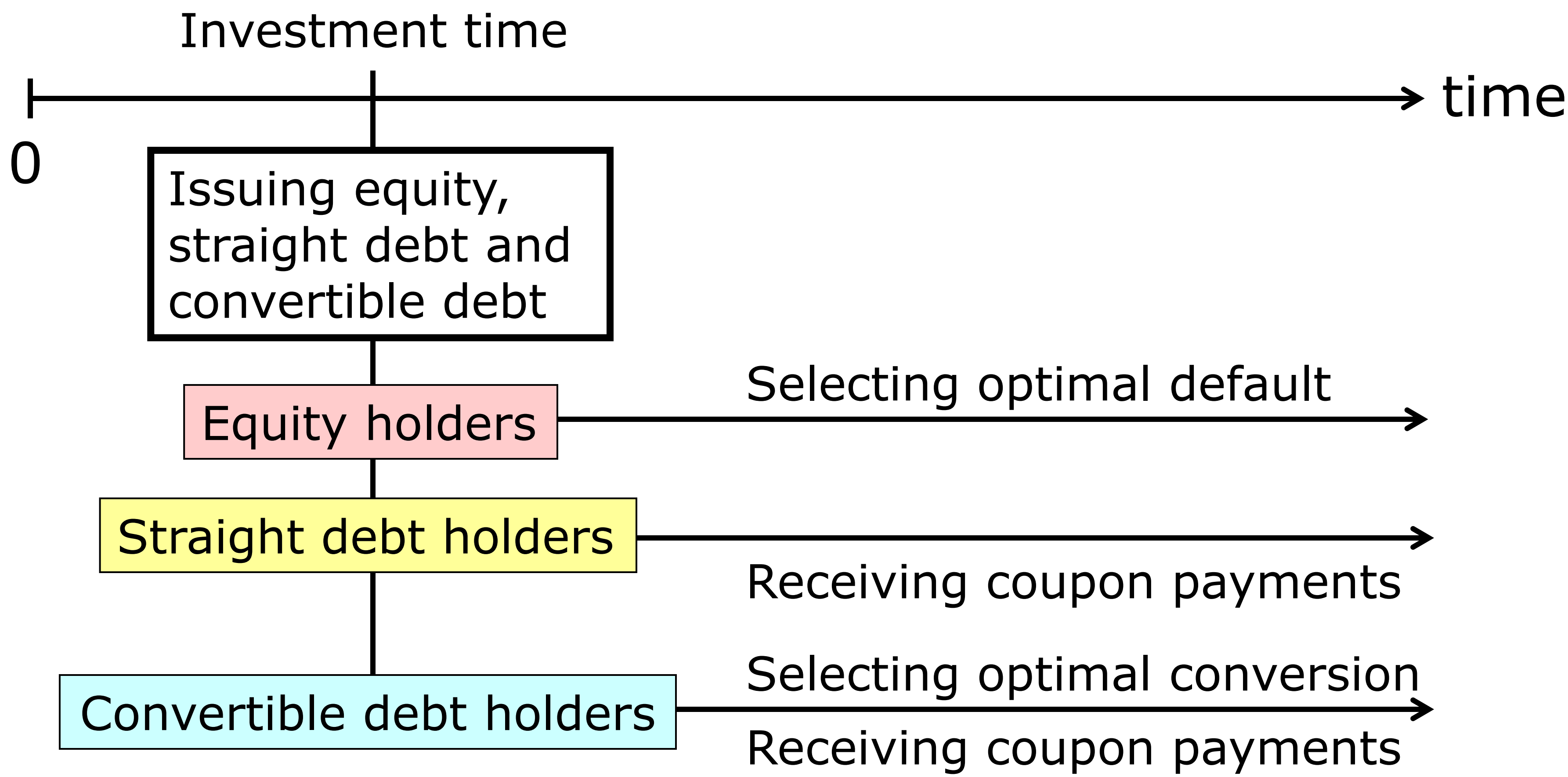
(Lyandres and Zhdanov(2006b), Yagi et al. (2008), Egami(2010))

Assumption : Same priority (Pari passu)

- Our study: the senior-sub structure of equity, straight debt and convertible debt

1. Introduction (5)

- The optimal investment policy for the firm financed by issuing equity, straight debt and convertible debt



1. Introduction (6)

Our model:

- The investment is financed with equity, straight debt and convertible debt with senior-sub structure.
- Straight debt and convertible debt are issued at par.

