Mean Field Games with a Common Noise

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Background

- Optimization over interacting particles/players
  - Interaction through the empirical measure of the system
  - Existence of Nash equilibria when number players $\to \infty$
- Standard theory $\rightsquigarrow$ players driven by independent noises
  - N players: dynamics of player number $1 \le i \le N$

$$dX_t^i = b(t, X_t^i, \bar{\mu}_t^N, \alpha_t^i)dt + \sigma dW_t^i,$$

  - $\bar{\mu}_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i} \qquad 0 \le t \le T$
  - Independence of B.M. $(W^i)_{1 \le i \le N}$
  - α_t^i prog. meas. w.r.t. $\sigma(W^1, \dots, W^N)$

Common noise

- New BM B, independent of $(W^i)_{1\leq i\leq N}$
  - Dynamics of player number $1 \leq i \leq N$

$$dX_t^i = b(t, X_t^i, \bar{\mu}_t^N, \alpha_t^i)dt + \sigma dW_t^i + \varsigma(t, X_t^i)dB_t$$

  - α_t^i prog. meas. w.r.t. $\sigma(W^1, \dots, W^N, B)$
  - $\varsigma \rightsquigarrow$ influence of B varies according to i
  - ex: CO_2 markets $\rightsquigarrow$ perceived emissions of the agents
- Nash equilibrium w.r.t.
  - $J^i = \mathbb{E}\Big[g(X_T^i, \bar{\mu}_T^N) + \int_0^T f(t, X_t^i, \bar{\mu}_t^N, \alpha_t^i)dt\Big]$
  - $(\alpha^{1,\star}, \dots, \alpha^{N,\star})$ is equilibrium if

$$J^i(\dots, \alpha^{i-1,\star}, \alpha^i, \alpha^{i+1,\star}, \dots) \geq J^i(\dots, \alpha^{i-1,\star}, \alpha^{i,\star}, \alpha^{i+1,\star}, \dots)$$

Conditional law of large numbers

- Find limit optimization problems as $N \uparrow +\infty$?
- 'Solvability' of the limit optimization problem?
  - Nash-equilibrium? Optimal control?
- Exchangeable equilibria $\rightsquigarrow$ conditional LLN

$$\bar{\mu}_t^N \sim_{N\uparrow+\infty} \mathcal{L}(X_t^1|B)$$

  - Dynamics of particle 1

$$dX_t^1 \underset{N\uparrow+\infty}{\sim} b(t, X_t^1, \mathcal{L}(X_t^1|B), \alpha_t^1)dt + \sigma dW_t^1$$

  - Cost of player 1

$$J^1 \underset{N\uparrow+\infty}{\sim} \mathbb{E}\left[g(X_T^1, \mathcal{L}(X_t^1|B)) + \int_0^T f(t, X_t^1, \mathcal{L}(X_t^1|B), \alpha_t^1)dt\right]$$

Notion of mean-field equilibrium

• When α^1 varies, the common conditional measure doesn't vary!

◦ Optimization is performed when the conditional measure is frozen

• Scheme

◦ Fix the flow of random measures $(\mu_t)_{0 \le t \le T}$ prog. meas. w.r.t. $\sigma(B)$!

◦ Optimize

$$dX_t^1 = b(t, X_t^1, \mu_t, \alpha_t^1)dt + \sigma W_t + \varsigma(t, X_t)dB_t$$

$$J^1 = \mathbb{E}\left[g(X_T^1, \mu_T) + \int_0^T f(t, X_t^1, \mu_t, \alpha_t^1)dt\right]$$

