The Arthur trace formula
and
spectral theory of automorphic forms

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0. Introduction

The main purpose for which Jim Arthur developed the trace formula is, of course, to establish functoriality. The trace formula, however, has also interesting applications in other fields such as analytic number theory and spectral theory of automorphic forms. From the beginning the Selberg trace formula has been an important tool to study spectral problems in the theory of automorphic forms. For some of the potential applications of the Arthur trace formula to spectral theory it is necessary to know that the spectral side of the trace formula is absolutely convergent.

The purpose of this talk is to discuss the problem of absolute convergence and some applications which are based on it.

1. The noninvariant trace formula

In this talk we will only be concerned with the noninvariant trace formula. The general framework is as follows:

- $\mathbb{A}$ ring of adèles of $\mathbb{Q}$

- G reductive algebraic group over $\mathbb{Q}$

$$G(\mathbb{A})^1 = \bigcap_{\chi \in X(G)_{\mathbb{Q}}} \ker |\chi|$$

$$G(\mathbb{A}) = G(\mathbb{A})^1 \cdot A_G(\mathbb{R})^0.$$

- $\mathfrak{X}$ equivalence classes of (M, ρ), M Levi factor of rational parabolic subgroup, ρ cuspidal automorphic representation of $M(\mathbb{A})^1$.

- $\mathfrak{X}$ set of cuspidal data

- $\mathfrak{O}$ set of equivalence classes of those elements in $G(\mathbb{Q})$ whose semisimple components are $G(\mathbb{Q})$-conjugate

The noninvariant trace formula is an identity between distributions on $G(\mathbb{A})^1$:

$$\sum_{\chi \in \mathfrak{X}} J_\chi(f) = \sum_{\mathfrak{o} \in \mathfrak{O}} J_\mathfrak{o}(f), \quad f \in C_0^\infty(G(\mathbb{A})^1)$$

spectral side = geometric side

- J_χ, $J_\mathfrak{o}$ distributions on $G(\mathbb{A})^1$

- $J_\mathfrak{o}$ weighted orbital integrals

The spectral side

- P_0 minimal $\mathbb{Q}$-parabolic subgroup, M_0 Levi subgroup of P_0

- $P \subset G$ $\mathbb{Q}$-parabolic subgroup, $P = M_P N_P$ Levi decomposition with $M_0 \subset M_P$

- $A_P \subset M_P$ split component of the center of M_P,

- $\mathfrak{a}_P = \mathrm{Lie}(A_P)$

- $\mathcal{A}^2(P)$ square integrable automorphic forms on

  $N_P(\mathbb{A}) M_P(\mathbb{Q}) \backslash G(\mathbb{A})$

• For $\chi \in \mathfrak{X}$, $\pi \in \Pi(M_P(\mathbb{A})^1)$, let $\mathcal{A}^2_{\chi,\pi}(P) \subset \mathcal{A}^2(P)$ be the subspace of all ϕ such that for each $x \in G(\mathbb{A})$, the function

$$\phi_x(m) = \phi(mx), \quad m \in M_P(\mathbb{A}),$$

belongs to $L^2(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1)_\chi$ and transforms under $M_P(\mathbb{A})$ according to π

• $\rho_{\chi,\pi}(P,\lambda)$ induced representation of $G(\mathbb{A})$ in $\overline{\mathcal{A}}^2_{\chi,\pi}(P)$

• $Q \subset G$ $\mathbb{Q}$-parabolic subgroup, $s \in W(\mathfrak{a}_P, \mathfrak{a}_Q)$

$$M_{Q|P}(s,\lambda)\colon \mathcal{A}^2(P) \to \mathcal{A}^2(Q), \quad \lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$$

intertwining operator, meromorphic function of λ.

