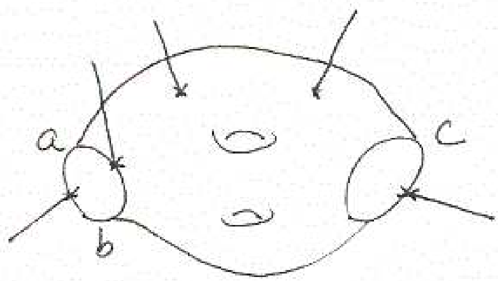
§ D-branes & Mirror Symmetry

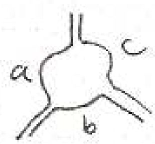
Overview

A D-brane is a boundary condition of string. One can consider correlation function on Riemann surfaces with boundary for a set of D-branes

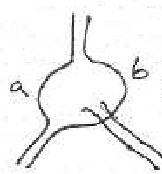


In string theory, D-branes introduces open strings

Open strings interact among themselves



, and also with closed strings



etc

(correlation functions compute the interaction)

Therefore D-branes themselves can be regarded as dynamical degrees of freedom.

- In superstring theory, we <u>need</u> to consider D-branes preserving diagonal $\mathcal{N}=1$ SUSY of $(1,1)$ SUSY. In $(2,2)$ SUSY theories, especially interesting ones are those preserving a diagonal $\mathcal{N}=2$ SUSY.

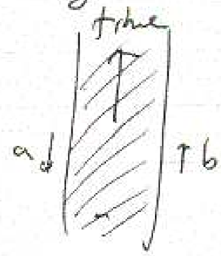
2 kinds of "diagonal"

$Q_A, \overline{Q}_A \Rightarrow$ A-branes — e.g. D-branes wrapped on Lagrangian submfd and supporting flat U(N) bdle.

$Q_B, \overline{Q}_B \Rightarrow$ B-branes — e.g. D-branes wrapped on Complex submfd and supporting holomorphic vector bundle.

- Consider the open string stretched between D-branes "a" & "b": $a \longrightarrow b$.

Open string states are states of the Quantum mechanics ~~formulated~~ obtained by formulating the theory on the strip:



state $\in \mathcal{H}_{a,b}$

If a & b preserves the same SUSY $\left(\begin{array}{c}\text{e.g. both} \\ Q_A\end{array}\right)$

$\mathcal{H}_{a,b}$ has operators $Q, Q^\dagger, H$ (& F)

obeying
$$\{Q, Q^\dagger\} = H$$
$$Q^2 = 0$$
$$[F, Q] = Q.$$

In particular, $(\mathcal{H}_{a,b}, Q, F)$ form a complex & SUSY ground states are in one to one correspondence with Q-cohomology classes.

e.g.

L_a, L_b : Lagrangians $\Rightarrow \mathcal{H}^{\cdot}_{SUSY} \cong HF^{\cdot}(L_a, L_b)$

Floer homology

E_a, E_b : holomorphic bundles $\Rightarrow \mathcal{H}^{\cdot}_{SUSY} \cong H^{0,\cdot}(X, E_a^* \otimes E_b)$

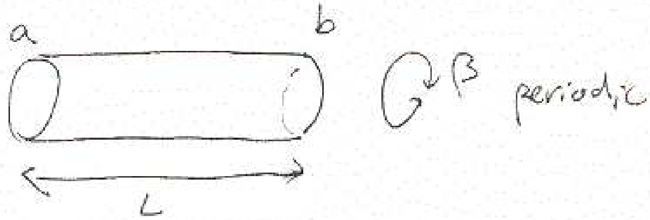
(or $\mathrm{Ext}^{\cdot}(E_a, E_b)$)

In string theory on $X \times \mathbb{R}^D$, SUSY ground states for pairs of D-branes in X corresponds to massless particles in $\mathbb{R}^D$

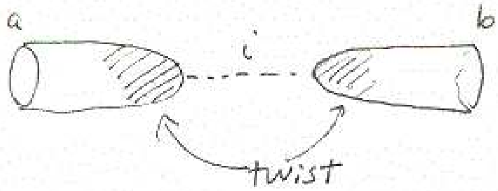
- Witten index of the SUSY QM $(\mathcal{H}_{a,b}, Q, F)$ is represented by periodic cylinder partition function.

