Field $\leftrightarrow$ State Correspondence

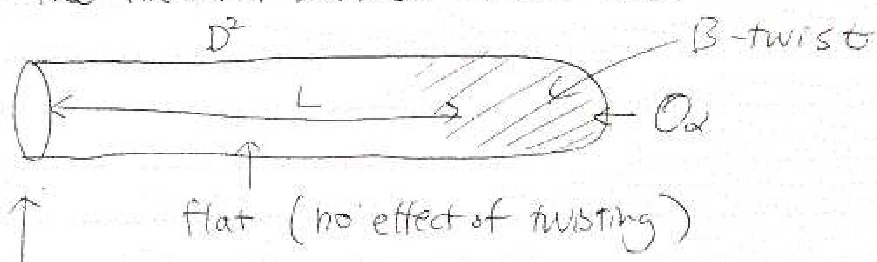
When a (2,2) theory is B-twistable,
(A- 〃)

the SUSY groundstates are in one to one correspondence with chiral ring elements
(twisted chiral ring)

i.e. Q_B - cohomology classes of fields
(Q_A - cohomology classes of fields).

Construction of the correspondence.

consider the Riemann surface like this:



a state $|\mathcal{O}_\alpha^{(L)}\rangle$ appears

i.e.
$$\Psi_{\alpha}^{(L)}(X_1) = \int_{X_1 = X|_{\partial D^2}} \mathcal{D}_{D^2} X \; e^{-S_{D^2}(X)} \, \mathcal{O}_\alpha(0)$$

Note 1. Fields are periodic along $S^1 = \partial D^2$ (called RR sector)

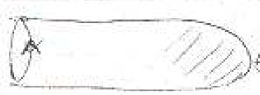
— the effect of twisting

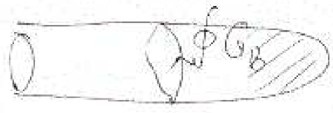
(before twist, fermions would have anti-periodic b.c. on $S^1 = \partial D^2$ (NS-NS sector))

2. $Q_B |\mathcal{O}_\alpha\rangle^{(L)} = 0$, $|\mathcal{O}_\alpha + [Q_B, \lambda]\rangle^{(L)} = |\mathcal{O}_\alpha\rangle^{(L)} + Q_B |\lambda\rangle^{(L)}$

Thus $\mathcal{O}_\alpha \mapsto |\mathcal{O}_\alpha\rangle^{(L)}$ defines a map $R_B \to H^*(Q_B)$

$Q_B = \oint_{S^1} G_B$ ← closed one form in B-twisted theory (supercurrent)

$Q_B|\mathcal{O}\rangle^{(L)} = \oint_{S^1} G_B$  $\mathcal{O}(0)$

$=$  $\oint G_B$ $\mathcal{O}(0)$

$=$  $\oint_0 G_B\, \mathcal{O}(0) = [Q_B, \mathcal{O}](0)$

$= |[Q_B, \mathcal{O}]\rangle^{(L)}$ //

3. By topological invariance $T_{\mu\nu} = \{Q_B, G_{\mu\nu}\}$
Q_B-cohomology class does not depend on the metric on D^2, in particular L.

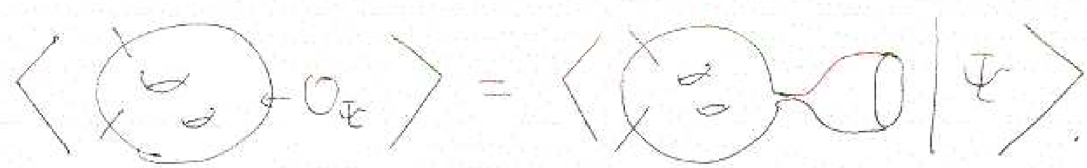
The limit $L \to \infty$ projects onto a SUSY ground state

$$|O_\alpha\rangle = \lim_{L\to\infty} |O_\alpha\rangle^{(L)} \in \mathcal{H}_{SUSY} \qquad \left(\lim_{L\to\infty} e^{-LH} = \begin{cases} 1 & \text{SUSY gnd} \\ 0 & E>0 \end{cases} \right)$$

