Outperforming The Market Portfolio With A Given Probability

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Outline

- Consider a financial market with a bond $B(\cdot) = 1$ and d stocks $X = (X_1, \cdots, X_d)$ which satisfy for $i = 1; \cdots d$,

$$dX_i(t) = X_i(t) \left(b_i(X(t))dt + \sum_{k=1}^{d} s_{ik}(X(t))dW_k(t) \right). \tag{1}$$

- Let $\mathcal{H}$ be the set of $\mathbb{F}$-progressively measurable processes $\pi : [0, T) \times \Omega \to \mathbb{R}^d$, which satisfies

$$\int_0^T \left(|\pi'(t)\mu(X(t))| + \pi'(t)\alpha(X(t))\pi(t) \right) dt < \infty, \quad \text{a.s.},$$

in which $\mu = (\mu_1, \cdots, \mu_d)$ and $\sigma = (\sigma_{ij})_{1 \le i,j \le d}$ with $\mu_i(x) = b_i(x)x_i$, $\sigma_{ik}(x) = s_{ik}(x)x_i$, and $\alpha(x) = \sigma(x)\sigma'(x)$.

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The Problem

- For each $\pi \in \mathcal{H}$ and initial wealth $y \geq 0$ the associated wealth process will be denoted by $Y^{y,\pi}(\cdot)$. This process solves

$$dY^{y,\pi}(t) = Y^{y,\pi}(t) \sum_{i=1}^{d} \pi_i(t) \frac{dX_i(t)}{X_i(t)}, \quad Y^{y,\pi}(0) = y.$$

- In this paper, we want to determine and characterize

The Problem

$$V(T,x,p) = \inf\{y > 0 | \exists \pi \in \mathcal{H} \text{ s.t.} \mathbb{P}\{Y^{y,\pi}(T) \geq g(X(T))\} \geq p\}$$

, where X(0)=x, $g : (0,\infty)^d \mapsto \mathbb{R}_+$ is a measurable function.

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Related Work

- In the case where $p = 1$ and $g(x) = x_1 + \cdots + x_d$,

$$V(T, x, 1) = \inf\{y > 0 | \exists \pi \in \mathcal{H} \text{ s.t. } Y^{y,\pi}(T) \geq g(X(T)) \text{ a.s.}\}.$$

In Fernholz and Karatzas (2008), a PDE characterization for $\tilde{V}(T, x, 1) := V(T, x, 1)/g(x)$ was derived when $V(T, x, 1)$ is assumed to be smooth.

- In Bouchard, Elie and Touzi (2009), a PDE characterization of $V(t, x, p)$ was derived.
  - Assumptions: rather strong, e.g. existence of a unique strong solution of (1);
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Related Work

- In our paper, we will also work towards a PDE characterization for $V(t,x,p)$, but
  - We only assume the existence of a weak solution of (1) that is unique in distribution;
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Assumptions

Assumption M

- Let $b_i : (0,\infty)^d \to \mathbb{R}$ and $s_{ik} : (0,\infty)^d \to \mathbb{R}$ be continuous functions and $b(\cdot) = (b_1(\cdot),\cdots,b_d(\cdot))'$ and $s(\cdot) = (s_{ij}(\cdot))_{1\le i,j\le d}$, which we assume to be invertible for all $x \in (0,\infty)^d$.
- We also assume that (1) has a weak solution that is unique in distribution for every initial value.
- Let $\theta(\cdot) := s^{-1}(\cdot)b(\cdot)$, $a_{ij}(\cdot) := \sum_{k=1}^{d} s_{ik}(\cdot)s_{jk}(\cdot)$ s atisfy

$$\sum_{i}^{d}\int_0^T \left(|b_i(X(t))| + a_{ii}(X(t)) + \theta_i^2(X(t))\right) < \infty. \qquad (2)$$

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Consequences of Assumptions

- We denote by $\mathbb{F}$ the augmentation of the natural filtration of $X(\cdot)$.
- Thanks to Assumption M,
  - every local martingale of $\mathbb{F}$ has the martingale representation property with respect to $W(\cdot)$ (adapted to $\mathbb{F}$).
  - the solution of (1) takes values in the positive orthant
  - the exponential local martingale

$$Z(t) := \exp\left\{-\int_0^t \theta(X(s))' dW(s) - \frac{1}{2}\int_0^t |\theta(X(s))|^2 ds\right\}, \tag{3}$$

    the so-called *deflator* is well defined. We do not exclude the possibility that $Z(\cdot)$ is a strict local martingale.

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What does the existence of a deflator entail?