By topological invariance, or by taking the long cylinder limit, it is expressed as bilinear form of disk (or hemisphere) diagrams:

$$\mathrm{Tr}_{\mathcal{H}_{ab}} (-1)^F e^{-\beta H} =$$

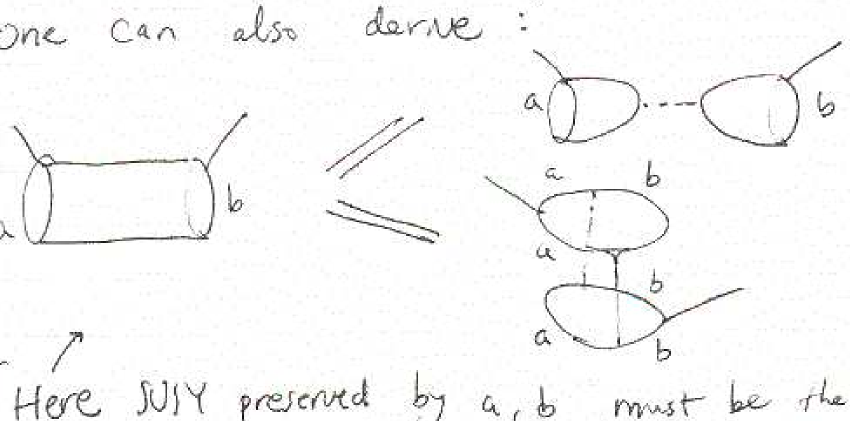


$$= \sum_{i:\ \text{SUSY ground states}}$$



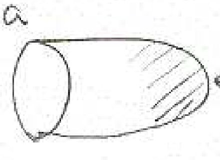
(Here, the SUSY preserved by a, b doesn't have to be correlated by the type of twist.)

From two point open string correlator on cylinder one can also derive:

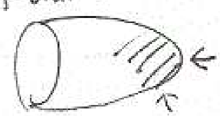


Called "Cardy Condition" in Open TFT.

(Here SUSY preserved by a, b must be the same as the type of twist)

• $\Pi_i^a =$  $\leftarrow \phi_i$ ((twisted) chiral ring element)

introduces a <u>linear structure</u> ~~on~~ the set of D-branes. In superstring theory ~~for~~ each ϕ_i corresponds to a spacetime gauge field C_i (RR field), & Π_i^a determines the <u>charge</u> of the D-brane a with respect to C_i. (RR-charge). This is also important to study the stability of branes. Π_i^a have special dependence on parameters of the theory

For A-brane  ← chiral ring

B-twist

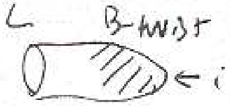
• indep of twisted chiral par.

• depends on chiral par. as

$$\nabla \Pi_i^a = 0.$$

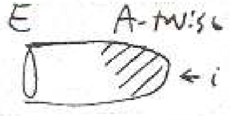
Here $\nabla_i = \partial_i + C_{ij}^k$ " is a flat connection of the bundle of RR ground states. Thus one can discuss about the <u>monodromy</u> of branes over ~~the~~ contours in the parameter space.

For $L \subset (X, \omega)$ Lagrangian (A-brane)



$$\leftarrow i \;=\; \int_L \Omega \cdot \phi_i \qquad \phi_i \in H^{0,\bullet}(X, \wedge^\bullet T_X)$$

For $E \to X$ holomorphic bdle (B-brane)



$$\leftarrow i \;=\; \int_X ch(E^*) \sqrt{Td(X)}\, e^{i\omega + B}\, \omega_i + \cdots \qquad \omega_i \in H^i_{DR}(X)$$

 $=$  Riemann's bilinear identity

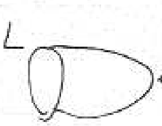
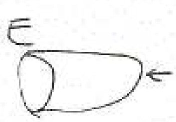
$$A: \quad \#(L_a \cap L_b) = \eta^{ij} \int_{L_a} \Omega \cdot \phi_i \int_{L_b} \Omega \cdot \phi_j$$

