Unwrapping $\varprojlim$:

If $\varprojlim F$ exists (i.e. if $\mathrm{Lim}\, F$ is representable), then $\exists\, Z \in C$ such that $\alpha: \mathrm{Hom}_C(-, Z) \cong \varprojlim_I F$ as presheaves

By Yoneda's lemma, $\{\text{Morphisms of functors } \mathrm{Hom}_C(-,Z) \to G\} \xleftrightarrow{1:1}$ elts in $G(Z)$

i.e. α is uniquely determined by an elt

$$\tilde{\alpha} \in \varprojlim_I F(Z) = \mathrm{Hom}_{\mathrm{Hom}(I,C)}(c_Z, F)$$

$$\tilde{\alpha}(a): c_Z(a) \to F(a) \qquad f: a \to b$$

$a \in I$, $c_Z(a) = Z$

$$\begin{tikzcd} Z = c_Z(a) \arrow[r, "\tilde{\alpha}(a)"] \arrow[d, "c_Z(f)"] & F(a) \arrow[d, "F(f)"] \\ Z = c_Z(b) \arrow[r, "\tilde{\alpha}(b)"] & F(b) \end{tikzcd}$$

$\tilde{\alpha}$ corresponds to a family of morphisms in C (one for each morphism in I):

$$(*) \quad \begin{tikzcd} & F(a) \arrow[d, "F(f)"] \\ Z \arrow[ur, "\tilde{\alpha}(a)"] \arrow[r, "\tilde{\alpha}(b)"'] & F(b) \end{tikzcd}$$

Namely $(\tilde{\alpha}(a))_a$ such that for each $f \in \mathrm{Mor}(I)$, $(*)$ commutes

What is the distinguishing feature of $(Z, \tilde{\alpha}(a)_{a \in \mathrm{Ob}\, I})$?

Assume given $(X, \beta(a)_{a \in \mathrm{Ob}\, I})$ such that each $(**)$ commutes

$$(**) \quad \begin{tikzcd} & & F(a) \arrow[dd, bend left, "F(f)"] \\ X \arrow[urr, "\beta(a)"] \arrow[r, "\exists! g"] \arrow[drr, "\beta(b)"'] & Z \arrow[ur, "\tilde{\alpha}(a)"] \arrow[dr, "\tilde{\alpha}(b)"'] & \\ & & F(b) \end{tikzcd}$$

then $\mathrm{Hom}_C(X, Z) \cong \varprojlim_I F(X)$

$\exists!$ morphism $g: X \to Z$ in C such that $\beta(a) = \tilde{\alpha}(a) g$ for each $a \in I$

Examples: ① I is discrete, i.e. I is a set with only identities as morphisms

A functor $F: I \to C$ is just a family of objects indexed by I

Given $Y \in \mathrm{Ob}\, C$, $\varprojlim_I F(Y) = \{(Y \xrightarrow{f_a} X_a)_a \mid f_a: Y \to X_a \in \mathrm{Mor}\, C\}$

$$= \prod_{a \in \mathrm{ob}\, I} \mathrm{Hom}_C(Y, X_a)$$

If $\varprojlim_I F$ exists, one writes representing object as $\prod_{a \in I} X_a$.

(cont.) NB The morphisms $\tilde{\alpha}(a) : \prod_{a \in I} X_a \to X_a$ are called the projections onto the factors X_a; $p_a := \tilde{\alpha}(a)$, and these morphisms in C satisfy

$$(*) \quad \begin{tikzcd} Y \arrow[rr, bend left, "q_a"] \arrow[r, dashed, "\exists ! q"'] & \prod_{a \in I} X_a \arrow[r, "p_a"'] & X_a \end{tikzcd}$$

For any family $q_a : Y \to X_a$ of morphisms, $\exists ! q : Y \to \prod_{a \in I} X_a$ such that $(*)$ commutes for each a.