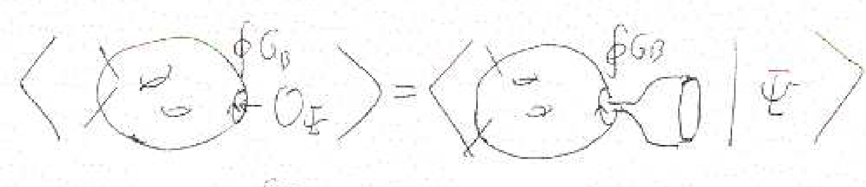
We obtain a map
$$R_B \to \mathcal{H}_{SUSY}$$
$$O_\alpha \mapsto |O_\alpha\rangle$$

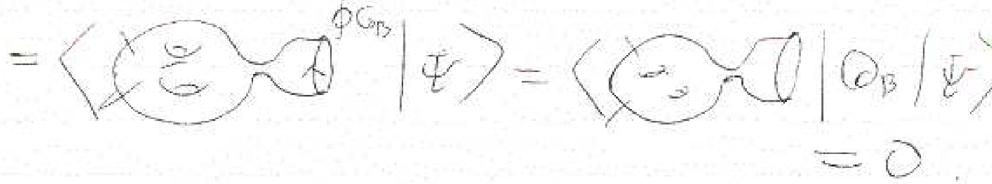
<u>The inverse</u>

Given a SUSY ground state Ψ, a field O_Ψ is obtained by



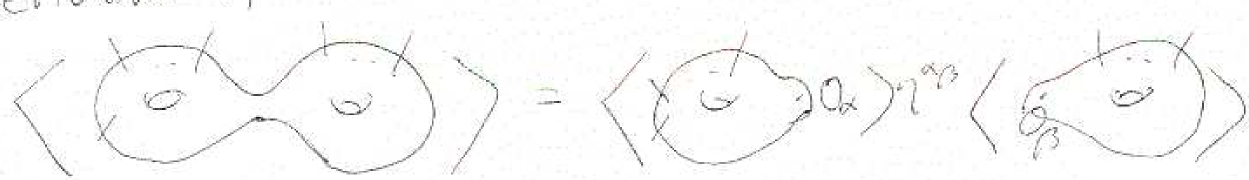
It is chiral,



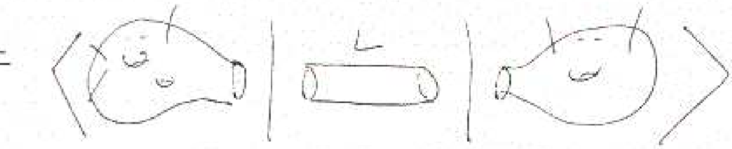


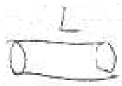
That it is the inverse of $O_\alpha \mapsto |O_\alpha\rangle$ is obvious.

Derivation of

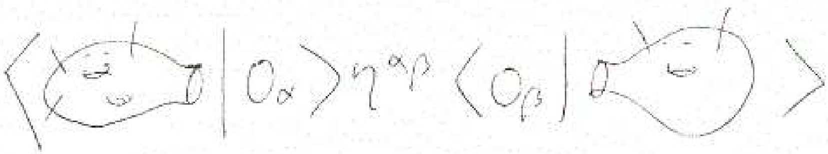


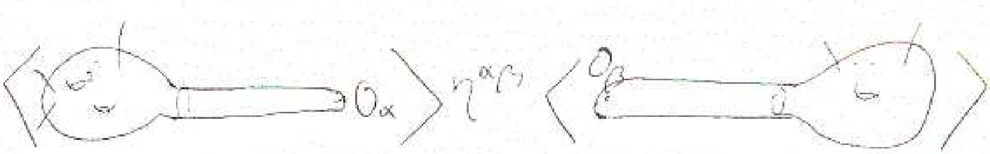
in the B-twisted model. $\eta^{\alpha\beta}$ inverse of $\eta_{\alpha\beta} = \langle O_\alpha \bigcirc O_\beta \rangle$

LHS = 

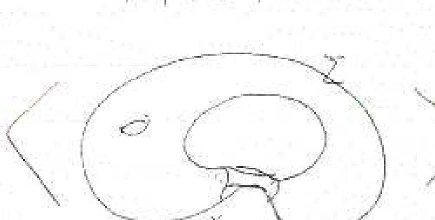
$\lim_{L\to\infty}$ 

= projection to the ground states $= |O_\alpha\rangle \eta^{\alpha\beta} \langle O_\beta|$

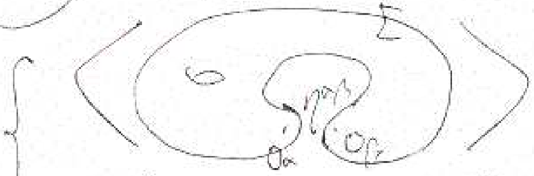
= 

= 

= RHS.