- While we do not assume the existence of equivalent local martingale measures, we assume the existence of a local martingale deflator (the $Z(\cdot)$ process). This is equivalent to the *No-Unbounded-Profit-with-Bounded-Risk* (NUPBR) condition, introduced in Karatzas and Kardaras (2007).
- By Kardaras (2010), NUPBR is equvalent to the non-existence of arbitrages of the first kind, arbitages that can be attained through nonegative wealth processes.
- So in our model, arbitrage may exist, but we cannot scale it up to make arbitrary amount of money.

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1. Introduction
   - The Problem
   - Related Work
   - The Model
2. On Quantile Hedging
3. The PDE Characterization
   - Stochastic Control Problem Formulation
   - Associated PDE

- Let $g : (0,\infty)^d \to \mathbb{R}_+$ be a measurable function satisfying

$$\mathbb{E}[Z(T)g(X(T))] < \infty. \tag{4}$$

- We want to determine

$$V(T,x,p) = \inf\{y > 0 | \exists \pi \in \mathcal{H} \text{ s.t. } \mathbb{P}\{Y^{y,\pi}(T) \geq g(X(T))\} \geq p\}, \tag{5}$$

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Lemma 3.1

We will present a probabilistic characterization of $V(T,x,p)$.

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Given $A \in \mathcal{F}_T$,

(I) if $\mathbb{P}(A) \geq p$, then

$$V(T,x,p) \leq \mathbb{E}[Z(T)g(X(T))1_A].$$

(II) if $\mathbb{P}(A) = p$ and

$$\operatorname{ess\,sup}_A\{Z(T)g(X(T))\} \leq \operatorname{ess\,inf}_{A^c}\{Z(T)g(X(T))\}, \quad (6)$$

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Proof of Lemma 3.1

(I) Assumption M implies that the market is complete. So $Z(T)g(X^{t,x}(T))1_A \in \mathcal{F}_T$ is replicable with initial capital $\mathbb{E}[Z(T)g(X^{t,x}(T))1_A]$. Since $\mathbb{P}(A) \geq p$, it follows that $V(T,x,p) \leq \mathbb{E}[Z(T)g(X^{t,x}(T))1_A]$.

(II) take arbitrary $y_0 > 0$ and $\pi_0 \in \mathcal{H}$ such that

$$\mathbb{P}\{B\} \geq p, \text{ where } B \triangleq \{Y^{y_0,\pi_0}(T) \geq g(X(T))\}.$$

To prove equality in (7), it's enough to show that

$$y_0 \geq \mathbb{E}[Z(T)g(X(T))1_A].$$

Observing that
$\mathbb{P}(A^c \cap B) = \mathbb{P}(A \cup B) - \mathbb{P}(A) \geq \mathbb{P}(A \cup B) - \mathbb{P}(B) = \mathbb{P}(B^c \cap A)$
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(I) Assumption M implies that the market is complete. So $Z(T)g(X^{t,x}(T))1_A \in \mathcal{F}_T$ is replicable with initial capital $\mathbb{E}[Z(T)g(X^{t,x}(T))1_A]$. Since $\mathbb{P}(A) \geq p$, it follows that $V(T,x,p) \leq \mathbb{E}[Z(T)g(X^{t,x}(T))1_A]$.

(II) take arbitrary $y_0 > 0$ and $\pi_0 \in \mathcal{H}$ such that

$$\mathbb{P}\{B\} \geq p, \text{ where } B \triangleq \{Y^{y_0,\pi_0}(T) \geq g(X(T))\}.$$

To prove equality in (7), it's enough to show that

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Proof of Lemma 3.1 (conti.)

$$
\begin{aligned}
y_0 &\geq \mathbb{E}[Z(T)Y^{y_0,\pi_0}(T)] \\
&= \mathbb{E}[Z(T)Y^{y_0,\pi_0}(T)1_B] + \mathbb{E}[Z(T)Y^{y_0,\pi_0}(T)1_{B^c}] \\
&\geq \mathbb{E}[Z(T)g(X(T))1_B] \\
&= \mathbb{E}[Z(T)g(X(T))1_{A\cap B}] + \mathbb{E}[Z(T)g(X(T))1_{A^c\cap B}] \\
&\geq \mathbb{E}[Z(T)g(X(T))1_{A\cap B}] + \mathbb{P}(A^c \cap B)\operatorname{ess\,inf}_{A^c\cap B}\{Z(T)g(X(T))\} \\
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&\geq \mathbb{E}[Z(T)g(X(T))1_{A\cap B}] + \mathbb{E}[Z(T)g(X(T))1_{A\cap B^c}] \\
&= \mathbb{E}[Z(T)g(X(T))1_A].
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Now we want show we can indeed find an $A \in \mathcal{F}_T$ satifying the coinditions in Lemma 3.1 (ii).