Our objectives:

- Senior-sub structure
  - Optimal policies for financing and investment
  - Optimal capital structure

2. The Model (1)

- A firm with an option to invest at any time.
- I : a fixed investment cost
- The firm finances the cost of investment with equity, straight debt and convertible debt.

Equity	Straight debt with face value F_s, coupon rate s	Convertible debt with face value F_c, coupon rate c

2. The Model (2)

Straight debt

Coupon payment
sF_s

Convertible debt

Coupon payment
cF_c

- τ : a constant corporate tax rate
- Coupon payments are tax-deductible.
- Straight debt and convertible debt are issued at par.

2. The Model (3)

- Suppose the firm observes the demand shock X_t for its product
  - X_t : a geometric Brownian motion

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x \tag{1}$$

  - μ : the risk-adjusted expected growth rate
  - σ : the volatility
  - W_t : a standard Brownian motion

- Once the investment option is exercised, the firm can receive the instantaneous profit

$$\pi(X_t, s, c) = (1-\lambda)(QX_t - sF_s - cF_c) \tag{2}$$

  - Q : the quality produced from the asset in place

2. The Model (4)

- The investment is financed with equity, straight debt and convertible debt.

1. Optimal investment policy
2. The values of equity, straight debt and convertible debt with same priority
3. The values of equity, straight debt and convertible debt with senior-sub structure
4. Optimal capital structure

2.1. Optimal Investment Policy (1)

- The equity holders of the firm which invests
  - to select the optimal investment timing, observing the demand shock X_t

- $V(x)$: The value of firm which is financed with equity, straight debt and convertible debt

$$V(x) = E(x) + D_s(x) + D_c(x) \tag{3}$$

  - $E(x)$: the value of equity
  - $D_s(x)$: the value of straight debt
  - $D_c(x)$: the value of convertible debt

2.1. Optimal Investment Policy (2)

- The value of investment partially financed with straight debt and convertible debt

$$\begin{aligned} F(x) &= \sup_{T \in \mathcal{T}_{0,\infty}} E_0^x \left[e^{-rT} \left(E(X_T) - (I - D_s(X_T) - D_c(X_T)) \right) \right] \\ &= \sup_{T \in \mathcal{T}_{0,\infty}} E_0^x \left[e^{-rT} \left(V(X_T) - I \right) \right] \end{aligned} \tag{4}$$

- x^* : the optimal investment threshold

- Since straight debt and convertible debt are issued at par, the coupon rates s and c are determined such that

$$D_s(x^*) = F_s \tag{5}$$

$$D_c(x^*) = F_c \tag{6}$$

2.2. Same priority (1)

- The values of equity, straight debt and convertible debt with the same priority

- The convertible debt holders can convert the debt into a fraction η of the original equity;

$$\eta = \alpha c F_c \tag{7}$$

  - α : constant

- The investment option has been exercised.

⇩

- From the issue of debt, the optimal default policy is established.

2.2. Same priority (2)

- The optimal default policy of the equity holders
  – to select the default time T_d, maximizing the equity value

- The optimal conversion policy of the convertible debt holders
  – to select the conversion time T_c, maximizing the value of convertible debt

- The optimal problems for the holders of equity and convertible debt must be solved simultaneously.

2.2. Same priority (3)

- At default time T_d
  - Equity holders cannot receive anything.
  - Straight debt holders receive

$$D_s(X_{T_d}) = \frac{F_s}{F_s + F_c}(1-\theta)\epsilon(X_{T_d}) \tag{8}$$

  - Convertible debt holders receive

$$D_c(X_{T_d}) = \frac{F_c}{F_s + F_c}(1-\theta)\epsilon(X_{T_d}) \tag{9}$$

> θ : the proportional bankruptcy cost
> $(0 \leq \theta \leq 1)$

2.2. Same priority (4)

- The value of equity at investment time t

$$E(x) = \sup_{T_d \in \mathcal{T}_{t,\infty}} E_t^x \Bigg[\int_t^{T_c^* \wedge T_d} e^{-r(u-t)} (1-\tau)(QX_u - sF_s - cF_c) du + 1_{\{T_c^* < T_d\}} e^{-r(T_c^* - t)} \frac{1}{1+\eta} E_a(X_{T_c^*}) \Bigg] \quad (10)$$