- Mirror symmetry. $Q_A \leftrightarrow Q_B$:

A-branes $\longleftrightarrow$ B-branes

Lagrangian $\longleftrightarrow$ holomorphic bdle

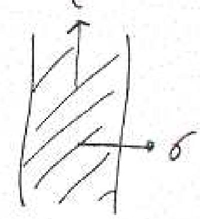
$HF^\bullet(L_a, L_b) \longleftrightarrow Ext^\bullet(E_a, E_b)$

 $\leftarrow i \longleftrightarrow$  $\leftarrow i$

Picard-Lefshetz monodromy ; "Some monodromy" ;

$\mathcal{M}_C$; $\mathcal{M}_K$

Boundary conditions

$\Sigma =$ 

Scalar field theory

$$S = \int_\Sigma \left(\frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\partial_\sigma\phi)^2 \right) dt\, d\sigma \qquad \phi: \mathbb{R}\text{-valued.}$$

Variation

$$\delta S = \int_\Sigma \delta\phi \left(-\partial_t^2\phi + \partial_\sigma^2\phi \right) dt\, d\sigma - \int_{\partial\Sigma} \delta\phi\, \partial_\sigma\phi\, dt$$

For EOM to remain $(-\partial_t^2 + \partial_\sigma^2)\phi = 0$, we need

$$\delta\phi\, \partial_\sigma\phi = 0 \qquad \text{at } \partial\Sigma.$$

Two solutions: ① $\partial_\sigma\phi = 0$ at $\partial\Sigma$: Neumann b.c.

② $\delta\phi = 0$ at $\partial\Sigma$: Dirichlet b.c.

Consider the theory as sigma model with target $= \mathbb{R}$:

① open string end point can be anywhere in $\mathbb{R}$

$$\dim \mathbb{R} = 1 \rightsquigarrow D1\text{-brane}$$

② open string end point must be at one point $\{p\}$ in $\mathbb{R}$

$$\dim\{p\} = 0 \rightsquigarrow D0\text{-brane}$$

general sigma-model :

$$\phi : \Sigma \to (M, g) \quad \text{Riemannian mfd.}$$

For $\gamma \subset M$ submfd, one can consider the b.c :

- $\phi(\partial\Sigma) \subset \gamma$
- $\partial_n \phi(p) \in N_{\phi(p)}\gamma \subset T_{\phi(p)} M \qquad p \in \partial\Sigma.$

This is called <u>the D-brane wrapped on γ</u>

$$\dim \gamma = p \quad \Rightarrow \quad Dp\text{-brane}.$$

<u>Chan - Paton factor</u>

One can consider $U(1)$ gauge field A on γ under which the string end point is electrically charged : Path-integral measure include a factor

$$e^{-i\int_{\partial\Sigma} \phi^* A} = e^{-i\int_{\partial\Sigma} A_I(\phi)\dot{\phi}^I dt}$$

One can also consider $U(n)$ vector bundle E with connection A

$$\to \quad P e^{-\int_{\partial\Sigma} \phi^* A}$$

These are called <u>Chan-Paton factor</u>

(γ, A) : <u>D-brane wrapped on γ and supporting (E, A)</u>

<u>$\mathcal{N}=1$ SUSY</u>

SUSY σ-model with target $= \mathbb{R}$

ϕ : $\mathbb{R}$-scalar , $\psi_\pm$: $\mathbb{R}$(Majorana)-scalar $\psi_\pm^* = \psi_\pm$

$$S = \int_\Sigma \left\{ \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\partial_\sigma\phi)^2 + \frac{i}{2}\psi_-(\partial_t + \partial_\sigma)\psi_- + \frac{i}{2}\psi_+(\partial_t - \partial_\sigma)\psi_+ \right\} dt\, d\sigma$$

$$\delta S = \int_\Sigma (\text{bulk}) + \int_{\partial\Sigma} \Big[\underbrace{-\delta\phi\, \partial_\sigma\phi}_{\overset{\parallel}{0}} + \underbrace{i(\psi_-\delta\psi_- - \psi_+\delta\psi_+)}_{\overset{\parallel}{0}} \Big] dt$$

$\Rightarrow \psi_+ = \pm\psi_-$ at $\partial\Sigma$.

