A general optimal stopping game with applications in finance

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Outline

- Motivations: game call options and callable stock loans
- Formulation of a general optimal stopping game
- Solutions: perpetual case
- Solutions: finite time horizon

Market model

Assuming a Black–Scholes market:

- A risk free bond B with a constant riskless interest rate r,

$$dB_t = rB_t dt,$$

- A stock with price process S, which under the risk neutral measure is governed by

$$dS_t = (r - d)\, S_t dt + \kappa S_t dW_t,$$

  where the interest rate $r \geq 0$, the dividend $d \geq 0$ and the volatility $\kappa > 0$ are constants and W is a one-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$ with $W_0 = 0$ almost surely.

game call options

- At time 0 the option holder pays a premium to the option writer and at any time t (before maturity) both the holder and the writer have the right to exercise the option. If the holder exercises the option at time t, he would claim the amount

$$Y_t = (S_t - K)^+$$

with strike price K. If the option writer exercises (or cancells) at time t, he is obliged to pay the holder the amount

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- What is the no arbitrage initial price for game call options?

game call options

- Initiated by Dynkin (1969) and later reformulated by Neveu (1975) to a more general set up.
- Game option by Kifer (2000).
- Kyprianou (2004): Perpetual game put options on stock without dividend payment:
  - When penalty is large: option writer should never exercise (cancel) the contract;
  - When penalty is small: exercising region for option writer is $\{K\}$.
- Kyprianou (2007): Finite game put options on stock without dividend payment.
- Kunita and Seko (2004): Finite game call options on stock paying dividend.

Callable stock loans

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  - paying the lender the principal amount and the loan interest, which is equal to $Le^{\gamma t}$ and hence redeeming his share of stock, or
  - surrendering the stock to the bank.
- The payoff of the client is $Y_t = (S_t - Le^{\gamma t})^+$ when he terminate the contract at time t.

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- Products with similar structure are traded on the financial markets under the name "callable repo".

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- initial value of callable stock loans: smallest initial capital for the lender of the loan to superhedge his position.
- The rational value of L and m should be such that the initial value of the callable stock loan is $(S_0 - L + m)$.
- What is the rationale values of L and m?

Literature review: Stock loans

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- Perpetual stock loans under Jump risk (Cai.N (2009))

A pereptual optimal stopping game

- Let X be a process under the risk neutral measure $\mathbb{Q}_x$ satisfying

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$$dX_t = (\rho - d)\, X_t dt + \kappa X_t dW_t, X_0 = x.$$

- Then the infinitesimal generator of the process $\left(e^{-\rho t} X_t\right)_{0 \le t < \infty}$ is given by

$$\mathcal{A} \triangleq \frac{\kappa^2}{2} x^2 \frac{d^2}{dx^2} + (\rho - d)\, x \frac{d}{dx} - \rho. \tag{1}$$

A pereptual optimal stopping game

- Define

$$g_1(x) \triangleq (x-q)^+, g_2(x) \triangleq \max(x-q+c, \theta c)^+$$

and

$$R_{s,t} \triangleq g_1(X_t)\mathbf{1}_{\{t \leq s\}} + g_2(X_t)\mathbf{1}_{\{s<t\}}.$$

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Problem

Find a function v and a pair of stopping times (σ^*, τ^*) such that the following holds

$$\begin{aligned} v(x) &= \sup_{\tau\geq 0}\inf_{\sigma\geq 0}\mathbb{E}_x\left[e^{-\rho\sigma\wedge\tau}R_{\sigma,\tau}\right] \\ &= \inf_{\sigma\geq 0}\sup_{\tau\geq 0}\mathbb{E}_x\left[e^{-\rho\sigma\wedge\tau}R_{\sigma,\tau}\right] = \mathbb{E}_x\left[e^{-\rho\sigma^*\wedge\tau^*}R_{\sigma^*,\tau^*}\right]. \end{aligned}$$

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- $\theta = 1$: game call options; $\theta = 0$: callable stock loans.

Solution to the perpetual case:

- Let $\lambda_1 > \lambda_2$ to be the roots of the quadratic equations

$$\frac{\kappa^2}{2}\lambda^2 + \left(\rho - d - \frac{\kappa^2}{2}\right)\lambda - \rho = 0.$$

and define

$$\lambda^* \triangleq \begin{cases} 1 & \text{if } d = 0 \text{ and } \rho \geq -\frac{\kappa^2}{2} \\ \frac{(\lambda_1 - 1)^{\lambda_1 - 1}}{\lambda_1^{\lambda_1}} < 1 & \text{if } d > 0, \text{or } d = 0 \text{ and } \rho < -\frac{\kappa^2}{2} \end{cases}.$$

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  - it is never optimal for the option seller to stop.
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- The explicit form of v in this case was given in Xia and Zhou (2007).