  - By converting, the equity value is diluted.
    - the dilution factor : $\dfrac{1}{1+\eta}$
  - $E_a(x)$: the value of equity after conversion

2.2. Same priority (5)

- The value of convertible debt at investment time t

$$D_c(x) = \sup_{T_c \in \mathcal{T}_{t,\infty}} E_t^x \Bigg[\int_t^{T_c \wedge T_d^*} e^{-r(u-t)} cF_c du \\ + 1_{\{T_c < T_d^*\}} e^{-r(T_c - t)} \frac{\eta}{1+\eta} E_a(X_{T_c}) \\ + 1_{\{T_d^* < T_c\}} e^{-r(T_d^* - t)} \frac{F_c}{F_s + F_c} (1-\theta)\epsilon(X_{T_d^*}) \Bigg] \quad (11)$$

- Optimal default time

$$T_d^* = \inf\{T_d \in [t, \infty) \mid X_{T_d} \le x_d\} \quad (12)$$

- Optimal conversion time

$$T_c^* = \inf\{T_c \in [t, \infty) \mid X_{T_c} \ge x_c\} \quad (13)$$

  – x_c : Optimal conversion threshold

2.2. Same priority (6)

- The value of straight debt at investment time t

$$D_s(x) = E_t^x\left[\int_t^{T_c^*\wedge T_d^*} e^{-r(u-t)} sF_s du + 1_{\{T_c^*<T_d^*\}} e^{-r(T_c^*-t)} D_{s,a}(X_{T_c^*}) + 1_{\{T_d^*<T_c^*\}} e^{-r(T_d^*-t)} \frac{F_s}{F_s+F_c}(1-\theta)\epsilon(X_{T_d^*})\right] \tag{14}$$

  – $D_{s,a}(x)$: the value of straight debt after conversion

2.2. Same priority (7)

- The post-conversion value of equity

$$E_a(x) = \epsilon(x) - \frac{(1-\tau)sF_s}{r} - \left(\frac{x}{x_{d,c}}\right)^{\beta_2}\left(\epsilon(x_{d,c}) - \frac{(1-\tau)sF_s}{r}\right) \tag{15}$$

- The post-conversion value of straight debt

$$D_{s,a}(x) = \frac{sF_s}{r} - \left(\frac{x}{x_{d,c}}\right)^{\beta_2}\left(\frac{sF_s}{r} - (1-\theta)\epsilon(x_{d,c})\right) \tag{16}$$

  – $\beta_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0$

- The post-conversion default threshold

$$x_{d,c} = \frac{\beta_2}{\beta_2 - 1}\frac{r-\mu}{r}\frac{sF_s}{Q} \tag{17}$$

2.3. Senior-Sub Structure (1)

- The values of equity, straight debt and convertible debt with the senior-sub structure

- At default time T_d
  - Straight debt holders receive

$$D_s(X_{T_d}) = \min(F_s, (1-\theta)\epsilon(X_{T_d})) \tag{18}$$

  - Convertible debt holders receive

$$D_c(X_{T_d}) = \min(F_c, \max((1-\theta)\epsilon(X_{T_d}) - F_s, 0)) \tag{19}$$

  - Equity holders receive

$$E(X_{T_d}) = \max((1-\theta)\epsilon(X_{T_d}) - F_s - F_c, 0) \tag{20}$$

2.3. Senior-Sub Structure (2)

- The value of equity at investment time t

$$E(x) = \sup_{T_d \in \mathcal{T}_{t,\infty}} E_t^x \Bigg[\int_t^{T_c^* \wedge T_d} e^{-r(u-t)}(1-\tau)(QX_u - sF_s - cF_c)du$$
$$+ 1_{\{T_c^* < T_d\}} e^{-r(T_c^* - t)} \frac{1}{1+\eta} E_a(X_{T_c^*})$$
$$+ 1_{\{T_d < T_c^*\}} e^{-r(T_d - t)} \max((1-\theta)\epsilon(X_{T_d}) - F_s - F_c, 0) \Bigg] \quad (21)$$