• most important ingredient of the spectral side

Let M be a Levi subgroup and $Q, P \in \mathcal{P}(M)$. Set

$$M_{Q|P}(\lambda) := M_{Q|P}(1,\lambda).$$

• $L \supset M_P$ Levi subgroup

$$\mathfrak{M}_L(P,\lambda) := \lim_{\Lambda\to 0}\Bigg(\sum_{Q_1\in\mathcal{P}(L)} \mathrm{vol}(\mathfrak{a}_{Q_1}^G/\mathbb{Z}(\Delta_{Q_1}^\vee)) \times M_{Q|P}(\lambda)^{-1}\frac{M_{Q|P}(\lambda+\Lambda)}{\prod_{\alpha\in\Delta_{Q_1}}\Lambda(\alpha^\vee)}\Bigg),$$

Given $Q_1 \in \mathcal{P}(L)$, $Q \in \mathcal{P}(M_P)$ is the a parabolic subgroup with $Q \subset Q_1$, λ and Λ are constrained to lie in $i\mathfrak{a}_L^*$.

Special case: $L = M$, $\dim \mathfrak{a}_L^G = 1$, α unique simple root of (P, A), $\tilde{\omega} \in (\mathfrak{a}_M^G)^*$, $\tilde{\omega}(\alpha^\vee) = 1$.

$$\mathfrak{M}_L(P, z\tilde{\omega}) = -\,\mathrm{vol}(\mathfrak{a}_M^G/\mathbb{Z}\alpha^\vee) \times M_{\overline{P}|P}(z\tilde{\omega})^{-1}\cdot\frac{d}{dz}M_{\overline{P}|P}(z\tilde{\omega}).$$

So $\mathfrak{M}_L(P,\lambda)$ can be regarded as "generalized logarithmic derivative"

• Let $f \in C_0^\infty(G(\mathbb{A})^1)$. Then Arthur has shown that the distribution J_χ has the following expression:

$$J_\chi(f) = \sum_{M_0 \subset M \subset L} \sum_{s \in W^L(\mathfrak{a}_M)_{\mathrm{reg}}} \sum_{\pi \in \Pi(M(\mathbb{A})^1)}$$
$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \sum_{P \in \mathcal{P}(M)} \mathrm{Tr}\left(\mathfrak{M}_L(P,\lambda) M_{P|P}(s,0) \rho_{\chi,\pi}(P,\lambda,f)\right) \, d\lambda.$$

Theorem(Arthur): $\sum_{\chi \in \mathfrak{X}} |J_\chi(f)| < \infty$

Problem: Absolute convergence of the spectral side w.r.t. the trace norm.

2. Absolute convergence of the spectral side

2.1 The discrete spectrum

- ξ unitary character of $A_G(\mathbb{R})^0$

$$L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}), \xi) = \bigoplus_{\pi \in \Pi(G(\mathbb{A}))} m(\pi) \mathcal{H}_\pi.$$

- R_{disc} regular representation of $G(\mathbb{A})$ in $L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}), \xi)$

Theorem(M.,'89): Let $f \in C^\infty(G(\mathbb{A}),\xi)$ be K-finite. Then $R_{\text{disc}}(f)$ is trace class.

$$\operatorname{Tr}(R_{\text{disc}}(f)) = \sum_{\pi\in\Pi(G(\mathbb{A}))} m(\pi)\operatorname{Tr}(\pi(f))$$

converges with respect to the trace norm

- L. Ji, M. '98: Removal of K-finiteness assumption.

2.2 The rank one cuspidal spectrum

Theorem(Langlands, '90): Let P be maximal parabolic. Then

$$\sum_{\chi\in\mathfrak{X}} \sum_{\pi\in\Pi_{\text{cusp}}(M_P(\mathbb{A})^1)}$$

$$\int_{-\infty}^{\infty} \operatorname{Tr}\left(M_{\overline{P}|P}(i\lambda,\pi)^{-1}\frac{d}{dz}M_{\overline{P}|P}(i\lambda,\pi)\rho_{\chi,\pi}(P,i\lambda,f)\right)\,d\lambda$$

converges absolutely w.r.t. trace norm.