If $\partial\Sigma = \phi$,

S has (1,1) SUSY $\begin{cases} \delta\phi = i\epsilon_-\psi_+ + i\epsilon_+\psi_- \\ \delta\psi_\pm = -\epsilon_\mp(\partial_t \pm \partial_\sigma)\phi \end{cases}$

We need b.c. to preserve diagonal $\mathcal{N}=1$, $\epsilon_+ = \epsilon_- = \epsilon$

$$\begin{cases} \delta\phi = i\epsilon(\psi_+ + \psi_-) \\ \delta(\psi_+ + \psi_-) = -2\epsilon\,\partial_t\phi \ , \ \delta(\psi_+ - \psi_-) = -2\epsilon\,\partial_\sigma\phi . \end{cases}$$

$\Rightarrow$ ① $\partial_\sigma\phi = 0$, $\psi_+ = \psi_-$

② $\delta\phi = 0$, $\psi_+ = -\psi_-$

$\phi : \Sigma \to M$ NLSM $\qquad \gamma \subset M$

- $\phi(\partial\Sigma) \subset \gamma$, $\partial_n \phi \in N_\phi \gamma$
- $\psi_+ + \psi_- \in T_\phi \gamma$, $\psi_+ - \psi_- \in N_\phi \gamma$.

— This is the D-brane wrapped on γ in SUSY σ-model

$\mathcal{N} = (2,2)$: $\epsilon_+, \bar{\epsilon}_+, \epsilon_-, \bar{\epsilon}_-$

$\cup$

diagonal $\mathcal{N} = 2$ $\left\langle \begin{array}{l} \mathcal{N} = 2_A : \quad \epsilon_+ = \bar{\epsilon}_- \\ \mathcal{N} = 2_B : \quad \epsilon_+ = -\epsilon_- \end{array} \right.$

NLSM on a Kähler mfd $X \supset \gamma$

Recall $\delta\phi^i = \epsilon_+ \psi_-^i - \epsilon_- \psi_+^i$

B-type $\quad \delta_B \phi^i = \epsilon_+ (\psi_-^i + \psi_+^i) \in T\gamma$

$\epsilon_+ \in \mathbb{C}$: one can multiply $i = \sqrt{-1}$.

$\therefore$ $T\gamma$ is invariant under $J : TX \to TX$.

i.e. $\gamma \subset X$ is a complex submfd

<u>A-type</u> $\quad \delta_A \phi^i = \epsilon_+ \psi_-^i - \bar{\epsilon}_+ \psi_+^i \qquad \epsilon_+ = \epsilon_1 + i\epsilon_2$

$$= \underbrace{\epsilon_1(\psi_-^i - \psi_+^i)}_{\substack{\parallel \\ i \cdot \underbrace{i\epsilon_1(\psi_+^i - \psi_-^i)}}} + i\epsilon_2(\psi_-^i + \psi_+^i) \quad \in T\gamma$$

$$\mathcal{N}=1 \Rightarrow \left.\begin{array}{l} i\epsilon(\psi_+ + \psi_-) \in T\gamma \\ i\epsilon(\psi_- - \psi_+) \in N\gamma \end{array}\right\}$$

$J: TX \to TX \quad$ maps $\quad N\gamma \to T\gamma$
$\qquad\qquad\qquad\qquad\quad T\gamma \to N\gamma$

$\overset{*}{\Leftrightarrow} \quad \gamma \subset (X, \omega)$ Lagrange submfd.

*) Lagrangian $\overset{\text{def}}{\equiv} \quad \omega|_\gamma = 0 \, ; \ \frac{1}{2}$ dim of X.