Examples:
- Sets: all products exist (set-indexed sets)
- Groups: products are direct products
- Abelian groups: ditto
  (the list goes on)
- Finite groups: finite products exist
- $(M, \cdot)$ an associative monoid, $\mathbb{B}M$ its classifying category. Given I a discrete category,

  $\{F : I \to \mathbb{B}M\} \overset{1:1}{\longleftrightarrow} \{\cdot_a\}_{a \in I}$

  $\varprojlim_I F(\cdot) = \prod_{a \in I} \mathrm{Hom}_{\mathbb{B}M}(\cdot, \cdot) = \prod_{a \in I} M$

  If $\varprojlim_I F(\cdot)$ exists, then $M = \mathrm{Hom}_{\mathbb{B}M}(\cdot, \cdot) \cong \prod_{a \in I} M$

  Thus $\varprojlim_I F(\cdot)$ exists iff I has one object or M has one element

② $I = \{\cdot \to *\}$, "one morphism category"

C arbitrary category. $\{F : I \to C\}$ = Morphisms in C

(More precisely, Def. $\mathrm{Mor}\, C := \mathrm{Mor}(I, C)$ as a category. Morphisms are commutative squares)

$\varprojlim_I F = Z$ such that

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ Z \arrow[u, "\alpha_X"] \arrow[r, equal] \arrow[ur, phantom, "\circlearrowleft"] & Z \arrow[u, "\alpha_Y"'] \end{tikzcd}$$

and for any U,

$$\begin{tikzcd} & X \arrow[r, "f"] & Y \\ & Z \arrow[u] \arrow[r, equal] & Z \arrow[u] \\ U \arrow[uur, bend left] \arrow[uurr, bend right] \arrow[ur, dashed] \arrow[urr, dashed] & & \end{tikzcd}$$

$\exists ! p$ making diagram commute

Taking $Z = X$, $\alpha_X = id_X$, $\alpha_Y = f$ suffices (cont). Better example: $I = (\cdot \rightrightarrows *)$, $F: I \to C$ is $\underset{F(\cdot)}{X} \overset{f}{\underset{g}{\rightrightarrows}} \underset{F(*)}{Y}$

Def. If $\varprojlim_I F$ exists, it is called the equalizer of F and G.

$$\varprojlim_I F(z) = \{h: z \to X \mid fh = gh\}$$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ Z \arrow[u, "h"] \arrow[r, shift left, "="] \arrow[r, shift right, "="'] & Z \arrow[u, "fh=gh"'] \end{tikzcd}$$

$G_z: Z \rightrightarrows Z$

Assume $C = Sets$. $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ Does equalizer exist?
Yes: $\varprojlim (X \overset{f}{\underset{g}{\rightrightarrows}} Y) \equiv \{x \in X \mid f(x) = g(x)\}$
Exercise: check this

Assume C has a zero object ε.
Then the equalizer of $(X \overset{f}{\underset{\varepsilon}{\rightrightarrows}} Y) = Ker(f)$

$Sets^*$:= Sets with basepoint
Then $Ker(f: (X, *) \to (Y, *)) = f^{-1}(*)$

③ Another way of writing equalizers:
$$I = \left(\begin{smallmatrix}\cdot \\ \cdot\end{smallmatrix} \rightrightarrows \cdot\right) \xrightarrow{F} \left(\begin{smallmatrix}X \\ Y\end{smallmatrix} \rightrightarrows Z\right)$$

Def. $\varprojlim_I F$ is called the fibre product of f and g
or the fibre product of X and Y over Z
or the pullback of X and Y (or f and g) over Z.

$\varprojlim_I F = X \times_Z Y$ where
$$\begin{tikzcd} U \arrow[dr, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_Z Y \arrow[r, "p_X"] \arrow[d, "p_Y"'] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"'] & Z \end{tikzcd}$$
commutes for all U

Diagramme called "Cartesian square" or "pullback diagramme"
p_X sometimes called the pullback of f along g, etc.

