# Geometric properties of random matrices with independent log-concave rows/columns


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Based on joint work with
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- $X$ is $\psi_\alpha$ ($\alpha \in [1, 2]$) with constant $C$ if for all $y \in \mathbb{R}^n$,

$$\|\langle X, y\rangle\|_{\psi_\alpha} \leq C|y|,$$

where

$$\|Y\|_{\psi_\alpha} = \inf\{a > 0 \colon \mathbb{E}\exp((Y/a)^\alpha) \leq 2\}$$

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**Fact**

*For every random vector $X$ not supported on any $n-1$ dimensional hyperplane, there exists an affine map $T: \mathbb{R}^n \to \mathbb{R}^n$ such that $TX$ is isotropic.*

If for a set $K \subseteq \mathbb{R}^n$ the random vector distributed uniformly on $K$ is isotropic, we say that $K$ is isotropic.

# Log-concavity

A random vector $X$ in $\mathbb{R}^n$ is log-concave if its law $\mu$ satisfies a Brunn-Minkowski type inequality

$$\mu(\theta A + (1-\theta)B) \geq \mu(A)^{\theta}\mu(B)^{1-\theta}.$$

**Theorem (Borell)**

*A random vector not supported on any $(n-1)$ dimensional hyperplane is log-concave iff it has density of the form $\exp(-V(x))$, where $V\colon \mathbb{R}^n \to (-\infty, \infty]$ is convex.*

**Lemma (Borell)**

*An isotropic log-concave random vector is $\psi_1$ with a universal constant $C$.*

# Examples

The following distributions are log-concave:

- Gaussian measures
- Uniform distributions on convex bodies
- Measures with density of the form $C \exp(-\|x\|)$, where $\|x\|$ is a norm.
- Products, affine images and convolutions of the above distributions.

# The basic model

## Definition

Let $\Gamma$ be an $n \times N$ matrix with columns $X_1, \ldots, X_N$, where $X_i$'s are independent isotropic log-concave random vectors in $\mathbb{R}^n$

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- When is $\Gamma^T$ close to a multiple of isometry?
- How does $\Gamma$ ($\Gamma^T$) act on sparse vectors?
- What is the smallest singular value of $\Gamma$?
- What is the distribution of singular values /eigenvalues of $\Gamma$?

# Motivations: sampling convex bodies

**Problem**

Let $K \subseteq \mathbb{R}^n$ be a convex body, s.t. $B_2^n \subseteq K \subseteq RB_2^n$. Assume we have access to an oracle (a black box), which given $x \in \mathbb{R}^n$ tells us whether $x \in K$.

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How to generate random points uniformly distributed in $K$?
How to compute the volume of $K$?

- This can be done by using Markov chains.
- Their speed of convergence depends on the position of the convex body.
- Preprocessing: First put $K$ in the isotropic position (again by randomized algorithms).

- Centering the body is not comp. difficult – takes $\mathcal{O}(n)$ steps.
- The question boils down to:

How to approximate the covariance matrix of $X$ - uniformly distributed on $K$ by the empirical covariance matrix

$$\frac{1}{N}\sum_{i=1}^{N} X_i \otimes X_i.$$

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Given an isotropic convex body in $\mathbb{R}^n$, how large $N$ should we take so that

$$\left\|\frac{1}{N}\sum_{i=1}^{N} X_i \otimes X_i - Id\right\|_{\ell_2 \to \ell_2} \leq \varepsilon$$

with high probability?

# Interpretation in terms of Γ.

We have

$$\left\|\frac{1}{N}\sum_{i=1}^{N} X_i \otimes X_i - Id\right\|_{\ell_2\to\ell_2} = \sup_{y\in S^{n-1}} \left|\frac{1}{N}\sum_{i=1}^{N}\langle X_i, y\rangle^2 - 1\right|$$

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**So the (geometric) question is**

Let $\Gamma$ be a matrix with independent columns $X_1, \ldots, X_N$ drawn from an isotropic convex body (log-concave measure) in $\mathbb{R}^n$.

How large should $N$ be so that $N^{-1/2}\Gamma^T : \mathbb{R}^n \to \mathbb{R}^N$ was an almost isometry?

# History of the problem

- Kannan, Lovasz, Simonovits (1995) – $N = \mathcal{O}(n^2)$
- Bourgain (1996) – $N = \mathcal{O}(n \log^3 n)$
- Rudelson (1999) – $N = \mathcal{O}(n \log^2 n)$
- Giannopoulos, Hartzoulaki, Tsolomitis (2005) – unconditional bodies: $N = \mathcal{O}(n \log n)$
- Aubrun (2006) – unconditional bodies: $N = \mathcal{O}(n)$
- Paouris (2006) – $N = \mathcal{O}(n \log n)$
- Litvak, Pajor, Tomczak-Jaegermann, R.A. (2008) – $N = \mathcal{O}(n)$

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For arbitrary isotropic random vectors, if you do not assume any uniform bound on $\langle X_i, y \rangle$, $y \in S^{n-1}$, you cannot remove the logarithm (the optimal bound $N = \mathcal{O}(n \log^\beta n)$ is due to M. Rudelson). Recently $N = O(n \log\log n)$ was proven under a uniform bound on $(4+\varepsilon)$-th moments of $\langle X_i, y \rangle$ (R. Vershynin).