2.3 The general case

- M Levi subgroup, $\pi \in \Pi(M(\mathbb{A}))$, $\pi = \otimes_v \pi_v$.

$$\left(I_P^G(\pi_\lambda),\mathcal{H}_P(\pi)\right) \cong \left(\otimes_v I_P^G(\pi_{v,\lambda}),\otimes_v\mathcal{H}_P(\pi_v)\right), \quad \lambda\in\mathfrak{a}^*_{M,\mathbb{C}}.$$

$$J_{Q|P}(\pi_v,\lambda)\colon \mathcal{H}_P^0(\pi_v)\to\mathcal{H}_Q^0(\pi_v),\quad \lambda\in\mathfrak{a}^*_{M,\mathbb{C}}$$

local intertwining operator

Arthur, Langlands: There exist normalizing factors $r_{Q|P}(\pi_v,\lambda)$, meromorphic functions of $\lambda\in\mathfrak{a}^*_{M,\mathbb{C}}$, such that

$$R_{Q|P}(\pi_v,\lambda):=r_{Q|P}(\pi_v,\lambda)^{-1}J_{Q|P}(\pi_v,\lambda)$$

satisfy axioms of normalized intertwining operators such as

- $R_{S|Q}(\pi_v,\lambda)R_{Q|P}(\pi_v,\lambda)=R_{S|P}(\pi_v,\lambda)$
- $R_{Q|P}(\pi-v,\lambda)^*=R_{P|Q}(\pi_v,-\overline{\lambda})$

Example: $G=\mathrm{GL}(n)$, P standard maximal parabolic, $M_P=\mathrm{GL}(n_1)\times\mathrm{GL}(n_2)$, $\pi_v=\pi_{v,1}\otimes\pi_{v,2}$,

$L(s,\pi_{v,1}\times\tilde{\pi}_{v,2})$ Rankin-Selberg L-function

$$r_{\overline{P}|P}(\pi_v,s)=\frac{L(s,\pi_{v,1}\times\tilde{\pi}_{v,2})}{L(1+s,\pi_{v,1}\times\tilde{\pi}_{v,2})\epsilon(s,\pi_{v,1}\times\tilde{\pi}_{v,2},\psi)}.$$

Set

$$r_{Q|P}(\pi,\lambda) := \prod_v r_{Q|P}(\pi_v,\lambda).$$

- converges in some chamber in $\mathfrak{a}^*_{P,\mathbb{C}}$.
- Let $\pi \in \Pi_{\text{disc}}(M(\mathbb{A})^1)$. $r_{Q|P}(\pi,\lambda)$ admits meromorphic extension.

$$N_{Q|P}(\pi,\lambda) := r_{Q|P}(\pi,\lambda)^{-1} M_{Q|P}(\pi,\lambda)$$

- normalized global intertwining operator.

- ξ unitary character of $A_M(\mathbb{R})^0$, $\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))_\xi$

$$\overline{\mathcal{A}}^2_\pi(P) \cong \mathcal{H}_P(\pi) \otimes \mathsf{Hom}_{M(\mathbb{A})}\left(\pi, I^{M(\mathbb{A})}_{M(\mathbb{Q})A_M(\mathbb{R})^0}(\xi)\right)$$

- $\mathcal{H}_P(\pi)$ space of the induced representation

$$N_{Q|P}(\pi,\lambda) = \left(\prod_v R_{Q|P}(\pi_v,\lambda) \otimes \mathsf{Id}\right)$$

• $K_f = \prod_{v<\infty} K_v \subset G(\mathbb{A}_f)$ open compact subgroup.

There exists finite set of primes S_0 such that

$$R_{Q|P}(\pi_v, \lambda)_{K_v} = \mathrm{Id}, \quad v \notin S_0, \quad \pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A})^1)$$

a) Special case: P maximal parabolic

$$M_{\overline{P}|P}(\pi, z)^{-1} \frac{d}{dz} M_{\overline{P}|P}(\pi, z) = \frac{r'_{\overline{P}|P}(\pi, z)}{r_{\overline{P}|P}(\pi, z)} + N_{\overline{P}|P}(\pi, z)^{-1} \frac{d}{dz} N_{\overline{P}|P}(\pi, z)$$

b) General case: Use Arthur's theory of (G, M)-families to reduce the investigation of generalized logarithmic derivatives of $M_{Q|P}(\pi, \lambda)$ to that of $r_{Q|P}(\pi, \lambda)$ and $N_{Q|P}(\pi, \lambda)$.