◦ Solve the matching problem $\mu_t = \mathcal{L}(X_t | B)$

Strong vs. weak equilibra

- Strong sense
  - Probability space is given
  - Canonical space: $\underbrace{\mathcal{C}([0,T],\mathbb{R})}_{\text{for } B} \times \underbrace{\mathcal{C}([0,T],\mathbb{R})}_{\text{for } W}$
  - $(\mu_t)_{0\le t\le T}$ is prog. meas. w.r.t. $\sigma(B)$ (function of the 1st coordinate)
- Weak sense: probability space is not given
  - $\exists$ 2 filtered probability spaces $(\Omega^i, \mathcal{F}^i, \mathbb{P}^i)$, $i = 1, 2$
  - $(B_t, \mu_t)_{0\le t\le T}$ is carried on Ω^1, $(W_t)_{0\le t\le T}$ on Ω^2
  - $\mu_t = \mathcal{L}(X_t|\mathcal{F}_t^1)$
- Yamada-Watanabe: strong ! + weak $\exists \Rightarrow$ strong $\exists$

Strong stochastic maximum principle

- Freeze $(\mu_t)_{0\le t\le T}$ as a $\sigma(B)$ prog. meas. process
  - Hamiltonian $H(t,x,y,z,\mu,\alpha)$

$$= b(t,x,y,\mu,\alpha)y + \varsigma(t,x)z + f(t,x,\mu,\alpha)$$

  - $\hat{\alpha}(t,x,y,\mu) = \text{argmin}_\alpha H(t,x,y,z,\mu,\alpha)$
  - Adjoint equations:

$$\begin{aligned}
dX_t &= b\big(t,X_t,\mu_t,\hat{\alpha}(t,X_t,Y_t,\mu_t)\big)dt + \sigma dW_t + \varsigma(t,X_t)dB_t \\
dY_t &= -\partial_x H\big(t,X_t,Y_t,Z_t,\mu_t,\hat{\alpha}(t,X_t,Y_t,\mu_t)\big)dt \\
&\quad + Z_t dB_t + \zeta_t dW_t \\
Y_T &= \partial_x g(X_T,\mu_T)
\end{aligned}$$

- Solve eq. with the constraint $\mu_t = \mathcal{L}(X_t|B)$: MKV FBSDE
  - H and g convex w.r.t. $(x,\alpha) \Rightarrow X$ equilibrium

Weak stochastic maximum principle

- Probability space $(\Omega^1 \times \Omega^2, \mathcal{F}^1 \otimes \mathcal{F}^2, \mathbb{P}^1 \otimes \mathbb{P}^2)$,
  - $1 \rightsquigarrow B$, $2 \rightsquigarrow W$
- Freeze $(\mu_t)_{0 \le t \le T}$ as an $\mathcal{F}^1$ prog. meas. process
  - Galtchouk-Kunita-Watanabe $\rightsquigarrow$ adjoint equations:

$$
\begin{aligned}
dX_t &= b\big(t, X_t, \mu_t, \hat{\alpha}(t, X_t, Y_t, \mu_t)\big)dt + \sigma dW_t + \varsigma(t, X_t)dB_t \\
dY_t &= -\partial_x H\big(t, X_t, Y_t, Z_t, \mu_t, \hat{\alpha}(t, X_t, Y_t, \mu_t)\big)dt \\
&\qquad\qquad + Z_t dB_t + \zeta_t dW_t + dN_t \\
Y_T &= \partial_x g(X_T, \mu_T)
\end{aligned}
$$

  - $\langle N, B \rangle = 0, \langle N, W \rangle = 0$
- Solve eq. with the constraint $\mu_t = \mathcal{L}(X_t | \mathcal{F}^1)$

Dynamics of X

- Decoupling random field $u : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$
  - $(u(t, \cdot))_{0 \le t \le T}$ is $\mathcal{F}^1$ prog. meas.
  - Representation formula $Y_t = u(t, X_t)$
- Dynamics of X at equilibrium

$$dX_t = \underbrace{b\big(t, X_t, \mu_t, \hat{\alpha}(t, X_t, u(t, X_t), \mu_t)\big)}_{\hat{b}(t, X_t)} dt + \sigma dW_t + \varsigma(t, X_t) dB_t$$