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- Exercising region for option holder:

$$D_1 = \{x : v(x) = g_1(x)\}.$$

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$$g_1(x) = g_2(x) = 0 \text{ for } x \leq q-c.$$

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- This reflects the fact that $q-c$ is the amount of the loan that the bank can at least get.

Solution to the perpetual case:

- Case 1: $\rho \geq 0$ and $d = 0$.

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$$D_2 = (0, \infty), D_1 = (0, q - c]$$

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- In this case,

$$\mathcal{A}g_i \geq 0 \text{ for } i = 1, 2.$$

The bank exercises immediately; while the client don't exercise (as long as $X_t > q - c$):

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- Intuitively the client should stop as long as X is large enough.
- For the bank, when (i) $\rho < 0$, or (ii) $\rho = 0$ and $d > 0$, or (iii) $\rho > 0$ and $d \geq \rho > 0$,

$$\mathcal{A}g_2 = -dx + r\,(q-c) \leq 0 \; \mathit{for} \; x > q - c.$$

The bank should wait as long as $X_s > q - c$, i.e.

$$D_2 = (0, q-c]\,.$$

Solution to the perpetual case:

- In the case $\rho \geq 0$ and $d = 0$,

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- case (iv) $\rho > 0$ and $\rho > d \geq d^*$: $D_2 = (0, q - c]$;
- case (v) $\rho > 0$ and $d^* > d > 0$: $D_2 = (0, b_1]$ with $b_1 \uparrow \infty$ as $d \downarrow 0$.

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$$D_2 = (0, q - c]\,, D_1 = (0, q - c] \cup [a_0, \infty)$$

Solution to the perpetual case:

- Plotting the value function:

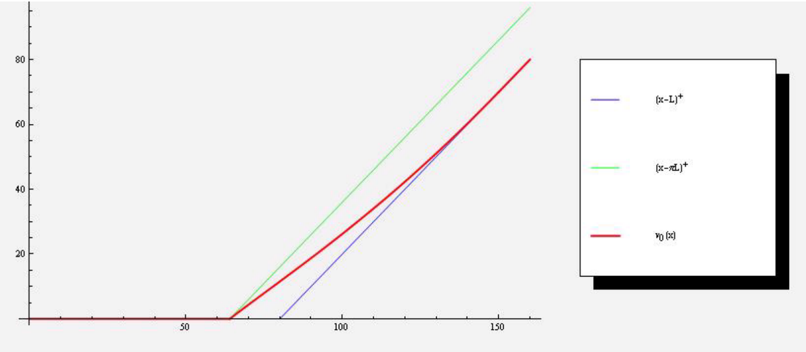

Figure: Figure 3: Graphical illustration of v with a market model $\rho = 0.03$, $\kappa = 0.15$, $d = 0.025$, $q = 80$ and $c = 16$. In this case $D_2 = (0, q - c] = (0, 64]$.

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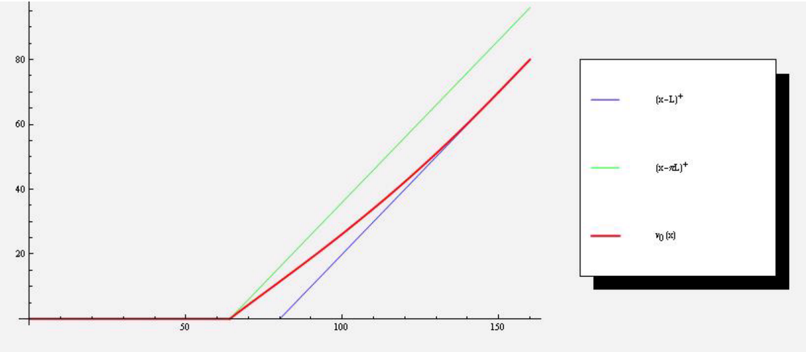

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- Smooth fit principle fails at the lower boundary $q - c$.