Also 

$= \int \mathcal{D}_{S^1}X_1\, \mathcal{D}_{S^1}X_2\; Z_\Sigma(X_1, X_2)\, \Psi_\alpha(X_2)\, \eta^{\alpha\beta}\, \Psi^*_\beta(X_1)$

= 

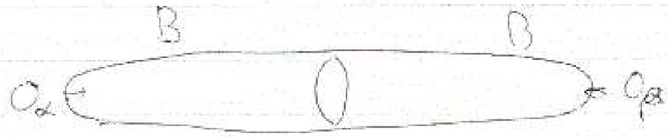
$= \mathrm{Tr}\big((-1)^F Z_\Sigma\big)$

or $\int \mathcal{D}_{S^1}X_1\, \mathcal{D}_{S^1}X_2\, (-1)^{|\alpha||\beta|}\, \Psi_\beta^*(X_1)\, Z_\Sigma(X_1, X_2)\, \Psi_\alpha(X_2)$

$|O_\alpha\rangle$ basis of $\mathcal{H}_{SUSY}$

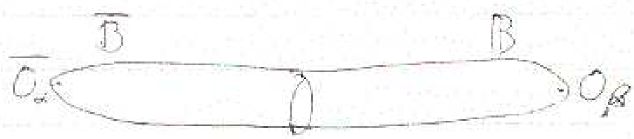
$\langle O_\beta|$ basis of $\check{\mathcal{H}}_{SUSY}$

Thm. $\underline{\eta_{\alpha\beta} = \langle O_\alpha | O_\beta \rangle \text{ is nondegenerate.}}$



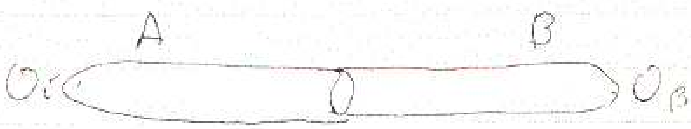
$= \eta_{\alpha\beta}$

topological metric



$= g_{\bar{\alpha}\beta}$

tt^* metric



$= ?$

(when both twistable)

~~Decoupling Theorem~~ Parameter dependence

Recall that SUSY Lagrangian can be written as

$$L = \int d^4\theta \; K(\Phi_i, \overline{\Phi}_i, \widetilde{\Phi}_p, \overline{\widetilde{\Phi}}_p ; \xi_A) + \left[\int d^2\theta \; W(\Phi_i ; t_a) + c.c.\right] + \left[\int d^2\widetilde{\theta} \; \widetilde{W}(\widetilde{\Phi}_p ; \widetilde{t}_a) + c.c.\right]$$

Φ_i : chiral superfield

$\widetilde{\Phi}_p$: twisted chiral superfield

$\xi_A, t_a, \widetilde{t}_a$ ... parameters

We call t_a ... chiral parameters

$\widetilde{t}_a$ ... twisted chiral parameters

decoupling thm

Correlation function of A-twisted model (when possible) depends holomorphically on twisted chiral parameters and are indep of chiral parameters.

Correlation function of B-twisted model (when possible) depend holomorphically on chiral parameters and are indep of twisted chiral parameters.

(physically ... no mixing of chiral & twisted chiral parts)

Let us show this (say, the latter part)

We want to show that correlators of B-twisted model are independent of $\bar{t}_a$, $\tilde{t}_\alpha$, $\bar{\tilde{t}}_\alpha$.

