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Frobenius Manifolds and Integrable Hierarchies

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Lecture 4

From Frobenius Manifolds to Integrable Systems of the Topological Type

Recall: we are classifying hierarchies of bi-hamiltonian systems of n PDEs

$$u^i_t = A^i_j(u) u^j_x + \epsilon \left(B^i_j(u) u^j_{xx} + \frac{1}{2} C^i_{jk}(u) u^j_x u^k_x \right) + O(\epsilon^2)$$

$i = 1, \dots, n$ wrt the group of Miura-type transformations

$$u^i \mapsto \tilde u^i = f^i_0(u) + \epsilon\, f^i_1(u; u_x) + \epsilon^2\, f^i_2(u; u_x, u_{xx}) + O(\epsilon^3)$$

$$\deg f^i_m(u; u_x, \dots, u^{(m)}) = m, \quad \det \left(\frac{\partial f^i_0(u)}{\partial u^j} \right) \neq 0$$

Main assumptions: triangular bihamiltonian recursion relation, existence of tau-function, certain genericity conditions.

Moduli (?):

- (calibrated) semisimple Frobenius manifold, $n(n-1)/2$ parameters

- n "central charges" $c_1, \ldots, c_n$

Hierarchy of the topological type:

$$c_1 = c_2 = \cdots = c_n = \frac{1}{24}$$

Main goal: for any n construct **universal** integrable hierarchy of the topological type

Recall Quasitriviality Theorem: there exists a substitution

$$w^\alpha = v^\alpha + \frac{\partial^2}{\partial x \partial t^{\alpha,0}} \sum_{k \geq 1} \epsilon^k F_{[k]}(v; v_x, \dots, v^{(n_k)}), \quad \alpha = 1, \dots, n$$

transforming dispersionless tau-structure to the full one. Here

$$F_{[1]} = 0, \quad F_{[2]} = G(v) + \sum_{i=1}^{n} c_i \log u_x^i$$

$F_{[k]}(v; v_x, \dots, v^{(n_k)})$ a **rational** function of v_x, $\dots$, $v^{(n_k)}$ for $k > 2$,

$$n_{2g}, \quad n_{2g+1} \leq 3g - 2$$

Virasoro symmetries

Recall: *monodromy at the origin* (μ, R) describes a basis of horizontal sections of $\tilde{\nabla}$ near $z = 0$

$$(1 + O(z))\ z^{\mu} z^{R}$$

Semisimple μ

$$\mu = \frac{2-d}{2} - \nabla E = \mathrm{diag}\,(\mu_1, \dots, \mu_n)$$

Triangular R (in case of *resonances*)

$$R = R_1 + R_2 + \dots, \quad (R_k)_{\alpha\beta} \neq 0 \text{ only if } \mu_\alpha - \mu_\beta = k$$

Free field realization for the Virasoro algebra

Heisenberg algebra with the generators

$$a_{\alpha,p}, \quad \alpha = 1, \ldots, n, \quad p \in \mathbf{Z} + \frac{1}{2}$$

$$[a_{\alpha,p}, a_{\beta,q}] = (-1)^{p-\frac{1}{2}} \eta_{\alpha\beta} \delta_{p+q,0}.$$

Introduce the row vectors

$$\mathbf{a}_p = (a_{1,p}, \ldots, a_{n,p})$$

and $\phi(\lambda) = (\phi_1(\lambda), \ldots, \phi_n(\lambda))$

$$\phi_(\lambda) = \int_0^\infty \frac{dz}{z} e^{-\lambda\, z} \sum_{p \in \mathbf{Z}+\frac{1}{2}} \mathbf{a}_p z^{p+\mu} z^R$$

$$T(\lambda) = \sum_{m \in \mathbf{Z}} \frac{L_m}{\lambda^{m+2}} = -\frac{1}{2} : \partial_\lambda \phi_\alpha \, G^{\alpha\beta} \partial_\lambda \phi_\beta :$$

$$+ \frac{1}{4\,\lambda^2} \mathrm{tr} \left(\frac{1}{4} - \mu^2 \right).$$

Here

$$G^{\alpha\beta} = -\frac{1}{2\,\pi} \left[\left(e^{\pi\, i\, R} e^{\pi\, i\, \mu} + e^{-\pi\, i\, R} e^{-\pi\, i\, \mu} \right) \eta^{-1} \right]^{\alpha\beta},$$

