

Unwrapping \varprojlim :

If $\varprojlim F$ exists (i.e. if $\varprojlim F$ is representable), then $\exists z \in C$ such that $\alpha: \text{Hom}_C(-, z) \cong \varprojlim F$ as presheaves

By Yoneda's lemma, $\{\text{Morphisms of functors } \text{Hom}_C(-, z) \rightarrow G\} \xleftrightarrow{1:1} \text{elts in } G(z)$
 i.e. α is uniquely determined by an elt

$$\tilde{\alpha} \in \varprojlim F(z) = \text{Hom}_{\text{Hom}(\mathcal{I}, C)}(c_z, F)$$

$$\tilde{\alpha}(a): c_z(a) \rightarrow F(a) \quad f: a \rightarrow b \quad \begin{array}{ccc} z = c_z(a) & \xrightarrow{\tilde{\alpha}(a)} & F(a) \\ \parallel & \downarrow G(f) & \downarrow F(f) \\ z = c_z(b) & \xrightarrow{\tilde{\alpha}(b)} & F(b) \end{array}$$

$\tilde{\alpha}$ corresponds to a family of morphisms in C (one for each morphism in \mathcal{I}):

$$\begin{array}{ccc} \tilde{\alpha}(a) & \rightarrow & F(a) \\ \downarrow & & \downarrow F(f) \\ \tilde{\alpha}(b) & \rightarrow & F(b) \end{array} \quad \text{Namely } (\tilde{\alpha}(a))_a \text{ such that for each } f \in \text{Mor}(\mathcal{I}), (*) \text{ commutes}$$

What is the distinguishing feature of $(z, \tilde{\alpha}(a)_{a \in \text{ob}(\mathcal{I})})$?

Assume given $(x, \beta(a)_{a \in \text{ob}(\mathcal{I})})$ such that each $(**)$ commutes

$$(**) \quad \begin{array}{ccc} & & F(a) \\ & \nearrow \beta(a) & \uparrow \tilde{\alpha}(a) \\ x & \xrightarrow{\exists! g} & z \\ & \searrow \beta(b) & \downarrow F(f) \\ & & F(b) \end{array}$$

then $\text{Hom}_C(x, z) \cong \varprojlim F(x)$

$\exists!$ morphism $g: x \rightarrow z$ in C such that $\beta(a) = \tilde{\alpha}(a)g$ for each $a \in \mathcal{I}$

Examples: (1) \mathcal{I} is discrete, i.e. \mathcal{I} is a set with only identities as morphisms.

A functor $F: \mathcal{I} \rightarrow C$ is just a family of objects indexed by \mathcal{I}

Given $Y \in \text{ob} C$, $\varprojlim_{\mathcal{I}} F(\mathcal{I}) = \{ (Y \xrightarrow{f_a} X_a)_a \mid f_a: Y \rightarrow X_a \in \text{Mor}(C) \}$

$$= \prod_{a \in \text{ob}(\mathcal{I})} \text{Hom}_C(Y, X_a)$$

If $\varprojlim F$ exists, one writes representing object as $\prod_{a \in \mathcal{I}} X_a$.

(cont.) NB. The morphisms $\tilde{\alpha}(a): \prod_{a \in I} X_a \rightarrow X_a$ are called the projections onto the factors X_a ; $p_a := \tilde{\alpha}(a)$, and these morphisms in \mathcal{C} satisfy

$$\begin{array}{ccc}
 & & X_a \\
 & \nearrow p_a & \\
 * & Y & \\
 & \searrow \exists! q & \\
 & \prod_{a \in I} X_a & \\
 & \nearrow p_a & \\
 & & X_a
 \end{array}$$

For any family $q_a: Y \rightarrow X_a$ of morphisms,
 $\exists! q: Y \rightarrow \prod_{a \in I} X_a$ such that (*) commutes for each a .

- Examples:
- Sets: all products exist (set-indexed sets)
 - Groups: products are direct products
 - Abelian groups: ditto (the list goes on)
 - Finite groups: finite products exist
 - (M, \cdot) an associative monoid, $\mathcal{B}M$ its classifying category. Given \mathcal{I} a discrete category, $\{F: \mathcal{I} \rightarrow \mathcal{B}M\} \xrightarrow{\cong} \{ \cdot \}_{a \in \mathcal{I}}$
- $$\lim_{\mathcal{I}} F(\cdot) = \prod_{a \in \mathcal{I}} \text{Hom}_{\mathcal{B}M}(\cdot, \cdot) = \prod_{a \in \mathcal{I}} M$$

If $\lim_{\mathcal{I}} F(\cdot)$ exists, then $M = \text{Hom}_{\mathcal{B}M}(\cdot, \cdot) \cong \prod_{a \in \mathcal{I}} M$

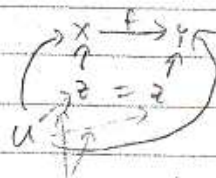
Thus $\lim_{\mathcal{I}} F(\cdot)$ exists iff \mathcal{I} has one object or M has one element.