- The value of convertible debt at investment time t

$$D_c(x) = \sup_{T_c \in \mathcal{T}_{t,\infty}} E_t^x \Bigg[\int_t^{T_c \wedge T_d^*} e^{-r(u-t)} cF_c du$$
$$+ 1_{\{T_c < T_d^*\}} e^{-r(T_c - t)} \frac{\eta}{1+\eta} E_a(X_{T_c})$$
$$+ 1_{\{T_d^* < T_c\}} e^{-r(T_d^* - t)} \min(F_c, \max((1-\theta)\epsilon(X_{T_d^*}) - F_s, 0)) \Bigg] \quad (22)$$

2.3. Senior-Sub Structure (3)

- The value of straight debt at investment time t

$$D_s(x) = E_t^x\left[\int_t^{T_c^* \wedge T_d^*} e^{-r(u-t)} sF_s du + 1_{\{T_c^* < T_d^*\}} e^{-r(T_c^*-t)} D_{s,a}(X_{T_c^*}) \right.$$

$$\left. + 1_{\{T_d^* < T_c^*\}} e^{-r(T_d^*-t)} \min(F_s, (1-\theta)\epsilon(X_{T_d^*}))\right] \quad (23)$$

- At post-conversion default time $T_{d,c}$
  - Straight debt holders receive

$$D_{s,a}(X_{T_{d,c}}) = \min(F_s, (1-\theta)\epsilon(X_{T_{d,c}})) \quad (24)$$

  - Equity holders receive

$$E(X_{T_{d,c}}) = \max((1-\theta)\epsilon(X_{T_{d,c}}) - F_s, 0) \quad (25)$$

2.3. Senior-Sub Structure (4)

- The post-conversion value of equity

$$E_a(x) = \sup_{T_{d,c} \in \mathcal{T}_{t,\infty}} E_t^x \left[\int_t^{T_{d,c}} e^{-r(u-t)}(1-\tau)(QX_u - sF_s)du \right.$$
$$\left. + e^{-r(T_{d,c}-t)} \max((1-\theta)\epsilon(X_{T_{d,c}}) - F_s, 0) \right] \tag{26}$$

- Optimal post-conversion default threshold

$$T_{d,c}^* = \inf\{T_{d,c} \in [t, \infty) \mid X_{T_{d,c}} \leq x_{d,c}\} \tag{27}$$

- The post-conversion value of convertible debt

$$D_{s,a}(x) = E_t^x \left[\int_t^{T_{d,c}^*} e^{-r(u-t)} sF_s du + e^{-r(T_{d,c}^*-t)} \min(F_s, (1-\theta)\epsilon(X_{T_{d,c}^*})) \right] \tag{28}$$

2.4. Optimal Capital Structure

- The optimal capital structure of the firm issuing equity, straight debt and convertible debt

- The face value of convertible debt F_c

$$\eta = \alpha c F_c \tag{7}$$

- The face value of straight debt F_s

$$F(x) = \sup_{T \in \mathcal{T}_{0,\infty}, F_s > 0} E_0^x \left[e^{-rT} \left(V(X_T) - I \right) \right] \qquad (5)'$$

3. Numerical Analysis

- Optimal face values for straight debt and convertible debt
- Coupon rates for straight debt and convertible debt
- Threshold for investment
- Optimal leverage ratio

solving nonlinear simultaneous equations.

Tab.1 Parameters

Quantity for X_t	$Q = 1$
Initial value of X_t	$x = 0.5$
Expected growth rate	$\mu = 0$
Volatility	$\sigma = 0.2$
Interest rate	$r = 0.05$
Investment cost	$I = 20$
Proportional constant on conversion	$\alpha = 1.5$
Proportional bankruptcy cost	$\theta = 0.3$
Corporate tax rate	$\tau = 0.3$

Tab.2 Optimal capital structure for the conversion ratio

	Senior-sub structure			Same priority		
η	0.5	1.0	1.5	0.5	1.0	1.5
F_c	6.6	11.7	16.3	7.3	13.2	18.4
F_s^*	20.6	16.7	12.9	19.7	15.0	10.4
c	0.051	0.057	0.061	0.046	0.051	0.054
s	0.064	0.061	0.057	0.067	0.067	0.067
x^*	2.320	2.355	2.382	2.316	2.346	2.368
$\frac{D_c(x^*)+D_s(x^*)}{V(x^*)}$	0.727	0.760	0.780	0.724	0.754	0.773

Tab.2.1 Optimal capital structure for the conversion ratio

	Senior-sub structure			Same priority			Increase -decrease
η	0.5	1.0	1.5	0.5	1.0	1.5	$\nearrow$
F_c	6.6	11.7	16.3	7.3	13.2	18.4	$\nearrow$
F_s^*	20.6	16.7	12.9	19.7	15.0	10.4	
c	0.051	0.057	0.061	0.046	0.051	0.054	$\nearrow$
s	0.064	0.061	0.057	0.067	0.067	0.067	

The conversion ratio η increases.