Solution to the perpetual case:

- $a_0 \triangleq \alpha_0 q$ and α_0 is the unique solution to either of the following equations.
  - for the case $d > 0$ or $\rho \neq -\frac{\kappa^2}{2}$,

$$(1-\lambda_1)\,\alpha^{1-\lambda_2} + \lambda_1 \alpha^{-\lambda_2} = \left(\frac{q-c}{q}\right)^{\lambda_1-\lambda_2} \left[(1-\lambda_2)\,\alpha^{1-\lambda_1} + \lambda_2 \alpha^{-\lambda_1}\right];$$

  - for the case $d = 0$ and $\rho = -\frac{\kappa^2}{2}$,

$$\alpha - \ln \alpha + \ln \frac{q-c}{q} - 1 = 0.$$

Solution to the perpetual case:

- When $\rho > 0$, define

$$v_B(x) \triangleq \sup_{\tau} \mathbb{E}_x \left[e^{-\rho\tau} g_1(X_\tau) \mathbf{1}_{\{\tau < \sigma_{q-c}\}} \right]$$

and

$$d^* \triangleq \inf \left\{ d > 0 : \frac{d}{dx} v_B((q-c)+) \leq 1 \right\},$$

where $\frac{d}{dx} u((q-c)+) = \lim_{x \downarrow (q-c+\theta c)} \frac{u(x)}{x-(q-c)}$.

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where $\frac{d}{dx} u((q-c)+) = \lim_{x \downarrow (q-c+\theta c)} \frac{u(x)}{x-(q-c)}$.

- v_B : the price of an American down-and-out call option with strike q and barrier $q-c$.
- d^* : the smallest dividend such that the delta of the American down-and-out call at the barrier is smaller than unity.

Solution to the perpetual case:

Case 2. b): When (v) $\rho > 0$ and $d^* > d > 0$,

$$D_2 = (0, b_1] \text{ and } D_1 = (0, q - c] \cup [a_1, \infty).$$

Solution to the perpetual case:

- Plotting the value function:

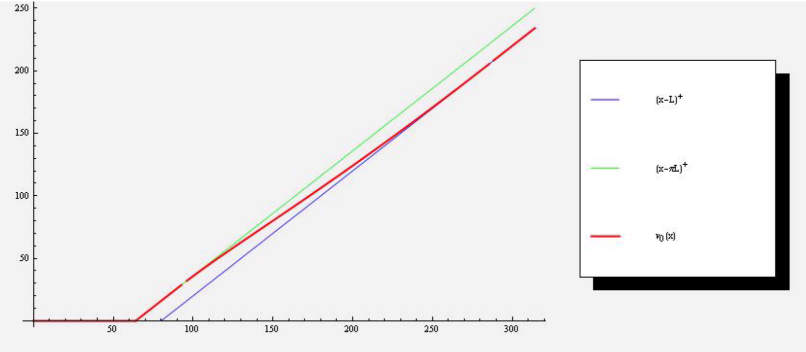

Figure: Figure 4: Graphical illustration of v with a market model $\rho = 0.03$, $\kappa = 0.15$, $d = 0.012$, $q = 80$ and $c = 16$. In this case $D_2 = (0, b_1] = (0, 92.77]$.

- Smooth fit principle holds at the lower boundary b_1.

Solution to the perpetual case:

- $(b_1, a_1) \triangleq (\beta_1 q, \alpha_1 q)$ and (β_1, α_1) is the unique pair of solutions to the system of equations

$$\left\{ \begin{array}{c} (1-\lambda_1)\,\alpha^{1-\lambda_2} + \lambda_1 \alpha^{-\lambda_2} + (\lambda_1 - \lambda_2)\left(\beta^{1-\lambda_2} - \frac{q-c}{q}\beta^{-\lambda_2}\right) \\ = \beta^{\lambda_1-\lambda_2}\left((1-\lambda_2)\,\alpha^{1-\lambda_1} + \lambda_2 \alpha^{-\lambda_1}\right), \\ (1-\lambda_1)\,\beta^{1-\lambda_2} + \lambda_1 \frac{q-c}{q}\beta^{-\lambda_2} + (\lambda_1-\lambda_2)\left(\alpha^{1-\lambda_2} - \alpha^{-\lambda_2}\right) \\ = \alpha^{\lambda_1-\lambda_2}\left((1-\lambda_2)\,\beta^{1-\lambda_1} + \lambda_2 \frac{q-c}{q}\beta^{-\lambda_1}\right). \end{array} \right. .$$

- Rational value of L and m: consider the case 2.b) $r - \gamma > 0$ and $d < d^*$:

Applications in finance

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- $X_0 \geq a_1$, i.e. $\frac{L}{X_0} \leq \frac{L}{a_1} = \frac{1}{\alpha_1}$, the loan-to-value is too small for the client, and the optimal time to redeem the stock is $\tau_{a_1} = 0$, thus there is actually no physical exchange between the bank and the client.