Eg. In Sets, $\varprojlim \left(Y \xrightarrow{g} Z \xleftarrow{f} X \right) \cong \{(x,y) \mid f(x)=g(y)\}$

Same in groups, abelian groups, rings, finite groups

Eg. In Groups, what is equalizer $(G \underset{\psi}{\overset{\varphi}{\rightrightarrows}} G')$, if it exists?

$H = \{g \in G \mid \varphi(g) = \psi(g)\}$, which is a group

So the equalizer exists

If G, G' are abelian then the equalizer of φ and ψ is $\ker(\varphi - \psi)$

Lemma. Let C be a category in which the product of any two objects exists. Then the association

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \longleftrightarrow \begin{array}{ccc} X \times_{X \times Y} X & \longrightarrow & X \\ \downarrow & & \downarrow (id_X, g) \\ X & \xrightarrow{(id_X, f)} & X \times Y \end{array}$$

gives a 1-1 correspondence between equalizers and fibred products.

Eg. in Sets, $X \times_{X \times Y} X = \{(x,x') : (x, f(x)) = (x', g(x'))\}$

$= \{x : f(x) = g(x)\}$ = equalizer of f and g

Pf. $\mathrm{Hom}_C(Z, X \times_{X \times Y} X) = \{(a : Z \to X;\ b : Z \to X) \mid (id_X, f)a = (id_X, g)b\}$

$\Leftrightarrow (a, fa) = (b, gb)$

$= \{c : Z \to X \mid fc = gc\}$

$= \mathrm{Hom}_C(Z, \text{Equalizer}(X \underset{g}{\overset{f}{\rightrightarrows}} Y))$

For the inverse,

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\pi_1} & Y \\ \pi_2 \downarrow & & \downarrow f \\ X & \xrightarrow{g} & Z \end{array}$$

In sets, $X \times_Z Y = \{(x,y) \in X \times Y \mid f(x) = g(y)\}$

$= Eq(X \times Y \underset{g \circ \pi_2}{\overset{f \circ \pi_1}{\rightrightarrows}} Z)$

Ex. Generalize this, as above. ☺

Thm-Def. A category C is closed under all (finite) limits if every (set-indexed) limit exists in C iff

(a) all (finite) products exist in C and (b) all pullbacks exist

iff (a) as above and (b') all equalizers exist

NB. Closed under all limits is sometimes called <u>complete</u>

E.g. Sets, Groups, Abelian groups, Rings are complete
Fields is not complete, because products do not always exist
Enlarging to the category of semi-simple (or separable) algebras gives completeness

Pf. b $\Leftarrow$ b' given in lemma
Given $F: I \to C$, want to show $\varprojlim_I F$ exists

$$\mathrm{Hom}_C\left(Z, \varprojlim_I F\right) = \left\{ \begin{tikzcd} & F(a) \arrow[d, "F(f)"] \\ Z \arrow[ur, "\alpha(a)"] \arrow[r, "\alpha(b)"'] & F(b) \end{tikzcd} \;\middle|\; \begin{array}{l} a \in \mathrm{Ob}\, I, \\ (\alpha(a): Z \to F(a))_{a \in \mathrm{Ob}\, I} \\ \text{requirement: For each} \\ f: a \to b \in I, \ * \text{ commutes} \end{array} \right\}$$

$*$

$$\text{Claim: } \sim \doteq \prod_{a \in I} \mathrm{Hom}_C(Z, F(a))$$

$$\sim = \mathrm{Eq}\left(\prod_{a \in I} \mathrm{Hom}_C(Z, F(a)) \overset{\varphi}{\underset{\psi}{\rightrightarrows}} \prod_{(f: a \to b) \in I} \mathrm{Hom}_C(Z, F(b)) \right)$$

$$(\alpha(a)_a) \longmapsto \varphi(\alpha(a)_a)_f := \alpha(b)$$
$$= \alpha(\text{codomain of } f)$$

$$(\alpha(a)_a) \longmapsto \psi(\alpha(a)_a)_f := F(f) \circ \alpha(a)$$

$\alpha(a)_a$ makes the triangle for $f: a \to b$ commute
iff $\varphi(\alpha)_f = \psi(\alpha)_f$.