$\omega(v, w) = g(v, Jw)$

- $J: N\gamma \rightleftarrows T\gamma \quad \Rightarrow \quad \omega(T_1, T_2) = g(T_1, JT_2) = 0$
  & $\frac{1}{2}$ dimensional

- Lagrangian $\Rightarrow \quad g(T_1, JT_2) = \omega(T_1, T_2) = 0$
  $\therefore J(T\gamma) \subset N\gamma \overset{\frac{1}{2}\dim}{\rightsquigarrow} J(T\gamma) = N\gamma$
  $\overset{J^2=-1}{\rightsquigarrow} \quad J(N\gamma) = T\gamma.$

more generally, D-brane wrapped on Y & supporting A is

<u>B-brane</u> iff $\quad Y \subset X$ complex submfd

A : holomorphic $\Big($ i.e. $F_A^{0,2} = 0$

so that $\bar{\partial}_A^2 = 0$ & $\bar{\partial}_A$ determines a holomorphic structure $\Big)$

<u>A-brane</u> iff
- $(TY)^\circ = \{ v \in TX \mid \omega(v, TY) = 0 \} \subset TY$ coisotropic
- $F_A = 0$ on $(TY)^\circ \times TY$
- $\omega + F_A \omega^{-1} F_A = 0$ on $TY/(TY)^\circ$

$\supset$ $\quad Y \subset X$ Lagrangian

A : flat $\qquad (F_A = 0)$.

(A flat $\Leftrightarrow$ $Y \subset X$ Lagrangian
under this condition)

LG model

$$S = S_{NLSM} + \underbrace{\int_\Sigma \left(-|W'|^2 - W'' \psi_+ \psi_- - \overline{W'' \psi_+ \psi_-} \right) dt\, d\sigma}_{\Downarrow \text{ extra condition}}$$

(γ, A) is

A-brane iff · ~~same~~ previous condition

+ $\underline{\mathrm{Im}\, W = \text{locally constant on } \gamma}$

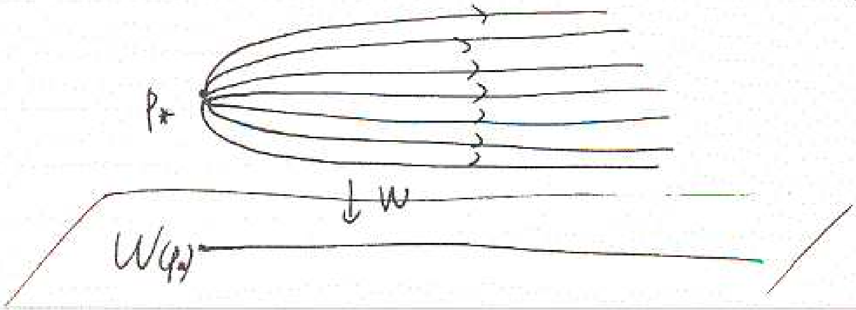
B-brane iff · previous condition

+ $\underline{W = \text{locally constant on } \gamma}$

- Example of A-branes in LG

$p_* \in \mathrm{Crit}(W)$, nondegenerate $\left(\det(\partial_i \partial_j W(p)) \neq 0 \right)$

$$L_{p_*} = \begin{bmatrix} \text{Stable mfd of } \mathrm{Re}(W) \\ \text{at } p_* \end{bmatrix} = \bigcup_{\substack{\phi: \mathbb{R} \to X \text{ grad } \mathrm{Re}(W) \text{ flow} \\ \phi(-\infty) = p_*}} \phi(\mathbb{R})$$



Claim L_{p_*} is a Lagrangian submfd on which $\mathrm{Im}\, W = \text{const.}$

$\because$ $v = \mathrm{grad}(W + \overline{W})$ $\quad v^i = g^{i\bar{j}} \partial_{\bar{j}} \overline{W}, \quad \bar{v}^{\bar{j}} = g^{i\bar{j}} \partial_i W$

$$i_v \omega = \frac{i}{2} g_{i\bar{j}} \left(v^i d\bar{z}^{\bar{j}} - dz^i v^{\bar{j}} \right)$$

$$= \frac{i}{2} \left(\partial_{\bar{j}} \overline{W}\, d\bar{z}^{\bar{j}} - dz^i\, \partial_i W \right) = d\, \mathrm{Im}\, W$$

- $\langle d\, \mathrm{Im}\, W, v \rangle = \omega(v, v) = 0$ $\quad$ $\mathrm{Im}\, W$ is constant along the gradient flow.