  - Convex Hamiltonian $\Rightarrow$ u Lipschitz in x
- Conditional path of X given $\mathcal{F}^1$ $\rightsquigarrow$ Freeze B
  - If $\varsigma(t, x) = \varsigma \Rightarrow \mathcal{L}(X_t | \mathcal{F}^1)_{|B = \beta} = \mathcal{L}(X_t^{\beta})$

$$X_t^{\beta} = x_0 + \int_0^t \hat{b}(s, X_s^{\beta}) ds + \sigma W_t + \varsigma \beta_t$$

A bit of rough paths

- When ς depends on (t,x) $\rightsquigarrow$

$$X_t^{\beta} = x_0 + \int_0^t \hat{b}(s, X_s^{\beta})\,ds + \sigma W_t + \int_0^t \varsigma(s, X_s^{\beta})\,d\beta_s$$

  - Notion of solution? $\rightsquigarrow$ iterated integrals $\int_0^{\cdot} \beta_s d\beta_s \ldots$

- When β piecewise affine $\rightsquigarrow$ well-defined!
  - β_N = affine interpolation of B with N nodes
  - ς smooth $\Rightarrow \exists$ universal subset $\subset \Omega$ s.t. X^{β_N} converges
  - $\lim_{N\to\infty} X^{\beta_N}$ = pathwise notion of solution
  - $\mathcal{L}(X_t|\mathcal{F}^1)_{|B=\beta} = \mathcal{L}(X_t^{\beta})$

$$X_t = x_0 + \int_0^t \hat{b}(s, X_s^{\beta})\,ds + \sigma W_t + \int_0^t \varsigma(t, X_t)\circ dB_t$$

PDE point of view: stochastic HJB

- No common noise $\Rightarrow$ MFG = HJB–Kolmogorov eq.
- Common noise $\Rightarrow$ Value function = random field

$$U(t,x) = \inf_{\alpha, X_t = x} \mathbb{E}\Big[g(X_T, \mu_T) + \int_t^T L(X_s, \mu_s, \alpha_s) ds \Big| \mathcal{F}_t^1 \Big]$$

- U adapted $\Rightarrow$ Backward Stochastic HJB

$$\begin{aligned} & d_t U(t,x) \\ & \quad + \Big(\underbrace{\mathcal{L}U(t,x)}_{\text{generator}} + \underbrace{\inf_\alpha \big[b(x, \mu_t, \alpha) \partial_x U(t,x) + L(x, \mu_t, \alpha) \big]}_{\text{standard Hamiltonian in HJB}} \\ & \quad\quad + \underbrace{\sigma(x) \partial_x V(t,x)}_{\text{Ito Wentzell cross term}} \Big) dt - \underbrace{V(t,x) dB_t + dN_t}_{\text{backward term}} = 0 \end{aligned}$$

- Maximum principle $\rightsquigarrow$ $\mathrm{Y}_t = \partial_x U(t, X_t)$ i.e. $u = \partial_x U$

Stochastic Kolmogorov

- Dynamics of the conditional law of X given B
- Replace $(B_t)_{0 \le t \le T}$ by a piecewise affine curve $(\beta_t)_{0 \le t \le T}$
  - Kolmogorov equation

$$
\begin{aligned}
d_t \mu_t = & -\mathrm{div}\Big(b(x, \mu_t, \hat{\alpha}(x, \mu_t, u(t,x)))\Big) dt \\
& + \frac{\sigma^2}{2} \partial^2_{xx} \mu_t dt - \mathrm{div}\Big(\mu_t \sigma(t,x)\Big) \dot{\beta}_t dt
\end{aligned}
$$

- Use $\beta = \beta^N =$ affine interpolation of B with N nodes

$$
\begin{aligned}
d_t \mu_t = & -\mathrm{div}\Big(b(x, \mu_t, \hat{\alpha}(x, \mu_t, u(t,x)))\Big) dt \\
& + \frac{\sigma^2}{2} \partial^2_{xx} \mu_t dt - \mathrm{div}\Big(\mu_t \sigma(t,x)\Big) \circ dB_t
\end{aligned}
$$

Lifted value function

- Representation of the value random function

$$U(t,x,\omega) = \mathcal{U}\big(t,x,\mu_t(\omega)\big),$$

  - $\mathcal{U} : [0,T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$
  - $\exists$ if strong uniqueness