Namely
$$\left\langle \int d^2x \left(\int d^2\bar{\theta} \, \frac{\partial}{\partial \bar{t}_a} \overline{W}(\bar{\Phi}_i, \bar{t}_a)\right) \mathcal{O}_1 \mathcal{O}_2 \cdots \right\rangle_\Sigma = 0$$

$$\left\langle \int d^2x \left(\int d^2\tilde{\theta} \, \frac{\partial}{\partial \tilde{t}_\alpha} \widetilde{W}(\tilde{\Phi}_p, \tilde{t}_\alpha)\right) \mathcal{O}_1 \mathcal{O}_2 \cdots \right\rangle_\Sigma = 0$$

$$\left\langle \int d^2x \left(\int d^2\bar{\tilde{\theta}} \, \frac{\partial}{\partial \bar{\tilde{t}}_\alpha} \overline{\widetilde{W}}(\bar{\tilde{\Phi}}_p, \bar{\tilde{t}}_\alpha)\right) \mathcal{O}_1 \mathcal{O}_2 \cdots \right\rangle_\Sigma = 0$$

One can show that ① $= [\bar{Q}_+, [\bar{Q}_-, \partial_{\bar{a}} \overline{W}(\bar{\phi}_i, \bar{t}_a)]]$

② $= [\bar{Q}_+, [Q_+, \partial_\alpha \widetilde{W}(\phi_p, \tilde{t}_\alpha)]]$

③ $= [\bar{Q}_+, [Q_-, \partial_{\bar{\alpha}} \overline{\widetilde{W}}(\bar{\phi}_p, \bar{\tilde{t}}_\alpha)]]$

They are all of the form $[Q_B, \text{—}]$

up to total derivative.

Thus they all vanish.

proof of the expressions ~~from~~ ①, ② ③ :

Here I prove the basic one

(this should have been done when we discussed superfields sorry!)

$$\int d^2\theta\, W(\Phi) = [Q_-, [Q_+, W(\phi)]]$$

(Others follow by complex conjugation or $Q_- \leftrightarrow \overline{Q}_-$).

Recall that the chiral superfield $W(\Phi)$ or simply Φ can be written as

$$\Phi = \phi(y) + \theta^+\psi_+(y) + \theta^-\psi_-(y) + \theta^+\theta^- F(y)$$

$$y^\pm = x^\pm - i\theta^\pm\overline{\theta}^\pm.$$

wrt $(y, \theta^\pm, \overline{\theta}^\pm)$, $Q_\pm = \frac{\partial}{\partial\theta^\pm}$, $\overline{Q}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} - 2i\theta^\pm\frac{\partial}{\partial y^\pm}$

$$[Q_\pm, \Phi] = Q_\pm\Phi$$

$$\Rightarrow [Q_\pm, \phi(y)] - \theta^+[Q_\pm, \psi_+(y)] - \theta^-[Q_\pm, \psi_-(y)] + \theta^+\theta^-[Q_\pm, F(y)]$$

$$= \psi_\pm(y) \pm \theta^\mp F(y)$$

$$\psi_\pm = [Q_\pm, \phi] \quad , F = -[Q_+, \psi_-] = [Q_-, \psi_+]$$

$$= [Q_-, [Q_+, \phi]]$$

$\int d^2\theta\, \Phi = F = [Q_-, [Q_+, \phi]]$. Same for $\Phi \to W(\Phi)$.

Note also

$$[\bar{Q}_\pm, \Phi] = \bar{\mathcal{D}}_\pm \Phi$$

$$= -2i\theta^\pm \partial_\pm \phi(y) - 2i\theta^\pm \theta^\mp \partial_\pm \psi_\mp(y)$$

$$\Rightarrow [\bar{Q}_\pm, \phi\cancel{(y)}] = 0, \quad [\bar{Q}_\pm, \psi_\pm] = 2i\partial_\pm \phi$$

$$[\bar{Q}_\pm, \psi_\mp] = 0$$

$$[\bar{Q}_\pm, F\cancel{(y)}] = \mp 2i\partial_\pm \psi_\mp$$

- lowest component of a chiral superfield is chiral.
- $\psi_\pm = [Q_\pm, \phi]$, $F = [Q_-, [Q_+, \phi]]$) — descent eqn.

  ... One can reconstruct a chiral superfield by a chiral field.