- $$\mathcal{L}_v \omega = d \underbrace{i_v \omega}_{\substack{\parallel \\ d\, \mathrm{Im} W}} + i_v \underbrace{d\omega}_{\substack{\parallel \\ 0}} = 0$$

The grad. flow $f_t : X \to X$ preserves ω.

$\forall q \in L_p, \quad \forall v_1, v_2 \in T_q L_p$

$$\omega_q(v_1, v_2) = \omega_{f_t(q)}(f_t v_1, f_t v_2) \quad \forall t$$

$$= \lim_{t \to -\infty} = \omega_p(0, 0) = 0.$$

isotropic.

- $$W = W(p) + \sum_{i=1}^{n} z_i^2 + \cdots \Rightarrow \begin{array}{l} \dot{z}_i = \bar{z}_i \\ \dot{\bar{z}}_i = z_i \end{array}$$

$$\Rightarrow z_i = z_i(p) + x_i e^t \quad \forall x_i \in \mathbb{R} \Rightarrow \frac{1}{2} \dim$$

- More general B-branes in LG

For Neumann b.c. on X,

$$\delta_B S_{LG} = \mathrm{Re}\int_{\partial\Sigma} dt\, d\theta\, \bar{\epsilon} W . \qquad \text{"Warner problem".}$$

Suppose $\quad W = f \cdot g \qquad$ (f, g holomorphic functions on X)

Let us introduce a fermionic superfield on the boundary $\Gamma(t, \theta, \bar{\theta})$ s.t. $\quad \bar{D}\Gamma = g(\Phi) \quad \left(\bar{D} = -\frac{\partial}{\partial\bar{\theta}} + i\theta\frac{\partial}{\partial t}\right)$

component: $\quad \Gamma = \eta + \theta G - \bar{\theta} g(\Phi) - i\theta\bar{\theta}\dot{\eta}$

If we add to S_{LG} the boundary term

$$S_{bdry} = \int_{\partial\Sigma} dt\, d\theta\, d\bar{\theta}\, \bar{\Gamma}\Gamma + \mathrm{Re}\int_{\partial\Sigma} dt\, d\theta\, \Gamma f(\Phi)$$

$$\delta_B S_{bdry} = -\mathrm{Re}\int_{\partial\Sigma} dt\, d\theta\, \bar{\epsilon}\bar{D}(\Gamma f(\Phi))$$
$$= -\mathrm{Re}\int_{\partial\Sigma} dt\, d\theta\, \bar{\epsilon}\, g(\Phi) f(\Phi)$$

So,
$$\delta_B(S_{LG} + S_{bdry}) = 0$$

in S_{bdy} η has kinetic term $i\bar{\eta}\dot{\eta}$

but G is an auxiliary field.

quantizing $\eta, \bar{\eta}$: $\eta^2 = \bar{\eta}^2 = 0$, $\{\eta, \bar{\eta}\} = 1$.

representation $\mathbb{C}^2 = \{|0\rangle, \bar{\eta}|0\rangle\}$ $\eta|0\rangle = 0$.

$$Q_B = Q_B^{bulk} + \underbrace{\eta f(\phi) + \bar{\eta} g(\phi)}$$

$$\hookrightarrow \begin{pmatrix} 0 & f(\phi) \\ g(\phi) & 0 \end{pmatrix} \text{ on } \mathbb{C}^2.$$

More generally, for $N \times N$ matrix valued holomorphic fcn $f(\phi)$, $g(\phi)$ s.t.

$$f(\phi) g(\phi) = g(\phi) f(\phi) = W(\phi)\, \mathrm{id}_N .$$

One can construct $\mathcal{N}=2_B$ SUSY boundary interaction.

$$Q_B = Q_B^{bulk} + \begin{pmatrix} 0 & f(\phi) \\ g(\phi) & 0 \end{pmatrix}$$