L_m for $m \geq -1$ are well defined; full Virasoro algebra only if Spec μ contains no half integers.

$$[L_k, L_l] = (k-l)L_{k+l} + n\frac{k(k^2-1)}{12}\delta_{k+l,0}$$

Realization

$$a_{\alpha,p} = \epsilon \frac{\partial}{\partial t^{\alpha,p-\frac{1}{2}}}, \quad p > 0,$$
$$a_{\alpha,p} = \epsilon^{-1}(-1)^{p+\frac{1}{2}}\eta_{\alpha\beta}t^{\beta,-p-\frac{1}{2}}, \quad p < 0$$

Example 1 $n = 1$ (KdV) $\mu = 0$, $R = 0$

$$L_m = \frac{\epsilon^2}{2} \sum_{k+l=m-1} \frac{(2k+1)!!\,(2l+1)!!}{2^{m+1}} \frac{\partial^2}{\partial t_k \partial t_l}$$

$$+ \sum_{k \geq 0} \frac{(2k+2m+1)!!}{2^{m+1}(2k-1)!!} t_k \frac{\partial}{\partial t_{k+m}} + \frac{1}{16}\delta_{m,0},$$

$$L_{-1} = \sum_{k \geq 1} t^k \frac{\partial}{\partial t_{k-1}} + \frac{1}{2\epsilon^2} t_0^2,$$

$$L_{-m} = \frac{1}{2\,\epsilon^2} \sum_{k+l=m-1} \frac{2^{m-1}}{(2k-1)!!(2l-1)!!} t_k\, t_l$$

$$+ \sum_{k \geq 0} \frac{2^{m-1}(2k+1)!!}{(2m+2k-1)!!} t_{k+m} \frac{\partial}{\partial t_k}, \quad m > 1.$$

Example 2 $M = QH^*(\mathbf{P}^1)$, $F = \frac{1}{2} u\, v^2 + e^u$

$$\mu = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

$$L_m = \epsilon^2 \sum_{k=1}^{m-1} k!\,(m-k)! \frac{\partial^2}{\partial t^{2,k-1} \partial t^{2,m-k-1}}$$

$$+ \sum_{k \geq 1} \frac{(m+k)!}{(k-1)!} \left(t^{1,k} \frac{\partial}{\partial t^{1,m+k}} + t^{2,k-1} \frac{\partial}{\partial t^{2,m+k-1}} \right)$$

$$+2 \sum_{k \geq 0} \alpha_m(k) t^{1,k} \frac{\partial}{\partial t^{2,m+k-1}}, \quad m > 0$$

$$L_0 = \sum_{k \geq 1} k \left(t^{1,k} \frac{\partial}{\partial t^{1,k}} + t^{2,k-1} \frac{\partial}{\partial t^{2,k-1}} \right)$$

$$+ \sum_{k \geq 1} 2\, t^{1,k} \frac{\partial}{\partial t^{2,k-1}} + \frac{1}{\epsilon^2} (t^{1,0})^2,$$

$$L_{-1} = \sum_{k \geq 1} t^{\alpha,k} \frac{\partial}{\partial t^{\alpha,k-1}} + \frac{1}{\epsilon^2} t^{1,0} t^{2,0}$$

Here the integer coefficients $\alpha_m(k)$ are defined by

$$\alpha_m(0) = m!, \quad \alpha_m(k) = \frac{(m+k)!}{(k-1)!} \sum_{j=k}^{m+k} \frac{1}{j}, \; k > 0.$$

Theorem For any semisimple Frobenius manifold there exists unique integrable hierarchy of the topological type s.t. the additional flows

$$\frac{\partial \tau}{\partial s_m} = L_m \tau, \quad m \geq -1$$

act by (infinitesimal) symmetries

$$\left[\frac{\partial}{\partial t^{\alpha,p}}, \frac{\partial}{\partial s_m}\right] = 0.$$