- Let $F(\cdot)$ be the cumulative distribution function of $Z(T)g(X(T))$.
- For any $a \in \mathbb{R}_+$ define

$$A_a := \{\omega : Z(T)g(X(T)) < a\}, \ \partial A_a := \{\omega : Z(T)g(X(T)) = a\},$$

  and let $\bar{A}_a$ denote $A_a \cup \partial A_a$.
- Taking $A = \bar{A}_a$ in Lemma 3.1, it follows that

$$V(T, x, F(a)) = \mathbb{E}[Z(T)g(X(T))1_{\bar{A}_a}]. \tag{8}$$

  On the other hand, taking $A = A_a$, we obtain that

$$V(T, x, F(a-)) = \mathbb{E}[Z(T)g(X(T))1_{A_a}]. \tag{9}$$

  The last two equalities imply the following relationship

$$\begin{aligned} V(T, x, F(a)) &= V(t, x, F(a-)) + a\mathbb{P}\{\partial A_a\} \\ &= V(t, x, F(a-)) + a(F(a) - F(a-)). \end{aligned} \tag{10}$$

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Next, we will determine $V(T,x,p)$ for $p \in (F(a-), F(a))$ when $F(a-) < F(a)$.

Proposition 3.1

Fix arbitrary $(t,x,p) \in (0,T) \times (0,\infty)^d \times [0,1]$

(I) There exists $A \in \mathcal{F}_T$ satisfying $\mathbb{P}(A) = p$ and (6). As a result, we have

$$V(T,x,p) = \mathbb{E}[Z(T)g(X(T))1_A].$$

(II) If $F^{-1}(p) := \{s \in \mathbb{R}_+ : F(s) = p\} = \emptyset$, then letting $a := \inf\{s \in \mathbb{R}_+ : F(s) > p\}$ we have

$$\begin{aligned} V(T,x,p) &= V(T,x,F(a-)) + a(p - F(a-)). \\ &= V(T,x,F(a)) - a(F(a) - p) \end{aligned} \tag{11}$$

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- Define $A := A_a \cup (\partial A_a \cap B_{b^*})$. Observe that $\mathbb{P}(A) = \mathbb{P}(A_a) + \mathbb{P}(\partial A_a \cap B_{b^*}) = p$, and A satisfies (6).

Proof of Proposition 3.1 (conti.)

For (ii), it follows immediately from (i),

$$\begin{aligned} V(T,x,p) &= \mathbb{E}[Z(T)g(X(T))1_A] \\ &= \mathbb{E}[Z(T)g(X(T))1_{A_a}] + \mathbb{E}[Z(T)g(X(T))1_{\partial A_a \cap B_{b^*}}] \\ &= V(T,x,F(a-)) + a\mathbb{P}(\partial A_a \cap B_{b^*}) \\ &= V(t,x,F(a-)) + a(p - F(a-)). \end{aligned}$$

Remark

When Z is a martingale:

- Using Neyman-Pearson Lemma, Follmer and Leukert (1999) showed that

$$V(T,x,p) = \inf_{\varphi \in \mathcal{M}} \mathbb{E}[Z(T)g(X(T))\varphi] = \mathbb{E}[Z(T)g(X(T))\varphi^*], \tag{12}$$

where

$$\mathcal{M} = \{\varphi : \Omega \to [0,1] \text{ is } \mathcal{F}_T \text{ measurable s.t. } \mathbb{E}[\varphi] \geq p\}. \tag{13}$$

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Example

- Consider a market with a single stock, whose dynamics follow a three-dimensional Bessel process, i.e.

$$dX(t) = \frac{1}{X(t)}dt + dW(t) \quad X_0 = x > 0,$$

  and let $g(x) = x$.
- In this case, $Z(t) = x/X(t)$, which is the classical example for a strict local martingale; see Johnson and Helms (1963). On the other hand, $Z(t)X(t) = x$ is a martingale.
- Thanks to Proposition 3.1 there exits a set $A \in \mathcal{F}_T$ with $\mathbb{P}(A) = p$ such that

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Proposition 3.3

Here, we will give alternative representation of V, which facilitates its PDE characterization in the next section. Recall that

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- Thanks to Proposition 3.1 there exists a set $A \in \mathcal{F}_T$ such that $V(T,x,p) = \mathbb{E}[Z(T)g(X(T))1_A]$. Since $1_A \in \mathcal{M}$, clearly

$$V(T,x,p) \geq \inf_{\varphi \in \mathcal{M}} \mathbb{E}[Z(T)g(X(T))\varphi].$$

- For the other direction, we will show that for any $\varphi \in \mathcal{M}$ and a given set $A \in \mathcal{F}_T$ satisfying $\mathbb{P}(A) = p$ and (6), we have