— ~~descent eqn~~

Example

for a non-linear sigma model on a Kähler mfd

① chiral parameters are complex structure parameters

② twisted chiral parameters are "complexified Kähler class parameters"

Kähler class $[\omega] \in H^2(X,\mathbb{R})$ $\quad \omega = \frac{i}{2} g_{i\bar{j}}\, dz^i \wedge d\bar{z}^{\bar{j}}$

Complexified Kähler class $[\omega] - i[B] \in H^2(X,\mathbb{C})$

flat B-field $(dB=0)$

— This is not obvious by the patch-by-patch description

$$\mathcal{L} = \int d^4\theta \; K(\Phi^i, \bar{\Phi}^{\bar{i}})$$

① sounds natural since Cplx str enters into the relation of chiral fields

$$\Phi^i = f^i(\Phi'^i, t_a) \quad \text{between different patches}$$

② looks mysterious, but one can show this by looking at A-topological correlators.

§ Localization Principle

Consider a system of n-bosonic & m-fermionic variables $X^1, \dots, X^n, \psi^1, \dots, \psi^m$.

with supersymmetry

$$\delta\left(d^n X \, d^m\psi \, e^{-S(X,\psi)} \right) = 0 .$$

Suppose $\cancel{\delta\psi^i \neq 0}$ $(\delta\psi^1, \dots, \delta\psi^m) \neq (0, \dots, 0)$ at any (X, ψ).

<u>In some case</u>, one can find a change of variables

$$\tilde{X}^i = f^i(X, \psi) \qquad i=1 \cdots n$$

$$\tilde{\psi}^j = g^j(X, \psi) \qquad j=1 \cdots m$$

s.t.

$$\begin{cases} \delta \tilde{X}^i = 0 \\ \delta \tilde{\psi}^1 = \epsilon \\ \delta \tilde{\psi}^j = 0 \qquad j \neq 1 \end{cases}$$

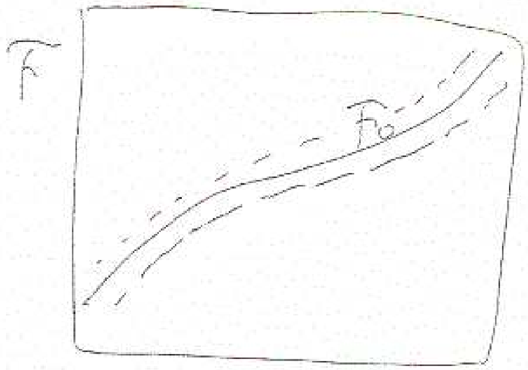
Then, by the invariance of the weighted measure

$$d^n X \, d^m\psi \, e^{-S(X,\psi)} = J(\tilde{X}^i, \tilde{\psi}^{j\neq 1}) \, d^n\tilde{X} \, d^m\tilde{\psi} \, e^{-\tilde{S}(\tilde{X}, \tilde{\psi}^{j\neq 1})}$$

$$\Rightarrow Z = \int d^n x \, d^m\psi \, e^{-S(X,\psi)} = \pm \int d\tilde{\psi}^1 \cdot 1 \cdot \int d^n\tilde{X} \, d^{m-1}\tilde{\psi} \, J \, e^{-\tilde{S}}$$

$$= 0 .$$

In general, $\delta\psi^i = 0 \ \forall i$ in some locus $\mathcal{F}_0$



$$Z = \int_{\mathcal{F}} e^{-S} = \int_{\mathcal{F}_0} e^{-S} + \underbrace{\int_{\mathcal{F} \setminus \mathcal{F}_0} e^{-S}}_{=0}$$

One can focus on the locus $\mathcal{F}_0$ where $\delta\psi^i = 0 \ \forall i$.

This is the <u>Localization Principle</u>

Under which case, $\exists$ such change of variable?

$$\boxed{\begin{array}{l} \text{We need } \ \delta\delta(\text{fields}) = 0. \\ \text{In fact } \ \delta^2 = 0 \ \text{ is sufficient.} \end{array}}$$

<u>proof</u> If $(\delta\psi^1, \cdots, \delta\psi^n) \neq (0, \cdots, 0)$ at any (X, ψ),

Choose the direction of $\delta(\cdots)$ as one fermionic coordinate & take the invariant ones as the rest of the coordinate. (Rectification).