② $\mathcal{I} = \{ \cdot \rightarrow \cdot \}$, "one morphism category"
 \mathcal{C} arbitrary category, $\{F: \mathcal{I} \rightarrow \mathcal{C}\} = \text{Morphisms in } \mathcal{C}$
 (More precisely, Def. $\text{Mor } \mathcal{C} := \text{Mor}(\mathcal{I}, \mathcal{C})$ as a category. Morphisms are commutative squares)

$\lim_{\mathcal{I}} F = Z$ such that

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \alpha_x \uparrow & \circlearrowleft & \uparrow \alpha_y \\
 Z & = & Z
 \end{array}$$

and for any U ,



Taking $Z = X$, $\alpha_x = \text{id}_X$, $\alpha_y = f$ suffices

$\exists! g$ making diagram commute

(cont). Better example: $I = (\cdot \rightrightarrows *)$, $F: I \rightarrow C$ is $Y \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} Y$

Def. If $\lim_I F$ exists, it is called the equalizer of F and G .

$$\begin{array}{ccc} X & \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} & Y \\ \uparrow h & & \uparrow fh = gh \\ G: Z & \begin{matrix} \xrightarrow{=} \\ \xrightarrow{=} \end{matrix} & Z \end{array} \quad \lim_I F = \{h: Z \rightarrow X \mid fh = gh\}$$

Assume $C = \text{Sets}$, $X \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} Y$ Does equalizer exist?

Yes: $\lim_I (X \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} Y) = \{x \in X \mid f(x) = g(x)\}$

Exercise: check this

Assume C has a zero object e .

Then the equalizer of $(X \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} Y) = \text{Ker}(f)$

Sets* := Sets with basepoint

Then $\text{Ker}(f: (X, *) \rightarrow (Y, *)) = f^{-1}(*)$

③ Another way of writing equalizers:

$$I = (\cdot \rightrightarrows \cdot) \xrightarrow{F} (X \rightrightarrows Y)$$

Def. $\lim_I F$ is called the fibre product of f and g
 or the fibre product of X and Y over Z
 or the pullback of X and Y (or f and g) over Z .

$$\lim_I F = X \times_Y Y \text{ where } \begin{array}{ccc} U & \begin{matrix} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{matrix} & X \\ \downarrow h & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array} \text{ commutes for all } U$$

Diagramme called "Cartesian square" or "pullback diagramme"
 p_1 sometimes called the pullback of f along g , etc.

Eg. In Sets, $\lim_{\leftarrow} (Y \rightrightarrows X) = \{(x, y) \mid f(x) = g(y)\}$
 Same in groups, abelian groups, rings, finite groups

Eg. In Groups, what is equalizer $(G \rightrightarrows G')$, if it exists?
 $H = \{g \in G \mid \varphi(g) = \psi(g)\}$, which is a group

So the equalizer exists

If G, G' are abelian, then the equalizer of φ and ψ is $\ker(\varphi - \psi)$

Lemma. Let C be a category in which the product of any two objects exists. Then the association

$$X \begin{array}{c} \xrightarrow{f} \\ \rightrightarrows \\ \xrightarrow{g} \end{array} Y \iff \begin{array}{ccc} X \times X & \longrightarrow & X \\ \downarrow & & \downarrow (id_x, g) \\ X & \xrightarrow{(id_x, f)} & X \times Y \end{array}$$

gives a 1-1 correspondence between equalizers and fibred products.

Eg. in Sets, $X \times_{X \times Y} X = \{(x, x') \mid (x, f(x)) = (x', g(x'))\}$
 $= \{x \mid f(x) = g(x)\} = \text{equalizer of } f \text{ and } g$

Pf. $\text{Hom}_C(Z, X \times_{X \times Y} X) = \{(a: Z \rightarrow X, b: Z \rightarrow X) \mid (id_x, f)a = (id_x, g)b\}$
 $\iff (a, fa) = (b, gb)$

$$= \{c: Z \rightarrow X \mid fc = gc\}$$

$$= \text{Hom}_C(Z, \text{Equalizer}(X \begin{array}{c} \xrightarrow{f} \\ \rightrightarrows \\ \xrightarrow{g} \end{array} Y))$$

For the inverse,

$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_1} & Y \\ \pi_2 \downarrow & & \downarrow f \\ X & \xrightarrow{g} & Z \end{array} \quad \text{In sets, } X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$$

$$= \text{Eq}(X \times Y \begin{array}{c} \xrightarrow{f \circ \pi_1} \\ \rightrightarrows \\ \xrightarrow{g \circ \pi_2} \end{array} Z)$$