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- Letting $M = \operatorname{ess\,sup}_A\{Z(T)g(X(T))\}$, we can write

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&\quad - \mathbb{E}[Z(T)g(X(T))1_A] \\
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Outline

1. Introduction
   - The Problem
   - Related Work
   - The Model
2. On Quantile Hedging
3. The PDE Characterization
   - Stochastic Control Problem Formulation
   - Associated PDE

An Additional State Variable

- Let us denote by $P_\alpha^p(\cdot)$ the solution of

$$dP(t) = P(t)(1 - P(t))\alpha'(t)dW(t), \; P(0) = p \in [0,1], \; (14)$$

where $\alpha(\cdot)$ is an $\mathbb{F}-$progressively measurable $\mathbb{R}^d$-valued process such that $\int_0^T \|\alpha(s)\|^2 ds < \infty$ $\mathbb{P}$-a.s. We will denote the class of such processes by $\mathcal{A}$.

- The next result obtains an alternative representation for V in terms of P.

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$V(T, x, p) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}[Z(T)g(X(T))P_\alpha^p(T)] < \infty.$

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- The finiteness follows from (4).
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- The inclusion $\widetilde{\mathcal{M}} \supset \{P^p_\alpha(T) : \alpha \in \mathcal{A}\}$ is clear. To show the other inclusion, use the martingale representation theorem: For any $\varphi \in \mathcal{F}_T$ there exists an $\mathbb{F}-$progressively measurable $\mathbb{R}^d$-valued process $\psi(\cdot)$ satisfying

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The Value Function U

- We denote by $X^{t,x}(\cdot)$ the solution of (1) starting from x at time t and by $P_\alpha^{t,p}(\cdot)$ the solution of (14) starting from p at time t. We also introduce $Z^{t,x,z}(\cdot)$ as the solution of

$$dZ(s) = -Z(s)\theta(X^{t,x}(s))' dW(s), \;\; Z(t) = z, \tag{15}$$

and the value function

$$U(t,x,p) := \inf_{\alpha \in \mathcal{A}} \mathbb{E}[Z^{t,x,1}(T) g(X^{t,x}(T)) P_\alpha^{t,p}(T)]. \tag{16}$$

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Express U under a new measure $\mathbb{Q}$

- First define

$$\begin{aligned}\Lambda(t,\cdot) &:= \frac{x_1+\cdots+x_d}{Z^{t,x,1}(\cdot)(X_1^{t,x}(\cdot)+\cdots+X_d^{t,x}(\cdot))} \\ &= \exp\left(\int_t^{\cdot}(\tilde{\theta}(X^{t,x}(u)))' d\widetilde{W}(u) - \frac{1}{2}\int_t^{\cdot}\|\tilde{\theta}(X^{t,x}(u))\|^2 du\right)\end{aligned}$$

in which $\tilde{\theta}(\cdot) := \theta(\cdot) - s'(\cdot)\mathfrak{m}(\cdot)$, where $\mathfrak{m}$ is defined by $\mathfrak{m}_i(x) = x_i/(x_1+\cdots+x_d)$, $i = 1,\cdots,d$, and

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in which $\tilde{\theta}(\cdot) := \theta(\cdot) - s'(\cdot)\mathfrak{m}(\cdot)$, where $\mathfrak{m}$ is defined by $\mathfrak{m}_i(x) = x_i/(x_1+\cdots+x_d)$, $i = 1,\cdots,d$, and

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Express U under a new measure $\mathbb{Q}$ (conti.)

- There exists a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ such that $d\mathbb{P} = \Lambda(t,T)d\mathbb{Q}$ on each $\mathcal{F}(T)$, for $T \in (t, \infty)$. Under $\mathbb{Q}$, $\widetilde{W}(\cdot)$ is a Brownian motion and we have that

$$\frac{\mathbb{E}[Z^{t,x,1}(T)(X_1^{t,x}(T) + \cdots + X_d^{t,x}(T))]}{x_1 + \cdots + x_n} = \mathbb{Q}(\mathcal{T} > T),$$

for all $T \in [0, \infty)$, where

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Express U under a new measure $\mathbb{Q}$ (conti.)

We will make the following assumption to obtain a representation of $\mathcal{T}$ in terms of X.

Assumption 4.1

$\|\theta\|^2 \leq C(1 + Trace(a))$.