- Write second-order PDE in infinite dimension
  - Derivatives on $\mathcal{P}_2(\mathbb{R}) \rightsquigarrow$ r.v. $\partial_\mu \mathcal{U}(t,x,\mu_t)(X_t)$
  - Connection with V in stochastic HJB equation

$$V(t,x,\omega) = \int \partial_\mu \mathcal{U}(t,x,\mu_t(\omega))(y)\varsigma(t,y)\mu_t(\omega,dy)$$

- Used in parametric models $\mu_t = \mu(q_t)$

$$\mathcal{U}(t,x,\mu_t) \rightsquigarrow \mathcal{U}(t,x,q_t)$$

Solvability conditions

- Convexity of the Hamiltonian
  - $b(t,x,\mu,\alpha) = b_0(t,\mu) + b_1(t)x + b_2(t)\alpha$
  - $\varsigma(t,x) = \varsigma_0(t) + \varsigma_1(t)x$
  - f convex in (x,α) (and strictly convex in α)
- Local Lipschitz bound

$$
\begin{aligned}
&\big|f(t,x',\mu',\alpha') - f(t,x,\mu,\alpha)\big| + \big|g(x',\mu') - g(x,\mu)\big| \\
&\le L\left[1 + |x'| + |x| + |\alpha'| + |\alpha| + \left(\int_{\mathbb{R}^d} |y|^2 d(\mu+\mu')(y)\right)^{1/2}\right] \\
&\quad \times \big[|(x',\alpha') - (x,\alpha)| + W_2(\mu',\mu)\big]
\end{aligned}
$$

- Mean-reverting
  - $\langle x, \partial_x f(t,0,\delta_x,0)\rangle,\ \langle x, \partial_x g(0,\delta_x)\rangle \ge -c(1+|x|)$
- Smoothness ($\mathcal{C}^1$ in (x,α) with Lip derivatives)...

Uniqueness

- Specific structure of b
  - $b_0(t,\mu) = b_0(t)$ (b doesn't depend on μ)
- Specific structure of f
  - $f(t,x,\mu,\alpha) = f_0(t,x,\mu) + f_1(t,x,\alpha)$ (μ and α are separated)
- Monotonicity property:

$$\int_{\mathbb{R}} \big(f_0(t,x,\mu) - f_0(t,x,\mu')\big)\, d\big(\mu - \mu'\big)(x) \geq 0,$$
$$\int_{\mathbb{R}} \big(g(x,\mu) - g(x,\mu')\big)\, d\big(\mu - \mu'\big)(x) \geq 0,$$

- Application: weak $\rightsquigarrow$ strong

Strategy of proof for solvability

- Forget strong vs. weak! Freeze the conditional measure

$$
\begin{aligned}
dX_t &= b\big(t, X_t, \mu_t, \hat{\alpha}(t, X_t, Y_t, \mu_t)\big)dt + \sigma W_t + \varsigma(t, X_t) \circ dB_t \\
dY_t &= -\partial_x H\big(t, X_t, Y_t, Z_t, \mu_t, \hat{\alpha}(t, X_t, Y_t, \mu_t)\big)dt + Z_t dB_t + Z_t' dW_t \\
Y_T &= \partial_x g(X_T, \mu_T)
\end{aligned}
$$

- Find a fixed point $\Phi : (\mu_t)_{0 \le t \le T} \mapsto (\mathcal{L}(X_t^\mu | B))_{0 \le t \le T}$
- If no common noise
  - Fixed point in $\mathcal{C}\big([0, T], \underbrace{\mathcal{P}(\mathbb{R})}_{\text{set of prob. meas.}}\big)$
  - Use Schauder's th: Φ continuous and range of Φ compact
- If common noise
  - Fixed point in subset of $\big(\mathcal{C}([0, T], \mathcal{P}(\mathbb{R}))\big)^{\Omega}$
  - Compactness?