Ex. Generalize this, as above. \square

Thm-Def. A category C is closed under all (finite) limits if every (set-indexed) limit exists in C iff

(a) all (finite) products exist in C and (b) all pullbacks exist
 iff (a) as above and (b) all equalizers exist

MB. Closed under all limits is sometimes called complete

E.g. Sets, Groups, Abelian groups, Rings are complete
 Fields is not complete, because products do not always exist
 Entering to the category of semi-simple (or separable) algebras gives completeness

Pf. $b \Leftrightarrow b'$ given in lemma

Given $F: I \rightarrow C$, want to show $\varinjlim_I F$ exists

$$\text{Hom}_C(\mathbb{Z}, \varinjlim_I F) = \left\{ \begin{array}{c} \mathbb{Z} \xrightarrow{\alpha(a)} F(a) \\ \downarrow F(f) \\ \mathbb{Z} \xrightarrow{\alpha(b)} F(b) \end{array} \mid \begin{array}{l} a \in \text{Ob } I, \\ (a, b) \in I \\ \text{requirement: For each } f: a \rightarrow b \in I, \alpha \text{ commutes} \end{array} \right\}$$

$$\text{Claim: } \sim \cong \prod_{a \in I} \text{Hom}_C(\mathbb{Z}, F(a))$$

$$\sim = \text{Eq}_{\mathbb{Z}} \left(\prod_{a \in I} \text{Hom}_C(\mathbb{Z}, F(a)) \xrightarrow[\varphi_{(f: a \rightarrow b) \in I}]{\psi} \prod_{(f: a \rightarrow b) \in I} \text{Hom}_C(\mathbb{Z}, F(b)) \right)$$

$$(\alpha(a)_a) \mapsto \psi(\alpha(a)_a)_f := \alpha(b) = \alpha(\text{codomain of } f)$$

$$(\alpha(a)_a) \mapsto \varphi(\alpha(a)_a)_f := F(f) \circ \alpha(a)$$

$\alpha(a)_a$ makes the triangle for $f: a \rightarrow b$ commute

iff $\psi(\alpha)_f = \varphi(\alpha)_f$

$$\sim = \text{Eq}_{\mathbb{Z}} \left(\text{Hom}_C(\mathbb{Z}, \varinjlim_{I_{\text{disc}}} F) \xrightarrow[\varphi]{\psi} \text{Hom}_C(\mathbb{Z}, \prod_{f \in \text{Mor } I} F(\text{codomain } f)) \right)$$

By Yoneda Lemma,

$$\sim = \text{Eq}_{\mathbb{Z}} \left(\text{Hom}_C(\mathbb{Z}, \varinjlim_{I_{\text{disc}}} F) \xrightarrow[\text{Hom}_C(\mathbb{Z}, \varphi)]{\text{Hom}_C(\mathbb{Z}, \psi)} \text{Hom}_C(\mathbb{Z}, \prod_{f \in \text{Mor } I} F(\text{codomain } f)) \right)$$

for some unique morphisms $\hat{\varphi}, \hat{\psi}: \prod_{a \in I} F(a) \rightarrow \prod_{f \in \text{Mor } I} F(\text{codomain } f)$

$$\sim = \text{Hom}(\mathbb{Z}, \text{Eq}_{\mathbb{Z}}(\hat{\varphi}, \hat{\psi})), \text{ by definition of the equalizer. } \square$$

Consider finite groups $\xrightarrow{\text{forget}}$ sets

We saw that it is not representable, but

$$F_n = \text{Hom}(\mathbb{Z}/n, ?): \text{finite groups} \rightarrow \text{sets}$$
$$G \mapsto \{g \in G \mid \text{order } g \mid n\}$$

$$\text{Then } G \mapsto \bigcup_n F_n(G) = G$$

$$\bigcup_n F_n(G) = \varinjlim_{n \in \mathbb{N}(\cdot)} F_n(G) = \varinjlim_{n \in \mathbb{N}(\cdot)} \text{Hom}_{\text{finite groups}}(\mathbb{Z}/n, G)$$

$$(\text{This is special, normally false}) = \text{Hom}(\varprojlim_{n \in \mathbb{N}(\cdot)} \mathbb{Z}/n, G) = G$$

$\varprojlim_{n \in \mathbb{N}(\cdot)} \mathbb{Z}/n$ is not a finite group, but a profinite group

Def. If C is a category, then the pro-objects over C are objects of the full subcategory of all presheaves

$\hat{C} = \text{Hom}(C^{\text{op}}, \text{Sets})$ of the form $\varprojlim_I F$ for $F: I \rightarrow C$ a functor on a category I .

E.g. pro-objects over finite groups are called profinite groups.

