On Securitization, market completion and equilibrium Risk transfer

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The question

- The underlying risk factors are typically non-tradable.
  - Weather/Climate (temperature, rain, wind speed, snow)
- Securitization: Transform non-tradable risk factors into tradable financial securities.
  - An example is a structured derivative issued to shift insurance risks to capital markets.

The question is how to price the structured derivative?

The underlyings

- A set $\mathcal{A}$ of agents are exposed to tradable and non-tradable risk factors:
  - The non-tradable risk process follows a diffusion with additive noise:

$$dR_t = \mu^R(t, R_t)dt + \sigma^R(t, R_t)dW_t^R,$$

  - A tradable asset (a stock or a commodity) whose price follows a positive diffusion process:

$$\frac{dS_t}{S_t} = \mu^S(t, R_t, S_t)dt + \sigma^S(t, R_t, S_t)dW_t^S.$$

The payoffs

- The agents receive random incomes at some terminal horizon T:

$$H^a = h^a(X_T) + \int_0^T \varphi_s^a(X_s)ds, \quad a \in \mathcal{A}$$

where $X_t = (R_t, S_t)$ is the state (forward process).

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The market is a priori incomplete!

- A bank issues a structured derivative whose payoff H^l depends on the state process

$$H^l = h^l(X_T) + \int_0^T \varphi_s^l(X_s)ds.$$

Pricing schemes

- The structured derivative is priced by market forces and hence by an arbitrage-free pricing scheme.

- Each such pricing scheme can be identified with a measure $\mathbb{Q} \approx \mathbb{P}$.

- The agents have no impact on the tradable asset price $\Rightarrow$ Hence stock prices must be $\mathbb{Q}$-martingales.

- In equilibrium the structured derivative price is given by

$$B_t^* = \mathbb{E}_{\mathbb{Q}^*}[H^I|\mathcal{F}_t] \quad \text{w.r.t. an endogenous measure } \mathbb{Q}^*.$$

Pricing rules

$\rightarrow$look for a linear pricing rule.

Look for a predictable $\theta = (\theta^S, \theta^R) \in L^2$ such that

$$\mathbb{Q} = \exp\Big(- \int_0^T \theta_s dW_s - \frac{1}{2}\int_0^T |\theta_s|^2 ds \Big)\mathbb{P}$$

defines a measure and $dW_t^\theta = dW_t + \theta_t dt$ are $\mathbb{Q}$-Brownian motion

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- $\theta = (\theta^S, \theta^R)$ is the market price of risk
  - $\theta^S = \mu^S/\sigma^S$ is exogenously given
  - θ^R is endogenously given by an equilibrium condition.

The Derivative's price process

- For a market price of risk θ the corresponding structured derivative price process

$$B_t^\theta = \mathbb{E}_{\mathbb{Q}_\theta}[H^I|\mathcal{F}_t] = \mathbb{E}^\theta[H^I] + \int_0^t \kappa_s^\theta dW_s^\theta$$

- Structured derivative volatility is $\kappa^\theta = (\kappa^{\theta,S}, \kappa^{\theta,R})$ and is endogenously given by equilibrium. We assume that

$$\kappa^{\theta,R} \neq 0.$$

fluctuations of the external risk translate into fluctuations of the bond price

The wealth process

- The gains or losses from trading according to $\pi^{a,\theta} = (\pi^{a,\theta,1}, \pi^{a,\theta,2})$ are

$$V_t^{a,\theta}(\pi^{a,\theta}) = \int_0^t \pi_s^{a,\theta,1} dS_s + \int_0^t \pi_s^{a,\theta,2} dB_s^\theta$$

and agent's a payoff at terminal horizon T from trading according to $\pi^{a,\theta}$ is

$$H^a + V_T^{a,\theta}(\pi^{a,\theta})$$

The set of pricing rules is identified by the market prices of non-tradable risk.