Study of normalizing factors:

• M., '00: Let P be maximal parabolic, $\phi, \psi \in \mathcal{A}^2(P)$. Then

$$s \in \mathbb{C} \mapsto \langle M_{\overline{P}|P}(\pi, s)\phi, \psi \rangle$$

is a meromorphic function of order $\leq n+2$, $n = \dim G(\mathbb{R})/K_\infty$.

• $R_{\overline{P}|P}(\pi_p, s)$ is a rational function of p^{-s}, if $p < \infty$, and of s, if $p = \infty$.

$\Rightarrow$ $r_{\overline{P}|P}(\pi, s)$ is a meromorphic function of order $\leq n+2$.

Theorem(M.,'01): Suppose that for all $M \in \mathcal{L}(M_0)$, $Q, P \in \mathcal{P}(M)$ and v the following holds:

1) $v < \infty$: For every $K_v \subset G(\mathbb{Q}_v)$ and every invariant differential operator D_λ on $\mathfrak{a}_M^*$ there exists $C > 0$ such that

$$\| D_\lambda R_{Q|P}(\pi_v, \lambda)_{K_v} \| \leq C$$

for all $\lambda \in i\mathfrak{a}_M^*$ and all $\pi_v \in \Pi_{\text{disc}}(M(\mathbb{Q}_v))$.

2) $v = \infty$: For every invariant differential operator D_λ on $\mathfrak{a}_M^*$ there exist $C > 0$ and $N \in \mathbb{N}$ such that

$$\| D_\lambda R_{Q|P}(\pi_\infty, \lambda)|_{\mathcal{H}_{\pi_\infty}(\sigma)} \| \leq C(1+ \| \lambda \| +|\lambda_{\pi_\infty}| + |\lambda_\sigma|)^N$$

for all $\lambda \in i\mathfrak{a}_M^*$, $\sigma \in \Pi(K_\infty)$ and $\pi_\infty \in \Pi_{\text{disc}}(M(\mathbb{R}))$.

Then for every $f \in \mathcal{C}^1(G(\mathbb{A})^1)$ the spectral side is absolutely convergent.

Proposition(Lapid, '03): Let Q, P be adjacent. Let

$$\delta = \min\left\{\frac{1}{2}, |\text{Re}(\rho)| : \rho \text{ pole of } R_{Q|P}(\pi_\infty, s)|_{\mathcal{H}_{\pi_\infty}(\sigma)}\right\}.$$

For every differential operator $D(s)$ with constant coefficients there exist constants C, k such that

$$\| D(s) R_{Q|P}(\pi_\infty, s)|_{\mathcal{H}_{\pi_\infty}(\sigma)} \| \leq C \left(\frac{1+ \| \sigma \|}{\delta}\right)^k$$

for all $\sigma \in \Pi(K_\infty)$ and $s \in i\mathbb{R}$.

- a similar result holds at finite places

- reduces problem of absolute convergence to the existence of pole free strips

Special case: $G = \text{GL}(n)$.

Luo-Rudnick-Sarnak: Weak version of Ramanujan conjecture for $\text{GL}(n)$.

• π_v unitary generic representation of $\mathrm{GL}(m, \mathbb{Q}_v)$.

Then π_v is fully induced:

$$\pi_v \cong I_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\tau_1[t_1] \otimes \cdots \otimes \tau_r[t_r])$$

$$\tau[t](g) = \tau(g)|\det(g)|^t, \quad g \in G(\mathbb{Q}_v).$$

P standard parabolic subgroup,

$$M_P = \mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_r),$$

τ_i tempered representation of $\mathrm{GL}(n_i, \mathbb{Q}_v)$,

$$t_1 > t_2 > \cdots > t_r, \quad |t_i| < 1/2.$$

H. Jacquet: $\pi = \otimes \pi_v$ cuspidal automorphic representation of $\mathrm{GL}(m, \mathbb{A})$. Then π_v is generic.