Here

$$L_m = \sum \epsilon^2 a_m^{\alpha,p;\beta,q} \frac{\partial^2}{\partial t^{\alpha,p} \partial t^{\beta,q}} + b_{m\,\alpha,p}^{\beta,q} t^{\alpha,p} \frac{\partial}{\partial t^{\beta,q}}$$
$$+ \frac{1}{\epsilon^2} c^m_{\alpha,p;\beta,q} t^{\alpha,p} t^{\beta,q} + c_0 \delta_{m,0}$$

$$[L_m, L_n] = (m-n) L_{m+n}$$

are the above Virasoro operators given in terms of the (part of) the monodromy data (μ, R) of the Frobenius manifold.

Any *regular* solution to the hierarchy is obtained from the vacuum solution τ^{vac}

$$\frac{\partial \tau^{\text{vac}}}{\partial s_m} = 0, \text{ i.e. } L_m \tau^{\text{vac}} = 0, \quad m \geq -1$$

by a shift

$$\tau(\mathbf{t};\epsilon) = \tau^{\mathsf{vac}}(\mathbf{t} - \mathbf{t_0}(\epsilon);\epsilon), \quad \mathbf{t_0}(\epsilon) = \left(t_0^{\alpha,p}(\epsilon)\right)$$

Motivation: all known relations for the topological correlators

$$\langle\langle \tau_{p_1}(\phi_{\alpha_1}) \dots \tau_{p_k}(\phi_{\alpha_k}) \rangle\rangle := \epsilon^k \frac{\partial^k \log \tau^{\mathsf{top}}(\mathbf{t};\epsilon)}{\partial t^{\alpha_1,p_1} \dots \partial t^{\alpha_k,p_k}}$$

reproduced for the topological solution specified by the shift

$$\tau^{\mathsf{top}} = \tau^{\mathsf{vac}}|_{t^{1,1} \mapsto t^{1,1} - 1}$$

(checked for low genera)

E.g., WDVV,

TRR for $g = 0$

$$\langle\langle \tau_p(\phi_\alpha)\tau_q(\phi_\beta)\tau_r(\phi_\gamma)\rangle\rangle_0$$
$$= \langle\langle \tau_{p-1}(\phi_\alpha)\tau_0(\phi_\nu)\rangle\rangle_0 \eta^{\nu\mu} \langle\langle \tau_0(\phi_\mu)\tau_q(\phi_\beta)\tau_r(\phi_\gamma)\rangle\rangle_0,$$

TRR for $g = 1$

$$\langle\langle \tau_p(\phi_\alpha)\rangle\rangle_1 = \langle\langle \tau_{p-1}(\phi_\alpha)\tau_0(\phi_\nu)\rangle\rangle_0 \eta^{\nu\mu} \langle\langle \tau_0(\phi_\mu)\rangle\rangle_1 + \frac{1}{24}\eta^{\nu\mu}\langle\langle \tau_{p-1}(\phi_\alpha)\tau_0(\phi_\nu)\tau_0(\phi_\mu)\rangle\rangle_0,$$

Getzler's defining relation for $g = 1$ etc.

Proof based on

Lemma For any semisimple Frobenius manifold M^n there exists a unique solution

$$\Delta\mathcal{F} = \sum_{g\geq 1} \epsilon^{2g-2}\mathcal{F}_g(v; v_x, \ldots, v^{(3g-2)})$$

to the system of Virasoro constraints

$$L_m\left(\tau_0^{\text{vac}}(\mathbf{t})e^{\Delta\mathcal{F}}\right) = 0, \quad m \geq -1$$

(cf. Virasoro conjecture by Eguchi, Hori, Jinzenji and Xiong and S.Katz).