The preferences

Assumption: Utilities of the agents generated by monetary dynamic convex risk measures $\rightarrow$ BSDE

A BSDE is an equation of the type:

$$Y_t = \xi - \int_t^T Z_s dW_s + \int_t^T f(s, Z_s) ds$$

- T, deterministic terminal time
- ξ, the terminal condition. An $\mathcal{F}_T$ adapted integrable R.V.
- $f : \Omega \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ we call generator

In El Karoui & Peng & Quenez (1997) an overview is given

The preferences

The agent's risk assessment

$$Y_t^a = -[H^a + V_T^{a,\theta}(\pi^{a,\theta})] - \int_t^T g^a(s, Z_s^a)ds - \int_t^T Z_s^a dW_s$$

- driver g^a specifies the risk preference $\rightarrow$a convex function
- We chose a class of monetary utilities: the entropic case. We obtain BSDEs with quadratic drivers

$$g^a(t,z) = \frac{1}{2\gamma_a}||z||^2, \quad \gamma_a > 0$$

(This choice leads to the same risk criterion as the exponential utility: $U(x) = -\exp(\gamma_a^{-1}x)$.)

Agent's optimization problem

- For a given market price of risk θ:
  the risk (Y_t^a) of the agent's a payoff

$$Y_t^a = -[H^a + V_T^{a,\theta}(\pi^{a,\theta})] - \int_t^T g^a(s, Z_s^a)ds - \int_t^T Z_s^a dW_s$$

- Agent's a goal is to pick a trading strategy $\tilde{\pi}^{a,\theta}$ to minimize the risk, i.e.,

$$\tilde{\pi}^{a,\theta} = \arg\min_{\pi^\theta} Y_0^a(\pi^{a,\theta}).$$

Agent's optimization problem

- For a given market price of risk θ:
  the risk (Y_t^a) of the agent's *a* payoff

$$Y_t^a = -[H^a + V_T^{a,\theta}(\pi^{a,\theta})] - \int_t^T g^a(s, Z_s^a)ds - \int_t^T Z_s^a dW_s$$

- Agent's *a* goal is to pick a trading strategy $\tilde{\pi}^{a,\theta}$ to minimize the risk, i.e.,

$$\tilde{\pi}^{a,\theta} = \arg\min_{\pi^\theta} Y_0^a(\pi^{a,\theta}).$$

- θ^* is an equilibrium market price of risk if

$$\sum_{a \in \mathcal{A}} \tilde{\pi}_t^{a,\theta^*,2} = 1 \quad (0 \le t \le T).$$

The representative agent I

In complete markets Pareto optimal allocation (hence competitive equilibria) can be supported by equilibria of the Representative agent

What is the Representative's agent risk measure?

Assume two agents $\{a, b\}$; with risk profile g^a and g^b. Define:

$$g^{ab}(t,z) = g^a \square g^b(t,z) = \inf_x \{g^a(t, z-x) + g^b(t,x)\}.$$

(Inf-convolution - El Karoui & Barrieu 2005)

The representative agent II

- The Rep. Ag. risk is given by

$$Y_t^{ab} = -[H^a + H^b + H^I + V_T^{ab,\theta}(\pi^\theta)] - \int_t^T g^{ab}(s, Z_s)ds - \int_t^T Z_s dW_s$$

- Her goal is to minimize the risk:

$$\min_{\pi^\theta} Y_0^{ab}(\pi^\theta)$$

Finding the equilibrium market price of risk

- Look for $\theta^* = (\theta^S, \theta^{*R})$ such that

$$\tilde{\pi}^{ab,\theta^*} \triangleq \arg\min_{\pi^{\theta^*}} Y_0^{ab}(\pi^{\theta^*}) = 0.$$

Then θ^* is an equilibrium market price of risk characterized by a BSDE.

Finding the equilibrium market price of risk

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  Then θ^* is an equilibrium market price of risk characterized by a BSDE.

- We work under the standing assumption that derivative's price volatility (of W^R) under equilibrium pricing measure $\mathbb{Q}_{\theta^*}$ does not vanish,

$$\kappa^{\theta^*,R} \neq 0.$$

- This assumption is verified as long as structured derivative payoff is monotonic with respect to the non-tradable risk.

Obtaining the Market price of external risk

Theorem

$$\theta^R = Z^2$$

Overview

- Solve the BSDE for the Rep. Ag. (Quadratic growth BSDE)

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- The Z part of the Rep. Ag.'s BSDE will be the θ^R
- Knowing θ^S and θ^R the other quantities follow:
  - Derivative price
  - Agent's risks assessments