Because groups is complete, a profinite group can be realized as fibre products of products of finite groups

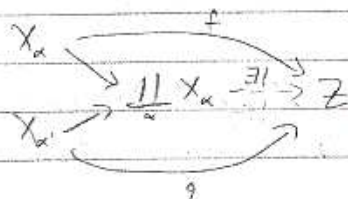
$\Leftrightarrow G$ is profinite iff it is the inverse limit over all homomorphisms $G \twoheadrightarrow H$ to finite groups H .

$$\text{Thm. } \varprojlim_{n \in \mathbb{N}(\cdot)} \mathbb{Z}/n =: \hat{\mathbb{Z}} = \prod_{p \text{ prime}} \hat{\mathbb{Z}}_p (= \text{the } p\text{-adic integers})$$

Remk. If $\text{finite groups}_p \subseteq \text{finite groups}$ denotes the full subcategory of all finite p -groups, then $\varprojlim \mathbb{Z}/p^n \cong \hat{\mathbb{Z}}_p$ and pro-represents the forgetful functor

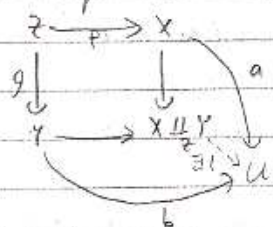
Dual statements: Formulate everything for colimits!

E.g. coproduct in Sets: $(X_\alpha)_\alpha$



Coproduct is the disjoint union

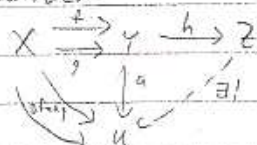
cofibre product



$$X \amalg_Z Y = X \dot{\cup} Y / (x \sim g(z))$$

disjoint union

coequalizer



$$Z = Y / (f(x) \sim g(x))$$

In groups, the coproduct of G and H is the free product $G * H$
 (i.e. all words of the form $g_1 h_1 g_2 h_2 \dots g_n h_n$, $g_i \in G$, $h_i \in H$)

Thm. $\mathbb{Z}/2 * \mathbb{Z}/2 \cong \text{PSL}(2, \mathbb{Z})$

In abelian groups, coproduct is direct sum (same as product!)

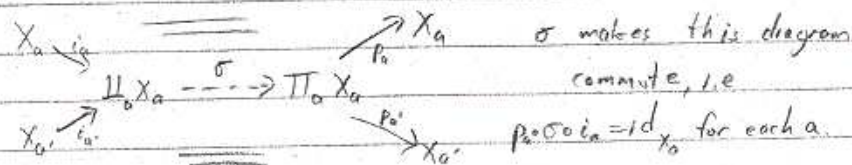
R, S rings: Does $R \amalg S$ exist?

Yes: $R \amalg S =$ free product of R, S

$$= \left\{ \sum_{i=1}^n r_i s_i \dots r_n s_n \right\}$$

If R, S are commutative, then $R \amalg S = R \otimes S$

Lemma. Assume both coproduct and product of a family $(X_a)_{a \in I}$ exist in \mathcal{C} and \mathcal{C} has a zero object. Then there exists a natural comparison morphism $\sigma: \coprod_a X_a \rightarrow \prod_a X_a$.



$$\text{Pf. } \text{Hom}_{\mathcal{C}}\left(\coprod_{b \in J} Y_b, \prod_{a \in I} X_a\right) \stackrel{\text{claim}}{=} \prod_{b \in J} \prod_{a \in I} \text{Hom}_{\mathcal{C}}(Y_b, X_a)$$

$$\text{For } I=J: X_a = Y_a, \text{ so } \leftarrow = \prod_{(a,b) \in I \times I} \text{Hom}_{\mathcal{C}}(X_b, X_a)$$

$$\sigma \in \prod_{(a,b) \in I \times I} \text{Hom}_{\mathcal{C}}(X_b, X_a) \text{ is such that } \sigma(a,a) = \text{id}_{X_a} \text{ and } \sigma(a,b) = 0. \square$$

Eg. • In sets, $(X, *)$, (Y, \cdot)

$$(X, *) \coprod (Y, \cdot) = X \cup Y / *, = X \cup Y$$

$$(X, *) \prod (Y, \cdot) = (X \times Y, (*, \cdot))$$

$$\sigma: x \mapsto (x, *)$$

$$y \mapsto (*, y)$$

$$\bullet \mathbb{Z}_2 * \mathbb{Z}_2 \xrightarrow{\sigma} \mathbb{Z}_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$$

Exercise: Describe σ in terms of matrices

Def. A category \mathcal{A} is additive if (a) it is preadditive, (b) there is a zero object, (c) product and coproduct of any two objects exist, (d) σ is an isomorphism for any $X, Y \in \mathcal{A}$.

Def. A category \mathcal{A} is abelian if it is additive and any morphism has an analysis.