Under this assumption, it follows $\mathbb{Q}-$a.e. that

$$\mathcal{T} = \min_{1 \leq i \leq d} \mathcal{T}_i, \qquad \text{in which} \quad \mathcal{T}_i = \inf\{s \geq t : X_i^{t,x}(s) = 0\}.$$

For these claims about the existence and the properties of the probability measure $\mathbb{Q}$ see Fernholz and Karatzas (2008, 2010), and the references therein.

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EXPRESS U UNDER A NEW MEASURE $\mathbb{Q}$ (CONTI.)

Now, U can be represented in terms of $\mathbb{Q}$ as

$$U(t,x,p) = (x_1 + \cdots + x_d) \inf_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{Q}} \left[\frac{g(X^{t,x}(T))}{X_1^{t,x}(T) + \cdots + X_d^{t,x}(T)} P_\alpha^{t,p}(T) 1_{\{\mathcal{T} > T\}} \right].$$

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Express U under a new measure $\mathbb{Q}$ (conti.)

The dynamics of $X^{t,x}$ and $P^{t,p}$ in terms of the $\mathbb{Q}$-Brownian motion $\widetilde{W}$ can be written as

$$dX_i^{t,x}(s) =$$
$$X_i^{t,x}(s)\left(\frac{\sum_{j=1}^{d} a_{ij}(X^{t,x}(s))X_j^{t,x}(s)}{X_1^{t,x}(s)+\cdots+X_d^{t,x}(s)}ds + \sum_{k=1}^{d} s_{ik}(X^{t,x}(s))d\widetilde{W}_k(s)\right),$$

for $i = 1, \cdots, d$, and

$$dP^{t,p}(s) = P^{t,p}(s)(1-P^{t,p}(s))\alpha'(s)(-\tilde{\theta}(X^{t,x})ds + d\widetilde{W}(s)). \quad (17)$$

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Dynamic Programming

To apply the dynamic programming principle due to Haussmann and Lepeltier (1990), we assume

Assumption 4.2

For all $y \in \mathbb{R}_+^d - \{0\}$ we have the following growth condition

$$\sum_{i=1}^{d} \sum_{j=1}^{d} y_i y_j |a_{ij}(y)| \leq C(1 + \|y\|).$$

for some constant C.

Assumption 4.3

The mapping $(t, x, p) \to \mathbb{E}[Z^{t,x,1}(T) g(X^{t,x}(T)) P_\alpha^{t,p}(T)]$ is lower semi-continuous on $t \in [0, T]$, $x \in \mathbb{R}_+^d$, $p \in [0, 1]$, for all $\alpha \in \mathcal{A}$.

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Dynamic Programming (Conti.)

Proposition 4.2

Under Assumption M, 4.1, 4.2 and 4.3,

(I) U^* is a viscosity subsolution of

$$\partial_t U^* + \frac{1}{2}\text{Trace}\left(\sigma\sigma' D_x^2 U^*\right) + \inf_{a\in\mathbb{R}^d}\left\{(D_{xp}U^*)'\sigma a + \frac{1}{2}|a|^2 D_p^2 U^* - \theta' a D_p U^*\right\} \geq 0,$$

with the boundary conditions
$U^*(t,x,1) = \mathbb{E}[Z^{t,x,1}(T)g(X^{t,x}(T))]$, $U^*(t,x,0) = 0$, and $U^*(T,x,p) \leq pg(x)$.

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Dynamic Programming (conti.)

Proposition 4.2 (conti.)

(ii) U^* is a viscosity supersolution of

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Remark

- Let us consider the PDE satisfied by the superhedging price $U(t,x,1)$:

$$0 = v_t + \frac{1}{2} Tr(\sigma\sigma' D_x^2 v), \quad \text{on } (0,T) \times (0,\infty)^d, \tag{19}$$

$$v(T-,x) = g(x), \quad \text{on } (0,\infty)^d. \tag{20}$$

Unless additional boundary conditions are specified, this PDE may have multiple solutions, see e.g. the volatility stabilized model of Fernholz and Karatzas (2008). Even when additional boundary conditions are specified, the growth of σ might lead to the loss of uniqueness. In the one-dimensional case one can determine an explicit condition which is sufficient and necessary for uniqueness (non-uniqueness), see e.g. Bayraktar and Xing (2010).

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Remark (conti.)

- Let $\Delta U(t,x,1)$ be the difference of two solutions of (19)-(20). Then both $U(t,x,p)$ and $U(t,x,p)+\Delta U(t,x,1)$ are viscosity supersolution of (18) (along with its boundary conditions). As a result when (19) and (20) has multiple solutions so does the PDE for the function U.

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Thank you very much for your attention!

Q & A