$$\sim = \mathrm{Eq}\left(\mathrm{Hom}_C\left(Z, \varprojlim_{I^{disc}} F|_{I^{disc}}\right) \overset{\varphi}{\underset{\psi}{\rightrightarrows}} \mathrm{Hom}_C\left(Z, \prod_{\mathrm{Mor}\, I} F(\text{codomain } f)\right) \right)$$

By Yoneda lemma,

$$\sim = \mathrm{Eq}\left(\mathrm{Hom}_C\left(Z, \varprojlim_{I^{disc}} F|_{I^{disc}}\right) \overset{\mathrm{Hom}_C(Z, \hat{\varphi})}{\underset{\mathrm{Hom}_C(Z, \hat{\psi})}{\rightrightarrows}} \mathrm{Hom}_C\left(Z, \prod_{f \in \mathrm{Mor}\, I} F(\text{codomain } f)\right) \right)$$

for some unique morphisms $\hat{\varphi}, \hat{\psi}: \prod_{a \in I} F(a) \longrightarrow \prod_{f \in \mathrm{Mor}\, I} F(\text{codomain } f)$

$$\sim = \mathrm{Hom}(Z, \mathrm{Eq}(\hat{\varphi}, \hat{\psi})), \text{ by definition of the equalizer. } \square$$

Consider finite groups $\xrightarrow{\text{forget}}$ sets

We saw that it is not representable, but

$$F_n = \mathrm{Hom}(\mathbb{Z}/n, ?) : \text{finite groups} \longrightarrow \text{sets}$$
$$G \longmapsto \{g \in G \mid \text{order } g \mid n\}$$

Then $G \longmapsto \bigcup_n F_n(G) = G$

$$\bigcup_n F_n(G) := \varinjlim_{n \in \mathbb{B}(\mathbb{N}, \cdot)} F_n(G) = \varinjlim_{n \in \mathbb{B}(\mathbb{N}, \cdot)} \mathrm{Hom}_{\text{finite groups}}(\mathbb{Z}/n, G)$$

(This is special, normally false) $= \mathrm{Hom}(\varprojlim_{n \in \mathbb{B}(\mathbb{N}, \cdot)} \mathbb{Z}/n, G) = G$

$\varprojlim_{n \in \mathbb{B}(\mathbb{N}, \cdot)} \mathbb{Z}/n$ is not a finite group, but a profinite group

Def. If C is a category, then the pro-objects over C are objects of the full subcategory of all presheaves $\hat{C} = \mathrm{Hom}(C^{op}, \text{Sets})$ of the form $\varprojlim_I F$ for $F : I \to C$ a functor on a category I

E.g. pro-objects over finite groups are called profinite groups.

Because groups is complete, a profinite group can be realized as fibre products of products of finite groups
$\Leftrightarrow$ G is profinite iff it is the inverse limit over all homomorphisms $G \twoheadrightarrow H$ to finite groups H.

Thm. $\varprojlim_{n \in \mathbb{B}(\mathbb{N}, \cdot)} \mathbb{Z}/n =: \hat{\mathbb{Z}} = \prod_{p \text{ prime}} \hat{\mathbb{Z}}_p$ (= the p-adic integers)

Rmk. If $\text{finite groups}_p \subseteq \text{finite groups}$ denotes the full subcategory of all finite p-groups, then $\varprojlim \mathbb{Z}/p^n \cong \hat{\mathbb{Z}}_p$ and pro-represent the forgetful functor

Dual statements: Formulate everything for colimits!