Discretization of the conditioning

- Discretization: $\mathcal{L}(X_t|B) \rightsquigarrow \mathcal{L}(X_t|\text{finitely supported process})$
  - Π projection mapping onto space grid $\{x_1, \ldots, x_M\} \subset \mathbb{R}$
  - $t_1, \ldots, t_N$ a finite time grid $\subset [0, T]$
  - $\hat{B}_{t_i} = \Pi(B_{t_i})$
- Forward-backward system with

$$\mathcal{L}(X_t|\hat{B}_{t_1}, \ldots, \hat{B}_{t_i}), \quad t_i \leq t < t_{i+1}$$

  - $(\hat{B}_{t_1}, \ldots, \hat{B}_{t_N})$ has finite support of size MN
  - Fixed point in $\big(\mathcal{C}([0, T], \mathcal{P}(\mathbb{R}))\big)^{MN}$
  - $\exists \underbrace{\hat{X}^{M,N}}_{\text{optimum}}, \underbrace{\hat{\mu}^{M,N}}_{\text{equilibrium}}, \underbrace{\hat{u}^{M,N}}_{\text{decoupling field}}$ s.t.

$$\hat{\mu}_t^{M,N} = \mathcal{L}(\hat{X}_t^{M,N}|\hat{B}_{t_1}, \ldots, \hat{B}_{t_i}), \quad \hat{Y}_t^{M,N} = \hat{u}^{M,N}(t, \hat{X}_t^{M,N})$$

Extraction of converging subsequence

- Conditional measure $\hat{\mu}^{M,N} \sim_{M,N\uparrow\infty} \mathcal{L}(\hat{X}_t^{M,N}|B)$
- Tightness $\hat{X}^{M,N}$ in $\mathcal{C}([0,T]\times\mathbb{R},\mathbb{R})$
  - Standard Kolmogorov criterion
- Tightness $\hat{u}^{M,N}$ in $\mathcal{C}([0,T]\times\mathbb{R},\mathbb{R})$
  - Convexity of Hamiltonian $\Rightarrow$ regularity of $\hat{u}^{M,N}$
- Tightness $\mathcal{L}(\hat{X}_t^{M,N}|B)$ in $\mathcal{C}([0,T],\mathcal{P}(\mathbb{R}))$
  - Given $B=\beta$, $\mathcal{L}(\hat{X}_t^{M,N}|B)$ is the law of

$$d\hat{X}_t^{\beta,M,N} = \hat{b}^{M,N}(t,\hat{X}_t^{\beta,M,N})dt + \sigma dW_t + \varsigma(t,\hat{X}_t^{\beta,M,N})d\beta_t$$

  - In rough paths sense (choose ς constant to simplify)
  - Law of $\hat{X}^{\beta,M,N}$ is explicitly controlled by β

Passage to the limit

- Limit is some 5-tuple (X, μ, u, W, B)

$$dX_t = b(t, X_t, \mu_t, \hat{\alpha}(t, X_t, u(t, X_t), \mu_t))dt + \sigma dW_t + \varsigma(t, X_t) \circ dB_t$$

  - (μ, u, W) independent of B and $\mu_t = \mathcal{L}(X_t | \mu, u, B)$

- Backward SDE $\rightsquigarrow$ $Y_t = u(t, X_t)$

$$Y_t = \partial_x g(X_T, \mu_T) + \int_t^T \underbrace{\partial_x H^0\big(s, X_s, Y_s, \mu_s, \hat{\alpha}(s, X_s, Y_s, \mu_s)\big)}_{\text{Hamiltonian without } z} ds$$

$$+ \underbrace{\Big\langle M, \int_0 \varsigma(s, X_s) dB_s \Big\rangle_T - \Big\langle M, \int_0 \varsigma(s, X_s) dB_s \Big\rangle_t}_{\text{remainder in the Hamiltonian}} + M_T - M_t$$

  - represent M by Galtchouk-Kunita-Watanabe
  - uniqueness to FBSDE $\Rightarrow$ u is $\sigma(\mu, B)$-prog. meas.