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- Rational value of L and m: consider the case 2.b) $r - \gamma > 0$ and $d < d^*$:
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- $X_0 \leq b_1$, i.e. $\frac{L}{X_0} \geq \frac{L}{b_1} = \frac{1}{\beta_1}$, the loan-to-value is too large for the bank, and the optimal call time is $\sigma_{b_1} = 0$, which also suggests that there is no exchange between the two parties again.
- $X_0 \in (b_1, a_1)$, both parties are willing to carry out the business and the fair fee charged is $m = v(X_0) - X_0 + L$, i.e. the loan is marketable if the loan-to-value ratio lies in $\left(\frac{L}{a_1}, \frac{L}{b_1}\right) = \left(\frac{1}{\alpha_1}, \frac{1}{\beta_1}\right)$.

optimal stopping game in finite time horizon

Problem

Find a function v and a pair of stopping times (σ^*, τ^*) such that the following holds

$$
\begin{aligned}
v(t,x) &= \sup_{\tau \leq T-t} \inf_{\sigma \leq T-t} \mathbb{E}_{t,x}\left[e^{-\rho \sigma \wedge \tau} R_{\sigma,\tau}\right] \\
&= \inf_{\sigma \leq T-t} \sup_{\tau \leq T-t} \mathbb{E}_{t,x}\left[e^{-\rho \sigma \wedge \tau} R_{\sigma,\tau}\right] = \mathbb{E}_{t,x}\left[e^{-\rho \sigma^* \wedge \tau^*} R_{\sigma^*,\tau^*}\right].
\end{aligned}
$$

Solution to the finite maturity problem

- When $\theta c \geq \lambda^* q^{1-\lambda_1} \left(q - c + \theta c\right)^{\lambda_1}$, the above problem becomes an optimal stopping problem, it is never for the option writer to stop.

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- When $\theta c < \lambda^* q^{1-\lambda_1} (q - c + \theta c)^{\lambda_1}$,
  - Case 1: $\rho \geq 0$ and $d = 0$.

$$v(t,x) = \begin{cases} \frac{\theta c}{q-c+\theta c} x & \textit{if} \quad x \leq q - c + \theta c \\ x - q + c & \textit{if} \quad x > q - c + \theta c \end{cases} .$$

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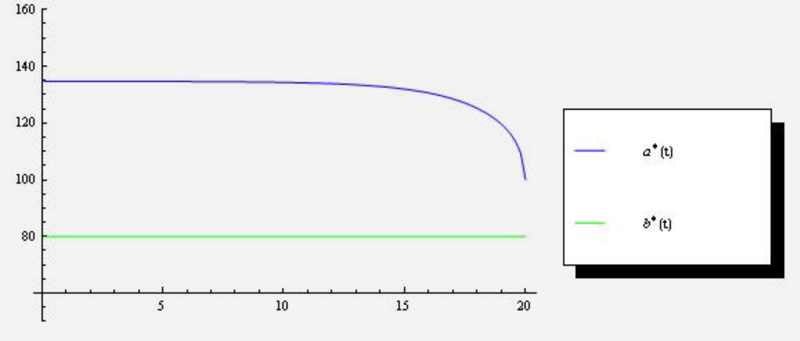
  - Case 2: $\rho < 0$ or $d > 0$.

Solution to the finite maturity problem

Take $\theta = 0$ as a example

- In case 2.a)



Plotting of optimal stopping boudaries in a market model with $\rho = -0.03$, $\kappa = 0.15$, $d = 0$, $q = 100$, $c = 16$ and $T = 20$.