E.g. coproduct in Sets: $(X_\alpha)_\alpha$

$$\begin{tikzcd}
X_\alpha \arrow[dr] \arrow[drr, bend left, "f"] & & \\
& \coprod_\alpha X_\alpha \arrow[r, dashed, "\exists!"] & Z \\
X_{\alpha'} \arrow[ur] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

Coproduct is the disjoint union

cofibre product

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d] \arrow[ddr, bend left, "a"] & \\
Y \arrow[r] \arrow[drr, bend right, "b"'] & X \amalg_Z Y \arrow[dr, dashed, "\exists!"] & \\
& & U
\end{tikzcd}$$

$X \amalg_Z Y = X \dot{\cup} Y \,/\, f(z) \sim g(z)$ ($\dot{\cup}$: disjoint union)

coequalizer

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, bend right] \arrow[dr, bend right=50] & Y \arrow[r, "h"] \arrow[d, "a"] & Z \arrow[dl, dashed, "\exists!"] \\
& U &
\end{tikzcd}$$

$Z = Y / f(x) \sim g(x)$

In groups, the coproduct of G and H is the free product $G * H$
(i.e. all words of the form $g_1 h_1 g_2 h_2 \cdots g_n h_n$, $g_i \in G$, $h_i \in H$)

Thm. $\mathbb{Z}/2 * \mathbb{Z}/3 \cong PSL(2, \mathbb{Z})$

In abelian groups, coproduct is direct sum (same as product!)

R, S rings: Does $R \amalg S$ exist?
Yes: $R \amalg S$ = free product of R, S
$= \{ \sum_{i,k} r_1 s_1 \cdots r_k s_k \}$
If R, S are commutative, then $R \amalg S = R \otimes_{\mathbb{Z}} S$

Lemma. Assume both coproduct and product of a family $(X_a)_{a\in I}$ exist in C and C has a zero object. Then there exists a natural comparison morphism $\sigma : \coprod_a X_a \longrightarrow \prod_a X_a$.

$$\begin{tikzcd}
X_a \arrow[dr, "i_a"] \arrow[rrrr, equal] & & & & X_a \\
& \coprod_a X_a \arrow[rr, dashed, "\sigma"] & & \prod_a X_a \arrow[ur, "p_a"] \arrow[dr, "p_{a'}"] & \\
X_{a'} \arrow[ur, "i_{a'}"] \arrow[rrrr, equal] & & & & X_{a'}
\end{tikzcd}$$

σ makes this diagram commute, i.e. $p_a \circ \sigma \circ i_a = id_{X_a}$ for each a.

Pf. $\mathrm{Hom}_C\left(\coprod_{b\in J} Y_b, \prod_{a\in I} X_a\right) \underset{\text{claim}}{=} \prod_{a\in I}\prod_{b\in J} \mathrm{Hom}(Y_b, X_a)$

For $I=J$: $X_a = Y_a$, so $\cdots = \prod_{(a,b)\in I\times I} \mathrm{Hom}_C(X_b, X_a)$

$\sigma \in \prod_{(a,b)\in I\times I} \mathrm{Hom}_C(X_b, X_a)$ is such that $\sigma(a,a) = id_{X_a}$, $\sigma_{(a,b),\, a\neq b} = 0$. □

E.g. • In $Sets_*$, $(X,*), (Y,*)$

$(X,*) \amalg (Y,*) = X \sqcup Y / *_x = *_y$

$(X,*) \prod (Y,*) = (X\times Y, (*_x, *_y))$

$\sigma : x \longmapsto (x, *_y)$

$\quad y \longmapsto (*_x, y)$

• $\mathbb{Z}/2 * \mathbb{Z}/2 \xrightarrow{\sigma} \mathbb{Z}_{16} = \mathbb{Z}/2 \times \mathbb{Z}/2$.

Exercise: Describe σ in terms of matrices

Def. A category A is *additive* if (a) it is preadditive, (b) there is a zero object, (c) product and coproduct of any two objects exist, (d) σ is an isomorphism for any $X, Y \in A$.

Def. A category A is *abelian* if it is additive and any morphism has an analysis.