A example

- Let (R_t) be the temperature process and (S_t) be the price of a share of an energy provider equity with dynamics

$$dR_t = a_t dt + 2.0\, dW_t^R, \qquad a_t = 4t$$
$$\frac{dS_t}{S_t} = \mu^S dt + \frac{1}{\sqrt{\Gamma(t,R_t)}}\, dW_t^S,$$

where

$$\Gamma(t,R_t) = 8(\arctan(-R_t) + \pi/2).$$

- A bank holding the stock may chose to hedge its financial risk as measured by the stock volatility by issuing a structured derivative that pays yield

$$\varphi^I(t,S_t,R_t) = \exp\left\{-M\left(\int_0^t a_s ds - R_t\right)^+\right\}, \quad (M>0).$$

A example

- Two more agents A and B with risk preferences are described by entropic utilities $\gamma_a = 1.0$ and $\gamma_b = 2.0$, have the incomes

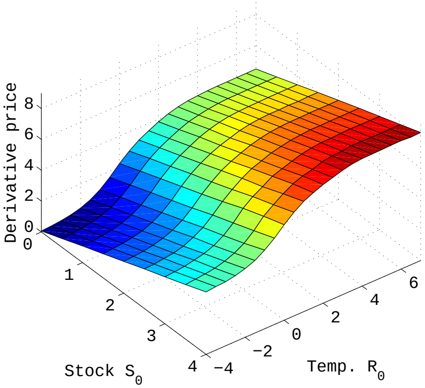
$$H^a = c^a S_T + \int_0^T \exp\{-M^a(R_t - R^a)^2\}dt,$$

$$H^b = c^b S_T + \int_0^T \exp\{-M^b(R_t - R^b)^2\}dt.$$

- The constants of our model are chosen as:

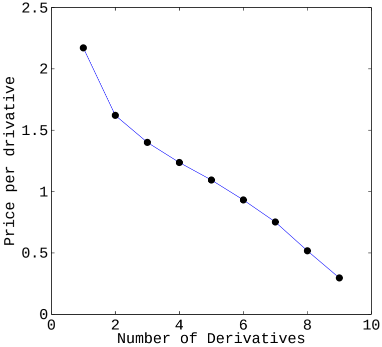
γ_a	γ_b	γ_R	M	M^a	M^b	c^a	c^b	R^a	R^b
1.0	2.0	3.0	2.0	0.5	0.5	0.5	0.5	4.0	-1.0

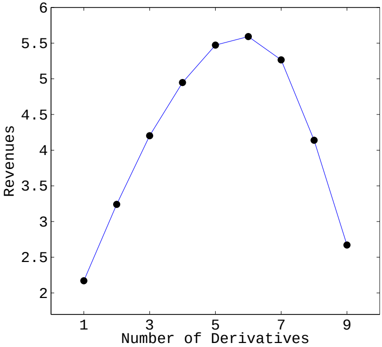
A example
Derivative price



Derivative prices as a function of the forward process.

A example
Price per share and Revenues





A example

Risk surfaces

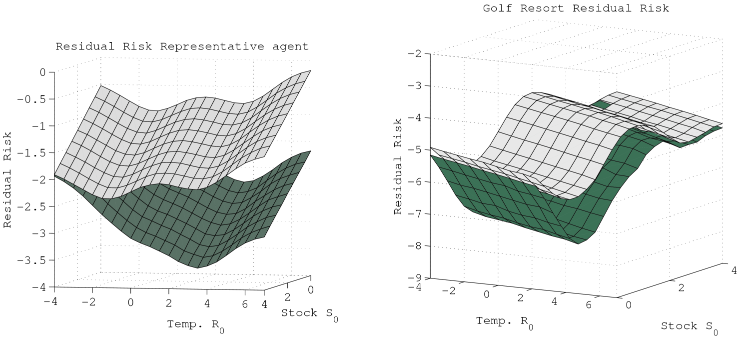


Figure: On the left the representative agent, on the right one of the agents.

Conclusion and Outlook

- Recap:
  - We proposed an equilibrium approach to pricing structured derivatives .
  - We derived sufficient conditions for market completeness (payoff's monotonicity with respect to the non-tradable risk).
  - Sensitivity analysis on the number of bonds and risk tolerance
  - We provide numerical results.

Thank you!

Thank you very much!

For Further Reading I

U. Horst, T. Pirvu and G. d. R.
On Securitization, Market Completion and Equilibrium Risk Transfer
Mathematics and finance economics 2010

P. Imkeller and G. d. R.
Path regularity and explicit convergence rate for BSDE with truncated quadratic growth
Stochastic processes and their applications 2010

N. Karoui, S. Peng and M. Quenez
BSDEs in finance
Mathematical Finance, Vol.7 (No. 1):1-71, 1997.