Theorem(Luo-Rudnick-Sarnak,'99): Let $\pi = \otimes \pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}(m, \mathbb{A})$, π_v unramified. Then

$$|t_i| < \frac{1}{2} - \frac{1}{m^2+1}, \quad i = 1, ..., r.$$

• **M., Speh:** Extension to ramified places.

• **Mœglin-Waldspurger:** Description of the residual spectrum of $\mathrm{GL}(n)$.

$\Rightarrow$ Extension to all automorphic representations in the discrete spectrum of $\mathrm{GL}(m, \mathbb{A})$

Theorem(M., Speh): $M = \mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_r)$ Levi subgroup of $\mathrm{GL}(m)$, $Q, P \in \mathcal{P}(M)$. For all place v and all $\pi_v \in \Pi_{\mathrm{disc}}(M(\mathbb{Q}_v))$, $R_{Q|P}(\pi_v, \mathrm{s})$ is holomorphic in

$$\left\{ \mathrm{s} \in \mathbb{C}^r \colon \mathrm{Re}(s_i - s_j) > -2/(1 + m^2), \quad 1 \leq i < j \leq r \right\}.$$

Theorem(Lapid, M., Speh): Let $G = \mathrm{GL}(n)$. Then the spectral side of the Arthur trace formula is absolutely convergent for all $f \in \mathcal{C}^1(G(\mathbb{A})^1)$.

General case. The above theorem suggests the following conjecture:

Conjecture 1: G reductive over $\mathbb{Q}$, P maximal $\mathbb{Q}$-parabolic subgroup of G. For all v there exists $\delta > 0$ such that $R_{\overline{P}|P}(\pi_v, s)$ is holomorphic for $|\mathrm{Re}(s)| < \delta$ and all $\pi_v \in \Pi_{\mathrm{disc}}(G(\mathbb{Q}_v))$.

• Conjecture 1 $\Rightarrow$ absolute convergence of spectral side.

Question: Is the conjecture compatible with Arthur's conjectures?

Arthur's conjectures: unramified case

• G split over $\mathbb{Q}$, $\mathcal{G}$ smooth model of G over $\mathbb{Z}$.

• p finite prime, $K_p = \mathcal{G}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$.

• π irreducible representation of $G(\mathbb{Q}_p)$, with K_p fixed vector. Then π is unique subquotient of

$$\rho = I_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\chi)$$

which has K_p-fixed vector.

• B Borel subgroup, χ quasi-character of $T(\mathbb{Q}_p) \cong (\mathbb{Q}_p^\times)^r$, $T \subset B$ maximal torus.

• $\widehat{G}$ complex dual group of G, $\widehat{T} \subset \widehat{G}$ maximal torus such that

$$X_*(T) = X^*(\widehat{T}).$$

$$\pi \mapsto t_\pi \in \widehat{T}/W, \quad W = W(G,T).$$

- $W_p = W_{\mathbb{Q}_p}$ Weil group

$$1 \to I_p \to W_p \to \mathbb{Z} \to 1$$

- frob_p inverse of the geometric Frobenius in W_p/I_p.

Arthur parameters: morphism

$$\psi\colon W_p \times \mathrm{SL}(2,\mathbb{C}) \to \widehat{G}$$

such that

1) $\psi|_{\mathrm{SL}(2,\mathbb{C})}$ is an algebraic representation

2) $\psi|_{W_p}$ is unramified and $\psi(\mathrm{frob}_p)$ belongs to maximal compact subgroup of $\widehat{G}$.

- $j\colon W_p \to \mathrm{SL}(2,\mathbb{C})$ map which is unramified and

$$\mathrm{frob}_p \mapsto \begin{pmatrix} |p|^{1/2} & 0 \\ 0 & |p|^{-1/2} \end{pmatrix}.$$

Conjecture 2(Arthur): If π_p is a local, unramified component of a representation occurring in $\mathcal{A}^2_G$, then

$$t_{\pi,p} = \psi(\mathrm{frob}_p, j(\mathrm{frob}_p))$$

for a parameter ψ satisfying 1) and 2).