E.g., for $n = 1$

$$\sum_r \frac{\partial \Delta \mathcal{F}}{\partial v^{(r)}} \partial_x^r \frac{1}{v-\lambda} + \sum_{r\geq 1} \frac{\partial \Delta \mathcal{F}}{\partial v^{(r)}} \sum_{k=1}^{r} \binom{r}{k} \partial_x^{k-1} \frac{1}{\sqrt{v-\lambda}} \partial_x^{r-k+1} \frac{1}{\sqrt{v-\lambda}}$$

$$= \frac{\epsilon^2}{2} \sum \left[\frac{\partial^2 \Delta \mathcal{F}}{\partial v^{(k)} \partial v^{(l)}} + \frac{\partial \Delta \mathcal{F}}{\partial v^{(k)}} \frac{\partial \Delta \mathcal{F}}{\partial v^{(l)}} \right] \partial_x^{k+1} \frac{1}{\sqrt{v-\lambda}} \partial_x^{l+1} \frac{1}{\sqrt{v-\lambda}}$$

$$- \frac{\epsilon^2}{16} \sum \frac{\partial \Delta \mathcal{F}}{\partial v^{(k)}} \partial_x^{k+2} \frac{1}{(v-\lambda)^2} - \frac{1}{16(v-\lambda)^2}$$

The substitution

$$v_\alpha \mapsto w_\alpha = v_\alpha + \epsilon^2 \frac{\partial^2 \Delta \mathcal{F}}{\partial t^{1,0} \partial t^{\alpha,0}}$$

transforms the Principal Hierarchy associated with M^n to the hierarchy of the topological type associated with M^n.

Theorem Let X be a smooth projective variety with $H^{\text{odd}}(X) = 0$ s.t.

- $QH^*(X)$ is semisimple
- Virasoro constraints hold true for the total GW potential

$$\mathcal{F}^X(\mathbf{t};\epsilon) = \sum_{g\geq 0} \epsilon^{2g-2}\mathcal{F}_g^X(\mathbf{t})$$

(e.g., $X = \mathbf{P}^d$, due to Givental)

Then

$$\tau = \exp \mathcal{F}^X$$

is the tau-function of the topological solution to the hierarchy of the topological type associated with the Frobenius manifold

$$M^n = QH^*(X), \quad n = \text{dim} H^*(X)$$

Application to Hermitean matrix integrals

$$Z_N(\lambda;\epsilon) = \frac{1}{\mathrm{Vol}(U_N)} \int_{N\times N} e^{-\frac{1}{\epsilon}\mathrm{Tr}\, V(A)}\, dA$$

$$V(A) = \frac{1}{2}A^2 - \sum_{k\geq 3} \lambda_k A^k$$

as function of $N = x/\epsilon$, λ
is a tau-function of extended Toda lattice

$$Z = \tau_{\mathrm{Toda}}^{\mathrm{vac}}(\mathbf{s}, \mathbf{t}; \epsilon)$$

for

$$\begin{aligned} &s_0 \equiv t^{1,0} = x, \quad s_p \equiv t^{1,p} = 0, \qquad p > 0 \\ &t_0 \equiv t^{2,0} = 0, \qquad t_1 \equiv t^{2,1} = -1 \\ &t_p \equiv t^{2,p} = (p+1)!\lambda_{p+1}, \qquad p \geq 2 \end{aligned}$$

Remark

$$\tau_{\mathrm{Toda}}^{\mathrm{vac}}(\mathbf{s}, \mathbf{t}; \epsilon) = \tau_{\mathrm{NLS}}^{\mathrm{vac}}(\mathbf{t}, \mathbf{s}; \epsilon)$$

Large N $\sim$ small ϵ expansion of

$$\tau_{\text{NLS}}^{\text{top}}(x, \mathbf{t}; \epsilon) = Z_N(\lambda; \epsilon)$$

