Fields–Schubert Wkshp
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Mathieu Willems, Basics of K-theory for G/B

$K(X)$: K-theory of X

Setup: $H \subset B \subseteq G$ $\mathbb{C}$ semisimple Lie gp
Cartan Borel

EG: $\left\{\begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix}\right\} \subseteq \left\{\begin{bmatrix} \diagdown & * \\ 0 & \diagdown \end{bmatrix}\right\} \subseteq SL(n,\mathbb{C})$.

so then G/B = flags in $\mathbb{C}^n$

Def 1) $K_{top}(X)$ Grothendieck gp of v. bundles on X;

$E \xrightarrow{\pi} X$ $\pi^{-1}(x)$ is a $\mathbb{C}$, f-dim'l vector space
locally trivial: $\mathcal{U} \times \mathbb{C}^n$ for $\mathcal{U}$ small

$K_{top}(X)$: abelian gp generated by all v. bdles over X, with relations
$E = E_1 + E_2$
if $0 \to E_1 \to E \to E_2 \to 0$.

$E \longmapsto [E] \in K_{top}(X)$

$E \oplus F \longmapsto [E \oplus F] = [E] + [F] \in K_{top}(X)$
$E \otimes F \longmapsto [E \otimes F] = [E] \cdot [F] \in K_{top}(X)$
} so a ring

and notice that $K_{top}(pt) = \mathbb{Z}$.

Def 2) X is a $\mathbb{C}$ alg. variety.
$K^0(X)$ now use algebraic bdles over X
$\varphi : K^0(X) \longrightarrow K_{top}(X)$

Prop: If $X = G/B$, φ is an isomorphism.

Def 3) X as in (2)

$K_0(X)$ coherent sheaves on X

$E \longmapsto H^0(E) =$ sheaf of sections of E

v bdle is a locally free sheaf (hence coherent)

so we have $\psi : K^0(X) \longrightarrow K_0(X)$

$E \longmapsto H^0(E)$

Prop: X smooth and projective: ψ an iso^m between $K^0(X) \xrightarrow{\cong} K_0(X)$.

Not all coherent sheaves are locally free, but all coherent sheaves (since X smooth) have finite resolution by locally free sheaves.

$$\cdots \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{F}$$

$$\downarrow$$

$$X$$

$\mathcal{F}_i$ = locally free.

$$\psi^{-1}(\mathcal{F}) = \sum (-1)^i [\mathcal{F}_i] \in K^0(X)$$ (indep of choice of resolution)

so in this case

$$K(X) := K_{top}(X) = K^0(X) = K_0(X).$$

all def^ns same ...

Example: $\mathbb{C}P^1$. trivial line bundle $[1] = \left[\begin{matrix}\mathbb{C}P^1 \times \mathbb{C} \\ \downarrow \\ \mathbb{C}P^1\end{matrix}\right]$

tautological line bdle $\left[\begin{matrix} E \\ \downarrow \pi \\ \mathbb{C}P^1 \end{matrix}\right]$ $\pi^{-1}[\ell] = \ell \subseteq \mathbb{C}^2$.

$K(\mathbb{C}P^1)$ = generated by $[1]$, $[E]$, with one relation

$$(E-1)^2 = 0.$$

$x = E - 1$: $K(\mathbb{C}P^1) = \mathbb{Z}[x]/x^2 \cong H^*(\mathbb{C}P^1)$ (as rings) in fact $K(\mathbb{C}P^n) = \mathbb{Z}[x]/x^{n+1} \overset{\text{Ring}}{\cong} H^*(\mathbb{C}P^n)$

Now look at $K(G/B)$:

1) Line bundles.