Clozel's version of this conjecture:

Conjecture 3(Clozel): If π_p is a local unramified component of an automorphic representation π, then for any isomorphism $\widehat{T} \cong (\mathbb{C}^\times)^r$:

$$t_{\pi,p} = (t_1, ..., t_r) \quad \text{with} \quad |t_i| = p^{w_i/2}, \quad w_i \in \mathbb{Z}.$$

• Using the properties satisfied by normalized intertwining operators, it follows that Conjecture 3 implies Conjecture 1 in the unramified case.

• There may be alternative approaches which avoid considerations of the existence of pole free regions of local intertwining operators.

3. Applications

We discuss a number of applications of the Arthur trace formula for which the absolute convergence of the spectral side is significant.

3.1 Existence of cusp forms and Weyl's law

- G connected semisimple algebraic group over $\mathbb{Q}$
- $\Gamma \subset G(\mathbb{Q})$ arithmetic subgroup
- $K_\infty \subset G(\mathbb{R})$ maximal compact subgroup
- $L^2_{\mathsf{cusp}}(\Gamma\backslash G(\mathbb{R})) \subset L^2(\Gamma\backslash G(\mathbb{R}))$ space of cusp forms
- $\sigma : K_\infty \to \mathsf{GL}(V_\sigma)$ irreducible unitary representation.

$$H^\Gamma_{\mathsf{cusp}}(\sigma) := \left(L^2_{\mathsf{cusp}}(\Gamma\backslash G(\mathbb{R})) \otimes V_\sigma\right)^{K_\infty}$$

Space of Γ-cusp forms of "weight" σ.

- $\Omega \in Z(\mathfrak{g}_\mathbb{C})$ Casimir element.
- ρ_∞ regular representation of $G(\mathbb{R})$ in $L^2_{\mathsf{cusp}}(\Gamma\backslash G(\mathbb{R}))$.
- Δ_σ selfadjoint operator in $H^\Gamma_{\mathsf{cusp}}(\sigma)$ induced by $-\rho_\infty(\Omega) \otimes \mathsf{Id}$.

Geometric interpretation: Assume that Γ is torsion free. Let

$$X = G(\mathbb{R})/K_\infty$$

be the Riemannian symmetric space and let $\tilde{E}_\sigma \to X$ be the homogeneous vector bundle attached to σ. Set

$$E_\sigma = \Gamma\backslash\tilde{E}_\sigma \to \Gamma\backslash X.$$

Then

$$\left(L^2(\Gamma\backslash G(\mathbb{R})) \otimes V_\sigma\right)^{K_\infty} \cong L^2(\Gamma\backslash X, E_\sigma)$$

and

$$\Delta_\sigma = (\nabla^\sigma)^* \nabla^\sigma - \lambda_\sigma \mathrm{Id},$$

where ∇^σ is the canonical invariant connection of $\tilde{E}_\sigma$ and λ_σ the Casimir eigenvalue of σ.

- Δ_σ has pure point spectrum in $H^\Gamma_{\text{cusp}}(\sigma)$:

$$\lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots \to \infty.$$

- cuspidal spectrum of "weight" σ.

Counting function:

$$N^\Gamma_{\text{cusp}}(\lambda, \sigma) := \#\left\{i\colon |\lambda_i| \le \lambda\right\}.$$

Let $d = \dim X$.

Weyl's constant:

$$C_\Gamma := \frac{\mathrm{Vol}(\Gamma\backslash X)}{(4\pi)^{d/2}\Gamma(d/2+1)}.$$

Conjecture(Sarnak, 1984): Assume that $\sigma|_{Z_\Gamma} = \mathrm{Id}$. Then

$$N^\Gamma_{\mathrm{cusp}}(\lambda,\sigma) \sim \dim(\sigma) C_\Gamma\, \lambda^{d/2}$$

as $\lambda \to \infty$.

The conjecture has been proved in the following cases:

- A. Selberg, 1954: $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ congruence subgroup, $\sigma = 1$.

- I. Efrat, 1987: $\Gamma \subset \mathrm{SL}(2,\mathbb{R})^n$ Hilbert modular group, $\sigma = 1$.

- A. Reznikov, 1993: $\Gamma \subset \mathrm{SO}_0(n,1)$ congruence subgroup, $\sigma = 1$.