Then
$$\begin{cases} \delta \hat{X}^i = 0 \\ \delta \hat{\psi}^1 = \epsilon\, u(\hat{X}, \hat{\psi}) & u(\hat{X}, \hat{\psi}) = 1 + \cdots \neq 0. \\ \delta \hat{\psi}^j = 0 \quad j \neq 1. \end{cases}$$

If $\delta^2 = 0$, then $u \dfrac{\partial u}{\partial \hat{\psi}_1} = 0 \overset{u\neq 0}{\Longrightarrow} \dfrac{\partial u}{\partial \hat{\psi}_1} = 0$

$\therefore\ u = u(\hat{X}, \hat{\psi}^{j\neq 1})$

Define $\tilde{\psi}^1 = u^{-1}(\hat{X}, \hat{\psi}^{j\neq 1})\, \hat{\psi}^1$ then $\delta\tilde{\psi}^1 = \epsilon$. //

Thus, for a supersymmetry with $\delta^2 = 0$ Localization Principle applies.

Computation of topological correlators

Basic tool — Localization.

A-twisted NLSM on a Kähler mfd X

$(\delta_A \phi^i = \bar{\epsilon}\psi_-^i \ , \ \delta_A \bar{\phi}^{\bar{i}} = \bar{\epsilon}\bar{\psi}_+^{\bar{i}})$

$\delta_A \psi_-^i = 0 \ , \ \delta_A \bar{\psi}_+^{\bar{i}} = 0$

$$\delta_A \psi_+^i = \bar{\epsilon}\left(i(\partial_t + \partial_\sigma)\phi^i + \Gamma^i_{jk}\psi_+^j \psi_-^k\right)$$

$$\delta_A \bar{\psi}_-^{\bar{i}} = \epsilon\left(i(\partial_t - \partial_\sigma)\bar{\phi}^{\bar{i}} + \Gamma^{\bar{i}}_{\bar{j}\bar{k}}\bar{\psi}_-^{\bar{k}}\bar{\psi}_+^{\bar{j}}\right)$$

$$\partial_t + \partial_\sigma \xrightarrow{\text{Wick}} i\partial_2 + \partial_1 = 2\frac{\partial}{\partial \bar{z}} \qquad z = x^1 + ix^2$$

$$\partial_t - \partial_\sigma \longrightarrow -2\frac{\partial}{\partial z}$$

RHS $= 0$ at $\psi_\pm = \bar{\psi}_\pm = 0$

$$\Leftarrow \quad \partial_{\bar{z}}\phi^i = 0$$

$\phi : \Sigma \longrightarrow X$ holomorphic map (bosonic part of) Path integral weight $= e^{-S_b}$ ~~$|_{\phi\,hol}$~~

$$S_b = \int_\Sigma g_{i\bar{j}}\left(\partial_z\phi^i\partial_{\bar{z}}\overline{\phi}^{\bar{j}} + \partial_{\bar{z}}\phi^i\partial_z\overline{\phi}^{\bar{j}}\right)d^2z - i\int_\Sigma \phi^*B$$

$$= \int_\Sigma\left(2g_{i\bar{j}}\,\partial_{\bar{z}}\phi^i\partial_z\overline{\phi}^{\bar{j}} + g_{i\bar{j}}\left(\partial_z\phi^i\partial_{\bar{z}}\overline{\phi}^{\bar{j}} - \partial_{\bar{z}}\phi^i\partial_z\overline{\phi}^{\bar{j}}\right)\right)d^2z$$

$$- i\int_\Sigma \phi^*B$$

$$= \underbrace{2\int_\Sigma g_{i\bar{j}}\,\partial_{\bar{z}}\phi^i\partial_z\overline{\phi}^{\bar{j}}\,d^2z}_{\geq 0} + \underbrace{\int_\Sigma \phi^*(\omega - iB)}_{\text{(locally) constant or topological}}$$

$= 0$ iff holomorphic

At $\phi : \Sigma \to X$ hol, $e^{-S_b} = e^{-\int_\Sigma \phi^*(\omega - iB)} = e^{-\int_{\phi_*[\Sigma]}(\omega - iB)}$.

The ∞-dim integral reduces to integration over the moduli space of holomorphic maps.