Let $X(H) :=$ group of characters of $H = \{$ morphisms $H \xrightarrow{\gamma} \mathbb{C}^* \}$

$$\left(H \cong (\mathbb{C}^\times)^\ell \quad (z_1, \dots, z_\ell) \right)$$

So then any $(k_1, \dots, k_\ell) \in \mathbb{Z}^\ell$ specifies $\gamma : (z_1, \dots, z_\ell) \mapsto z_1^{k_1} \cdots z_\ell^{k_\ell}$

so $\mathbb{Z}^\ell \xrightarrow[\cong]{} X(H)$

$\mathrm{Lie}(H) = \mathfrak{h} \cong \mathbb{C}^\ell$, $\mathfrak{h}^* \cong \mathbb{C}^\ell$

$$\left.\begin{matrix} \mathbb{Z}^\ell & \mathfrak{h}^*_{\mathbb{Z}} \\ \cap & \cap \\ \mathbb{C}^\ell & \subseteq \mathfrak{h}^* \end{matrix}\right\} \text{weight lattice}$$

$$(\mathfrak{h}^*_{\mathbb{Z}}, +) \cong X(H)$$
$$\lambda \longmapsto e^\lambda$$

↖ character assoc. to λ

$\lambda \in \mathfrak{h}^*_{\mathbb{Z}} \longmapsto e^\lambda : H \longrightarrow \mathbb{C}^*$

have $\rho : B \longrightarrow H$ "take diagonals"

$e^\lambda : B \rightsquigarrow \mathbb{C}^*$

$$\boxed{e^\lambda(b) = e^\lambda(\rho(b))}$$

so given a character λ, have associated line bundle $\mathcal{L}_\lambda$

$$\mathcal{L}_\lambda = G \times_B \mathbb{C} = \{ [g, v] / [g, v] = [gb, e^\lambda(b^{-1}) v] \}.$$

$$\downarrow$$

$$X = G/B$$

Facts: 1) $\mathcal{L}_{\lambda_1 + \lambda_2} \cong \mathcal{L}_{\lambda_1} \otimes \mathcal{L}_{\lambda_2}$

2) We get all line bundles this way

3) $K(X)$ is generated by line bundles $\{\mathcal{L}_\lambda\}$.

→ gives a description by generators and relations (similar to cohomology)

=

2) Structure Sheaves

$Y \subset X$ subvariety

$[\mathcal{O}_Y] \in K_0(X)$ "sheaf of functions on Y"

a.k.a. "structure sheaf of Y"

Recall $X^H \cong W = N_G(H)/H$ (for $G = SL_n$, $W = S_n$)

Look at B-orbit: $w \in W$ $X_w^\circ = BwB/B \cong \mathbb{C}^{\ell(w)}$

$$X_w := \overline{X_w^\circ} \subset X.$$

$$X = \bigsqcup_w X_w^\circ$$

Prop: $\{[\mathcal{O}_{X_w}]\}_{w\in W}$ is a basis (over $\mathbb{Z}$) of $K(X)$

Q1 $[\mathcal{O}_{X_u}] \times [\mathcal{O}_{X_v}] = \sum c_{uv}^w [\mathcal{O}_{X_w}]$ (☆)

$c_{uv}^w \in \mathbb{Z}$

We know: • $c_{uv}^w = 0$ unless $u \leq w$, $v \leq w$ in Bruhat order

• the sign of $c_{uv}^w = (-1)^{N\binom{w}{u,v}}$

$$N\binom{w}{u,v} = \operatorname{codim}(X_w) - \operatorname{codim}(X_u) - \operatorname{codim}(X_v)$$

• Link with cohomology:

$$[X_u][X_v] = \sum a_{uv}^w [X_w] \in H^*(X, \mathbb{Z})$$

$a_{uv}^w = 0$ unless $N\binom{w}{uv} = 0$.

if $N\binom{w}{uv} = 0$, then $a_{uv}^{w} = c_{uv}^{w}$.

Chern character : $ch : K(X) \otimes \mathbb{Q} \longrightarrow H^{*}(X; \mathbb{Q})$ isomorphism.

$\mathcal{L}_\lambda \longmapsto e^{c_1(\mathcal{L}_\lambda)}$ finite sum.

c_1 = first Chern class $\in H^2(X; \mathbb{Z})$

Prop : 1) ch is an isomorphism

2) $ch([\mathcal{O}_{X_u}]) = [X_u]$ + higher degree terms.

then 3) is lower-degree terms of (*)

Q2 Chevalley formula?

$$[\mathcal{L}_\lambda] \cdot [\mathcal{O}_{X_u}] = \sum b_{\lambda,u}^{w} [\mathcal{O}_{X_w}] \quad (\text{compute } b_{\lambda,u}^{w} \ldots)$$

(Answer: Pittie-Ram, L-S paths...)