- St. Miller, 2001: $\Gamma = \mathrm{SL}(3,\mathbb{Z})$, $\sigma = 1$.

Theorem(M.,'03): Let $G = \mathsf{SL}(n)$ and let $\Gamma \subset \mathsf{SL}(n,\mathbb{Z})$ be a principal congruence subgroup. Then

$$N^{\Gamma}_{\mathsf{cusp}}(\lambda,\sigma) \sim \dim(\sigma) C_{\Gamma}\, \lambda^{d/2}$$

for all $\sigma \in \Pi(\mathsf{SO}(n))$ such that $\sigma|_{Z_\Gamma} = \mathrm{Id}$.

Method: Combination of trace formual and heat equation method.

Assume: $\sigma = 1$.

Choice of test function:

$h_t \in \mathcal{C}^1(G(\mathbb{R}))$, $t > 0$, kernel of $e^{-t\Delta}$ on X.

$K_f \subset G(\mathbb{A}_f)$ open compact subgroup, $\Gamma = K_f \cap G(\mathbb{Q})$.

$$\phi_t(g) = h_t(g_\infty)\frac{1}{\mathrm{vol}(K_f)}\chi_{K_f}(g_f), \quad g = g_\infty g_f \in G(\mathbb{A}).$$

$K_f = K_f(N)$ congruence subgroup.

$$J_{\mathsf{spec}}(\phi_t) = \varphi(N)\sum_i e^{-t\lambda_i} + E(t),$$

$E(t)$ contribution of the Eisenstein series.

$$J_{\text{geom}}(\phi_t) = \varphi(N)\frac{\text{vol}(\Gamma(N)\backslash X)}{(4\pi)^{d/2}}t^{-d/2} + o(t^{-d/2})$$

as $t \to 0+$, $X = G(\mathbb{R})/K_\infty$, $d = \dim X$.

$$J_{\text{spec}}(\phi_t) = J_{\text{geom}}(\phi_t).$$

Problem: behaviour of $E(t)$ as $t \to 0+$.

Main result: Let $d = \dim G(\mathbb{R})/K_\infty$. Then

$$E(t) = O(t^{-(d-1)/2}), \quad t \to 0+.$$

For the proof we use the following results:

• Absolute convergence of the spectral side of the trace formula

• Weak version of Ramanujan conjecture (Luo-Rudnick-Sarnak)

• Analytic properties of Rankin-Selberg L-functions (Jacquet, Piatetski-Shapiro, Shalika, Shahidi, Mœglin, Waldspurger,...), bounds on the logarithmic derivatives.

A weaker result suffices: Let π_i, $i = 1, 2$, be a cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A})$. Set

$$\Lambda(s, \pi_1, \pi_2) = \frac{L(s, \pi_1 \times \tilde{\pi}_2)}{L(1 + s, \pi_1 \times \tilde{\pi}_2)\epsilon(s, \pi_1 \times \tilde{\pi}_2)}.$$

Then

$$\int_{-T}^{T} \left| \frac{\Lambda'}{\Lambda}(ir, \pi_1, \pi_2) \right| dr \leq C\, T \log(T + \nu(\pi_1 \times \tilde{\pi}_2))$$

for $T > 0$, where $\nu(\pi_1 \times \tilde{\pi}_2)$ is the analytic conductor.

• Description of the residual spectrum (Mœglin, Waldspurger).

• At present these results are only available for $\mathrm{GL}(n)$.

Estimation of remainder term:

Write

$$N^{\Gamma}_{\mathrm{cusp}}(\lambda, \sigma) = N_{\mathrm{smooth}}(\lambda, \sigma) + N_{\mathrm{osc}}(\lambda, \sigma)$$

Problem: Estimation of $N_{\mathrm{osc}}(\lambda)$

General case and weaker versions of Weyl's law:

Theorem(Piatetski-Shapiro). Let $\sigma = 1$. For every Γ there exists a normal subgroup of finite index Γ' of Γ such that

$$\lim_{\lambda \to \infty} N^{\Gamma'}_{\text{cusp}}(\lambda, 1) = \infty.$$

- A.B. Venkov, $G = \text{SL}(2)$.