$$x = \frac{N}{\epsilon}, \quad t_k = (k+1)!\lambda_{k+1}$$

has the form

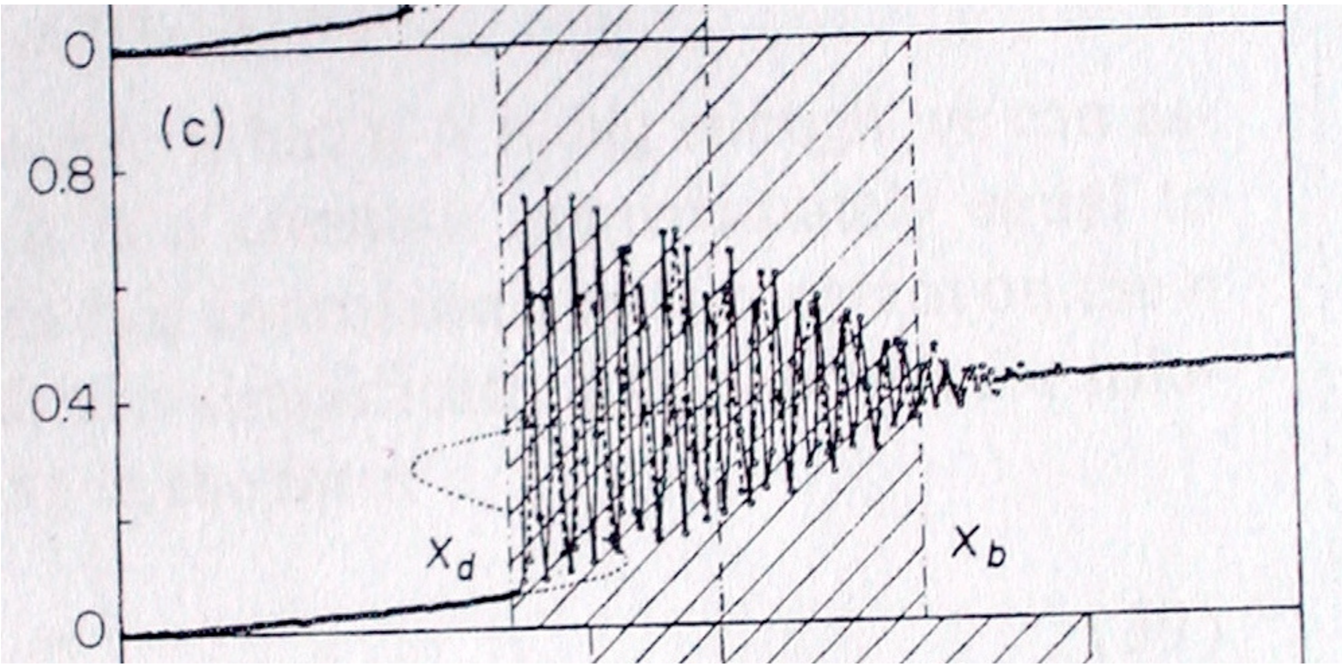
$$\log \tau_{\text{NLS}}^{\text{top}} = \frac{x^2}{2\epsilon^2}\left(\log x - \frac{3}{2}\right) - \frac{1}{12}\log x$$

$$+ \sum_{g\geq 2} \left(\frac{\epsilon}{x}\right)^{2g-2} \frac{B_{2g}}{2g(2g-2)} + \sum_{g\geq 0} \epsilon^{2g-2} F_g(x; \lambda_3, \lambda_4, \dots)$$

$F_g(x, \lambda)$ = generating function of numbers of fat graphs on genus g Riemann surfaces

Corresponds to the one-cut asymptotic distribution of the eigenvalues of the large size Hermitean random matrix A

Multicut case: G gaps in the asymptotic distribution of eigenvalues of random matrices $\Rightarrow$ singular behaviour of the correlation functions (terms $\sim e^{\frac{iat}{\epsilon}}$ arise)



(from Jurkiewicz, Phys. Lett. B, 1991)

Smoothed correlation functions: average out the singular terms

Question: Which integrable PDEs describe the large N expansion of *smoothed* correlation functions?

Claim (B.D., T.Grava, in progress): The full large N expansion of the smoothed correlation functions is given via the topological tau-function associated with the Frobenius structure M^n, $n = 2G + 2$ on the Hurwitz space of hyperelliptic curves

$$\mu^2 = \prod_{i=1}^{2G+2} (\lambda - u_i)$$

Recall the general construction: Frobenius structure on the Hurwitz space M^n = moduli of branched coverings

$$\lambda : \Sigma_G \to \mathbf{P}^1$$

fixed degree, genus G, ramification type at infinity, basis of a- and b-cycles (n = number of branch points $\lambda = u_i$ for generic covering).