Solution to the finite maturity problem

Consider $\theta = 0$ and case 2.a)

- $b^*(t) \equiv \pi L$ and $a^*(t) = a_0(t)$ is the unique solution to the integral equation

$$I(t, a(t)) = a(t) - L + \int_0^{T-t} K_1(t, a(t), s, a(t+s))\, ds$$

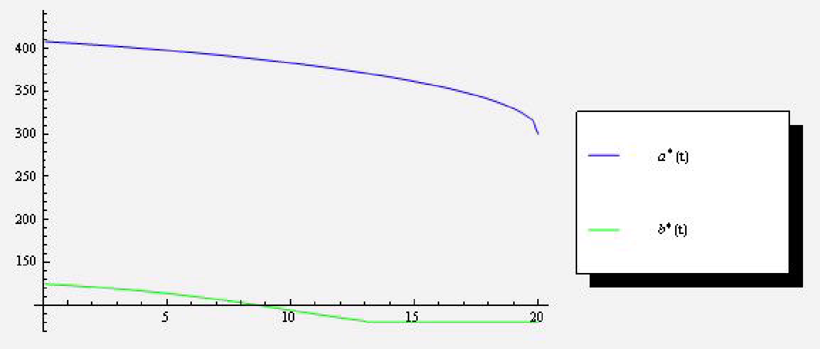
with terminal condition $a(T) = L$ if $\widetilde{r} < 0$ or $d \geq \widetilde{r}$ and $a(T) = \frac{\widetilde{r}}{d} L$ otherwise, where

$$\begin{aligned} I(t, x) &= \mathbb{E}_{t,x}\left[e^{-\widetilde{r}(T-t)} (X_T - L)^+ \mathbf{1}_{\{\tau_{\pi L} > T-t\}}\right], \\ K_1(t, x, s, y) &= \mathbb{E}_{t,x}\left[e^{-\widetilde{r}s} (-dX_{t+s} + \widetilde{r}L) \mathbf{1}_{\{X_{t+s} > y\}} \mathbf{1}_{\{\tau_{\pi L} > s\}}\right], \end{aligned}$$

for $t \in [0, T]$ and $s \in [0, T-t]$.

Solution to the finite maturity problem

- In case 2. b).



Plotting of optimal stopping boundaries in a market model with $\rho = 0.02$, $d = 0.01$, $\kappa = 0.15$, $d = 0.014$, $c = 16$, $q = 100$ and $T = 20$.

- The lower boundary is time dependent for $t < t^* = 13.3$.

Solution to the finite maturity problem

Consider $\theta = 0$ and case 2.b)

- Define

$$v_B(s,x) \triangleq \sup_{\tau \in \mathcal{T}_{0,u}} \mathbb{E}_x \left[e^{-\rho\tau} (X_\tau - q)^+ \mathbf{1}_{\{\tau \le \tau_{q-c}\}} \right], \tag{2}$$

and

$$s^* = \sup \left\{ u > 0 : \frac{d}{dx} v_B(s, (q-c)+) \le 1 \right\}. \tag{3}$$

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- $v_B(s,x)$ is the price of a American down and out call option with *time to maturity* s, strike q and barrier $q - c$.
- s^* is well-defined and $0 < s^* < \infty$. Define $t^* = (T - s^*) \vee 0$.

- For $t > t^*$, $b^*(t) \equiv q - c$ and $a^*(t) = a_0(t)$ as in the case 2.a).

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- For $t \leq t^*$, $b^*(t) = b_1(t) > q - c$ and $a^*(t) = \alpha_1(t)$, where (b_1, a_1) is the unique solution to the system of equations:

$$
\begin{aligned}
J(t, a(t)) &= a(t) - q + \int_0^{T-t} K_1(t, a(t), s, a(t+s))\, ds \\
&\quad + \int_0^{T-t} K_2(t, a(t), s, b(t+s))\, ds \\
J(t, b(t)) &= b(t) - (q - \delta) + \int_0^{T-t} K_1(t, b(t), s, a(t+s))\, ds \\
&\quad + \int_0^{T-t} K_2(t, b(t), s, b(t+s))\, ds
\end{aligned}
$$

with terminal condition $a_1(t^*) = a_0(t^*)$ and $b_1(t^*) = q - c$.

Solution to the finite maturity problem

- The function J and K_2 are defined as

$$
\begin{aligned}
J(t,x) &= \mathbb{E}_{t,x}\left[e^{-\widetilde{r}(t^*-t)}u\left(t^*-t, X_{t^*}\right)\mathbf{1}_{\{\tau_{\pi L}>t^*-t\}}\right], \\
K_2(t,x,s,y) &= \mathbb{E}_{t,x}\left[e^{-\widetilde{r}s}\left(-dX_{t+s}+\widetilde{r}\pi L\right)\mathbf{1}_{\{X_{t+s}<y\}}\mathbf{1}_{\{\tau_{\pi L}>s\}}\right],
\end{aligned}
$$

for $t \in [0, t^*]$ and $s \in [0, t^* - t]$.