Theorem(Labesse-M.). Let G be almost simple, connected and simply connected such that $G(\mathbb{R})$ is non compact. Let S be a finite set of primes containing at least two finite primes. There exists $C_\Gamma(S) \leq 1$ such that for every congruence subgroup $\Gamma \subset G(\mathbb{R})$ and every σ such that $\sigma|_{Z_\Gamma} = \text{Id}$ we have

$$\dim(\sigma) C_\Gamma C_\Gamma(S) \leq \liminf_{\lambda \to \infty} \frac{N^{\Gamma}_{\text{cusp}}(\lambda, \sigma)}{\lambda^{d/2}}.$$

- $0 < C_\Gamma(S)$ for Γ a deep enough congruence subgroup.

- The proof uses a simple form of the trace formula.

3.2. The tempered spectrum

- S. Miller: $\Gamma = \mathrm{SL}(3,\mathbb{Z})$.

- Δ Laplacian on functions of $\Gamma\backslash\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$.

$$\lim_{T\to\infty} \frac{\#\{\lambda_j \le T \mid \Delta\phi_j = \lambda_j\phi_j,\ \phi_j \text{ tempered}\}}{\#\{\lambda_j \le T\}} = 1.$$

Problem: Extension to congruence subgroups $\Gamma \subset \mathrm{SL}(n,\mathbb{Z})$.

3.3. Limit multiplicities

$$L^2_{\mathrm{cusp}}(\Gamma\backslash G(\mathbb{R})) = \bigoplus_{\pi\in\widehat{G(\mathbb{R})}} N_\Gamma(\pi)\mathcal{H}_\pi$$

$$\mu_\Gamma^{\mathrm{cusp}} = \frac{1}{\mathrm{vol}(\Gamma\backslash G(\mathbb{R}))} \sum_{\pi\in\widehat{G(\mathbb{R})}} N_\Gamma(\pi)\delta_\pi$$

- δ_π delta distribution

- $\mu_\Gamma^{\mathrm{cusp}}$ measure on $\widehat{G}(\mathbb{R})$.

$$\Gamma = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n \supset \cdots, \quad \cap_j \Gamma_j = \{e\},$$

tower of normal subgroups of finite index.

• μ Plancherel measure of $G(\mathbb{R})$.

Conjecture 4: Let (Γ_j) be a tower of normal subgroups of finite index (and of bounded depth). For every open relatively compact subset $U \subset \widehat{G(\mathbb{R})}$, which is regular for the Plancherel measure ($\mu(\overline{U}) = \mu(U)$), one has

$$\lim_{j\to\infty} \mu_{\Gamma_j}^{\text{cusp}}(U) = \mu(U).$$

Known results:

Γ cocompact: de George, Wallach, Delorme

Γ co-finite: Savin: Let $\pi \in \widehat{G(\mathbb{R})}_d$. Then

$$\mu_{\Gamma_j}^{\text{cusp}}(\{\pi\}) = \frac{N_{\Gamma_j}(\pi)}{\text{vol}(\Gamma_j \backslash G(\mathbb{R}))} \to d(\pi) = \mu(\{\pi\}).$$

• It is likely that Conjecture 4 can be proved for $G = \text{SL}(n)$ with the methods used to prove Weyl's law.

Let G be semisimple over $\mathbb{Q}$.

$\widehat{G(\mathbb{Q}_v)}_{\text{cusp}} \subset \widehat{G(\mathbb{Q}_v)}$ closure in Fell topology of the set of π_v's such that $\pi = \otimes_v \pi_v$ occurs in $L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$.

- cuspidal automorphic dual.

- Conjecture 4 $\Rightarrow$

$$\widehat{G(\mathbb{R})}_{\text{temp}} \subset \widehat{G(\mathbb{R})}_{\text{cusp}}.$$

- Ramanujan conjecture for $G = \mathrm{GL}(n)$: $'' ='' $ holds.

3.4. Asymptotics of class numbers

Recent work of Deitmar and Hoffmann on the asymptotic of class numbers for cubic number fields