Must choose a primary differential dp (say, holomorphic differential with constant a-periods)

Then, for any two vector fields ∂_1, ∂_2 on M^n the inner product

$$\langle \partial_1, \partial_2 \rangle = \sum_{i=1}^{n} \mathrm{res}_{\lambda=u_i} \frac{\partial_1(\lambda dp)\partial_2(\lambda dp)}{d\lambda}$$

for any three vector fields ∂_1, ∂_2, ∂_3 on M^n

$$\langle \partial_1 \cdot \partial_2, \partial_3 \rangle = -\sum_{i=1}^{n} \mathrm{res}_{\lambda=u_i} \frac{\partial_1(\lambda dp)\partial_2(\lambda dp)\partial_3(\lambda dp)}{d\lambda\, dp}$$

Example $G = 1$ (two-cut case). Here $n = 4$. Flat coordinates on the Hurwitz space of elliptic double coverings with 4 branch points are u, v, w, τ. Can be described by the superpotential (= symbol of Lax operator)

$$\lambda(p) = v + u \left(\log \frac{\theta_1(p - w|\tau)}{\theta_1(p + w|\tau)} \right)'$$

The Frobenius structure given by

$$F = \frac{i}{4\pi} \tau\, v^2 - 2\, u\, v\, w + u^2 \log \left[\frac{1}{\pi\, u} \frac{\theta_1(2w|\tau)}{\theta_1'(0|\tau)} \right]$$

Recall

$$\log \left[\frac{\theta_1(x|\tau)}{\pi\, \theta_1'(0|\tau)} \right] = \log \sin \pi\, x + 4 \sum_{m=1}^{\infty} \frac{q^{2m}}{1 - q^{2m}} \frac{\sin^2 \pi\, m\, x}{m}$$

$$q = e^{i\, \pi\, \tau}$$

Corresponding integrable hierarchy of the topological type for the functions u, v, w, τ, four infinite chains of times
$t^{u,p}$, $t^{v,p}$, $t^{w,p}$, $t^{\tau,p}$. Then

$$Z \sim \tau^{\text{vac}}$$

with $t^{w,1} \mapsto t^{w,1}-1$, $t^{w,0} = 0$, $t^{w,k} = (k+1)!\lambda_{k+1}$

$$t^{u,0} = x$$

other couplings = 0.

The *solution* is given in implicit form $\operatorname{grad}\Phi = 0$

$$\Phi = x\,w - u\,v + u^2 P_1(2w|\tau)$$

$$+3\lambda_3 u\,\left[v^2 - 2\,u\,v\,P_1(2w|\tau) + u^2\left(P_1^2(2w|\tau) - P_2(2w|\tau) + 4\pi\,i(\log\eta(\tau))'\right)\right]$$

$$+2\lambda_4 u\,\left[2v^3 - 6u\,v^2 P_1(2w|\tau) + 6u^2 v\,\left(P_1^2(2w|\tau) - P_2(2w|\tau) + 4\pi\,i(\log\eta(\tau))'\right)\right.$$

$$\left.-u^3\left[P_3(2w|\tau) + 2P_1(2w|\tau)\left(P_1(2w|\tau)^2 - 3P_2(2w|\tau) + 12\pi\,i\,(\log\eta(\tau))'\right)\right]\right]$$

$$+\ldots$$

where

$$P_k(x|\tau) := \partial_x^k \log\theta_1(x|\tau), \quad k = 1, 2, 3$$

Canonical coordinates (branch points)

$$u_i = v - 2u\,[\log\theta_i(v|\tau)]', \quad i = 1, \ldots, 4.$$

ϵ^2-correction

$$\mathcal{F}_1 = -\frac{1}{6}\log u - \log\eta(2\tau) + \frac{1}{24}\sum_{i=1}^{4}\log u_i'$$