$\Downarrow$

Face value F_c and coupon rate c for convertible debt increase.

➢ Convertible debt holders can receive more equity in conversion.

Tab.2.1 Optimal capital structure for the conversion ratio

	Senior-sub structure			Same priority			Increase -decrease
η	0.5	1.0	1.5	0.5	1.0	1.5	↗
F_c	6.6	11.7	16.3	7.3	13.2	18.4	
F_s^*	20.6	16.7	12.9	19.7	15.0	10.4	↘
c	0.051	0.057	0.061	0.046	0.051	0.054	
s	0.064	0.061	0.057	0.067	0.067	0.067	↘

The conversion ratio η increases.

⇩

Face value F_s^* and coupon rate s for straight debt decrease.

➢ Balance between convertible debt and straight debt

Tab.2.2 Optimal capital structure for the conversion ratio

	Senior-sub structure			Same priority			Increase -decrease
η	0.5	1.0	1.5	0.5	1.0	1.5	$\nearrow$
x^*	2.320	2.355	2.382	2.316	2.346	2.368	$\nearrow$
$\frac{D_c(x^*)+D_s(x^*)}{V(x^*)}$	0.727	0.760	0.780	0.724	0.754	0.773	$\nearrow$

The conversion ratio η increases.

$\Downarrow$

Investment threshold x^* and leverage ratio $\frac{D_c(x^*)+D_s(x^*)}{V(x^*)}$ increase.

➢ Dilution for equity value becomes larger.

Tab.2.3 Optimal capital structure for the conversion ratio

	Senior-sub structure			Comparison	Same priority		
η	0.5	1.0	1.5		0.5	1.0	1.5
F_c	6.6	11.7	16.3	$<$	7.3	13.2	18.4
F_s^*	20.6	16.7	12.9	$>$	19.7	15.0	10.4
c	0.051	0.057	0.061	$>$	0.046	0.051	0.054
s	0.064	0.061	0.057	$<$	0.067	0.067	0.067

Face value for convertible debt is smaller.
Coupon rate for convertible debt is larger.

- Possibility that convertible debt holders cannot receive anything at default

Tab.2.3 Optimal capital structure for the conversion ratio

	Senior-sub structure			Comparison	Same priority		
η	0.5	1.0	1.5		0.5	1.0	1.5
F_c	6.6	11.7	16.3	$<$	7.3	13.2	18.4
F_s^*	20.6	16.7	12.9	$>$	19.7	15.0	10.4
c	0.051	0.057	0.061	$>$	0.046	0.051	0.054
s	0.064	0.061	0.057	$<$	0.067	0.067	0.067

Face value for straight debt is larger.
Coupon rate for straight debt is smaller.

- Possibility that payoff to straight debt holders at default is higher.

Tab.2.4 Optimal capital structure for the conversion ratio

	Senior-sub structure			Comparison	Same priority		
η	0.5	1.0	1.5		0.5	1.0	1.5
x^*	2.320	2.355	2.382	$>$	2.316	2.346	2.368
$\frac{D_c(x^*)+D_s(x^*)}{V(x^*)}$	0.727	0.760	0.780	$>$	0.724	0.754	0.773

Investment threshold and leverage ratio are higher.

- Possibility that convertible debt holders cannot receive anything at default
- Conversion occurs earlier.
- Equity value decreases from dilution.

4. Summary

- The optimal investment policy of the firm financed by issuing equity, straight debt and convertible debt
  - **Senior-sub structure**
  - **Straight debt and convertible debt are issued at par.**
- The senior-sub structure
  - Convertible debt
    - Face value is larger and coupon rate is smaller.
  - Straight debt
    - Face value is smaller and coupon rate is larger.
  - Investment occurs later.
  - Leverage ratio is higher.

Thank you.

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