[actually $\mathcal{L}_\lambda$ dominant...]

Equivariant K-theory : again, maybe easier to compute...

$H = (\mathbb{C}^{*})^{\ell}$

$X = G/B$

Def^n : $K^0(H, X)$ = H-equivariant (algebraic) v. bundles $/\sim$.

$G \circlearrowright \left(\begin{array}{c} E \\ \pi \downarrow \\ X \end{array} \right)$

- algebraic bundles
- H acts on E
- $\pi(he) = h\pi(e)$
- $\forall h, x \quad E_x \longrightarrow E_{hx}$ linear.

Def: $K_0(H, X)$ use H-eqvt. coherent sheaves

again, X smooth projective $\Rightarrow$ $K_0(H, X) \cong K^0(H, X)$.

$$X = \bigsqcup X_w^\circ, \quad X_w^\circ \text{ } H\text{-stable}$$

Prop: $\left\{ \left[\mathcal{O}_{X_w}\right]_H \right\}$ is a basis as a $\underbrace{K(H, pt)}$-module of $K(H, X)$.

↑ H-eqvt classes

$= R(H)$ ring of rep'ns.
$= \mathbb{Z}[X(H)]$

$$K(H, X) \cong R(H) \otimes_{\mathbb{Z}} K(X) \quad \text{as modules}$$

$$\varepsilon : K(H, X) \longrightarrow K(X)$$

$$v_0 : R(H) \longrightarrow \mathbb{Z}$$
$$\chi \in X(H) \longmapsto 1$$

get a map $\tilde{\varepsilon} : K(H, X) \otimes_{R(H)} \mathbb{Z} \longrightarrow K(H)$ is an isom.

$\mathbb{Z}$ is a $R(H)$-module under v_0

$$\sum R_w [\mathcal{O}_{X_w}]_H \longmapsto \sum v_0(R_w) [\mathcal{O}_{X_w}]$$

so $v_0(\text{structure constants in } K(H, X)) =$ structure const's in $K(X)$.

But easier (maybe) to compute str. constants upstairs!!

⇝ restriction to fixed points. $X^H \overset{i_H}{\hookrightarrow} X$

$$i_H^* : K(H, X) \longrightarrow K(H, X^H)$$

Restriction means:

$$\begin{array}{ccc} E & & i_H^* E \\ \pi \downarrow & & \downarrow \pi_0 \\ X & \overset{i_{X^H}}{\hookleftarrow} & X^H \ni p \end{array} \qquad \pi_0^{-1}(p) = \pi^{-1}(p).$$

$$K(H, X^H) = \prod_{p \in X^H} R[H] \simeq F(W, R(H))$$

(with $X^H \cong W$)

all functions $W \longrightarrow R(H)$.

prop : $i_H^* : K(H, X) \longrightarrow F(W, R(H))$ is injective.

Q1 $\left(i_H^*[\mathcal{O}_{X_w}]\right)(u) \in R(H)$?

Q2 description of image of i_H^* ?

$\Delta \subset \mathfrak{h}_{\mathbb{Z}}^*$ (roots), $\Delta = \Delta^+ \sqcup \Delta^-$

$\alpha \in \Delta^+ \longleftrightarrow s_\alpha \in W$

$\longleftrightarrow e^\alpha \in X(H)$

$$i_H^*(K(H, X)) = \left\{ f \in F(W, R(H)),\ \forall \alpha \in \Delta^+,\ \forall w \in W \ \ \frac{f(ws_\alpha) - f(w)}{1 - e^\alpha} \in R(H) \right\}$$

Example : $\mathbb{C}P^1$, $H = \mathbb{C}^*$

2 fixed points : $[0:1] = s_1$, $[1:0] = 1$

$i_H^*(1)$: $\bullet\ 1$, $\bullet\ 1$ — restrictions to fixed pts

$i_H^*(E)$: $\bullet\ 1$, $\bullet\ e^{\alpha_1}$ $\quad e^{\alpha_1} : z \mapsto z$.

since i_H^* injective, $(E-1)(E-e^{\alpha_1}) = 0 \xrightarrow{\epsilon} (E-1)^2 = 0 \in K(X)$