$\beta \in H_2(X,\mathbb{Z})$

$$\mathcal{M}_\Sigma(X,\beta) = \left\{\phi : \Sigma \to X \;\middle|\; \begin{array}{l}\text{holomorphic}\\ \phi_*[\Sigma] = \beta\end{array}\right\}$$

$\beta = 0$ $\mathcal{M}_\Sigma(X,\beta) = X$

$$\mathcal{O}_i \leftrightarrow \omega_i \in H^{\cdot}_{DR}(X)$$

$$\langle \mathcal{O}_1^{(x_1)} \cdots \mathcal{O}_s^{(x_s)} \rangle_\Sigma = \sum_{\beta \in H_2(X,\mathbb{Z})} e^{-\int_\beta (\omega - iB)} \int_{\mathcal{M}_\Sigma(X,\beta)} ev_1^* \omega_1 \wedge \cdots \wedge ev_s^* \omega_s$$

$\uparrow$
$\beta = 0 : \int_X \omega_1 \wedge \cdots \wedge \omega_s$

$$ev_i : \mathcal{M}_\Sigma(X,\beta) \longrightarrow X$$
$$[\phi : \Sigma \to X] \mapsto \phi(x_i)$$

if $D_i \subset X$ Poincaré dual of ω_i

$\omega_i = \delta_{D_i}$ δ-function supported on D_i

$$\int_{\mathcal{M}_\Sigma(X,\beta)} ev_1^* \delta_{D_1} \wedge \cdots \wedge ev_s^* \delta_{D_s} = \# \left\{ \phi : \Sigma \to X \;\middle|\; \begin{array}{l} \text{holomorphic} \\ \phi(x_i) \in D_i \ \forall i \\ \phi_*(\Sigma) = \beta \end{array} \right\}$$

$$= n_{\beta, D_1 \cdots D_s}$$

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_s(x_s) \rangle_\Sigma = \sum_{\beta \in H_2(X,\mathbb{Z})} n_{\beta, D_1 \cdots D_s} e^{-\int_\beta (\omega - iB)}$$

e.g. $\langle \mathcal{O}_1(0) \mathcal{O}_2(1) \mathcal{O}_3(\infty) \rangle_{\mathbb{CP}^1} = \sum_{\beta \in H_2(X,\mathbb{Z})} n_{\beta, D_1 \cdots D_s} e^{-\int_\beta (\omega - iB)}$

$\longrightarrow$ quantum cohomology ring.
$\beta = 0$: classical part.

Important. they depends on the class

$$[\omega - iB] \in H^2(X, \mathbb{C})$$

They must be holomophic twisted chiral parameters of the system.

— We see it more explicitly in Linear Sigma Model.

B-twisted NLSM on a CY mfd X

$$\left(\delta_B \phi^i = 0 , \quad \delta_B \bar{\phi}^{\bar{i}} = \epsilon \bar{\eta}^{\bar{i}} \right) \qquad \bar{\eta}^{\bar{i}} = -(\bar{\psi}_-^{\bar{i}} + \bar{\psi}_+^{\bar{i}})$$

$$\theta_i = g_{i\bar{j}} (\bar{\psi}_-^{\bar{j}} - \bar{\psi}_+^{\bar{j}})$$

$$\delta_B \theta_i = 0 \qquad \delta_B \bar{\eta}^{\bar{i}} = 0$$

$$\delta_B \psi_\pm^i = -i \bar{\epsilon} (\partial_t \pm \partial_\sigma) \phi^i$$

$$\pm 2 \frac{\partial}{\partial \bar{z}} , \; -2 \frac{\partial}{\partial z}$$

RHS $= 0$ at $\partial_z \phi^i = \partial_{\bar{z}} \phi^i = 0$

i.e. $\phi : \Sigma \to X$ constant map (map to a point)

The path integral reduces to

integration on the moduli space of constant map

which is X !

Also $S_b \big|_{\delta \psi = 0} = \int \underbrace{g_{i\bar{j}} \partial^z \phi^i \partial_z \bar{\phi}^{\bar{j}}}_{=0} d^2 z - i \int \underbrace{\phi^* B}_{=0} = 0$

The result:

$$O_i \Leftrightarrow \omega_i \in H^{0,p_i}(X, \Lambda^{q_i} T_X)$$

$$\langle O_1 \cdots O_s \rangle_{\mathbb{CP}^1} = \int_X (\omega_1 \wedge \cdots \wedge \omega_s, \Omega) \wedge \Omega$$

X: CY 3-fold $\omega_i \in H^1(X, T_X)$

$$\langle O_1 O_2 O_3 \rangle_{\mathbb{CP}^1} = \int_X \omega_1^i \wedge \omega_2^j \wedge \omega_3^k \, \Omega_{ijk} \wedge \Omega$$