**Theorem (A refined estimate, ALPT 2010)**

*Assume that $N \geq n$. Then with probability at least $1 - \exp(-c\sqrt{n})$ one has*

$$\Big\| \frac{1}{N} \sum_{i=1}^{N} X_i \otimes X_i - 1 \Big\|_{\ell_2 \to \ell_2} \leq C \sqrt{\frac{n}{N}}.$$

**Remark**

Previous estimates (ALPT 2008) had an additional $\log(N/n)$ factor.

## Remark

If $\frac{1}{\sqrt{N}}\Gamma^T$ is an almost isometry then obviously $\|\Gamma\| \leq C\sqrt{N}$, so the KLS question and the question about $\|\Gamma\|$ are related.

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**Theorem (Litvak, Pajor, Tomczak-Jaegermann, R.A. (2008))**

*If $N \leq \exp(c\sqrt{n})$ and the vectors $X_i$ are log-concave then for $t > 1$, with probability at least $1 - \exp(-ct\sqrt{n})$,*

$$\forall_{m \leq N}\ A_m \leq Ct\left(\sqrt{n} + \sqrt{m}\log\left(\frac{2N}{m}\right)\right).$$

*In particular, with high probability $\|\Gamma\| \leq C(\sqrt{n} + \sqrt{N})$.*

# Sketch of the proof

A modification of Bourgain's approach. One approximates an arbitrary vector $z$ with $|\mathrm{supp}\, z| \le m$ by $x_0 + x_1 + \ldots + x_l$ ($l < \log_2 m$), where

$$|\mathrm{supp}\, x_i| \simeq m/2^i,\ \|x_i\|_\infty \simeq \sqrt{2^i/m},\ i \ge 1$$
$$|\mathrm{supp}\, x_0| \simeq m/2^l,\ \|x_0\|_\infty \le 1$$

and $x_i$ comes from a $2^{-i}$–net in the set of sparse vectors of support at most $m/2^i$.

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Then using the $\psi_1$ condition one shows that with high probability

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Thus $\max_i |X_i| \le C\sqrt{n}$ with high probability and we can solve the inequality for $A_m$.

# Compressed sensing and neighbourly polytopes

Imagine we have a vector $x \in \mathbb{R}^N$ ($N$ large), which is supported on a small number of coordinates (say $|\mathrm{supp}\ \mathrm{x}| = m << N$).

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What if we don't know the support?

**Answer** (Donoho, Candes, Tao, Romberg) Take measurements in random directions $Y_1, \ldots, Y_n$ and set

$$\hat{x} = \mathrm{argmin}\ \{\|y\|_1 : \langle Y_i, y\rangle = \langle Y_i, x\rangle\}$$

# Compressed sensing and neighbourly polytopes

**Definition**

*A polytope $K \subseteq \mathbb{R}^n$ is called $m$-**neighbourly** if any set of vertices of $K$ of cardinality at most $m+1$ is the vertex set of a face.*

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*A (centraly symetric) polytope $K \subseteq \mathbb{R}^n$ is called* $m$**-(symmetric)-neighbourly** *if any set of vertices of $K$ of cardinality at most $m+1$ (containing no opposite pairs) is the vertex set of a face.*

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## Theorem (Donoho)

*Let $\Gamma$ be an $n \times N$ matrix with columns $X_1, \dots, X_N$. The following conditions are equivalent*

(i) *For any $x \in \mathbb{R}^N$ with $|\mathrm{supp}\, x| \le m$, $x$ is the unique solution of the minimization problem*

$$\min \|t\|_1, \quad \Gamma t = \Gamma x.$$

(ii) *The polytope $K(\Gamma) = \mathrm{conv}(\pm X_1, \dots, \pm X_N)$ has $2N$ vertices and is $m$-symmetric-neighbourly.*

# Compressed sensing and neighbourly polytopes

## Definition (Restricted Isometry Property (Candes, Tao))

For an $n \times N$ matrix $\Gamma$ define the **isometry constant** $\delta_m = \delta_m(\Gamma)$ as the smallest number such that

$$(1 - \delta_m)|x|^2 \leq |\Gamma x|^2 \leq (1 + \delta_m)|x|^2$$

for all $m$-sparse vectors $x \in \mathbb{R}^N$.

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## Theorem (Candes)

*If $\delta_{2m}(\Gamma) < \sqrt{2} - 1$ then for every m-sparse $x \in \mathbb{R}^n$, $x$ is the unique solution to*

$$\min \|t\|_1, \quad \Gamma t = \Gamma x.$$

*In consequence, the polytope $K(\Gamma)$ (resp. $K_+(\Gamma) = \mathrm{conv}(X_1, \dots, X_N)$) is m-symmetric-neighbourly (resp. m-neighbourly)*

# History

The following matrices satisfy RIP

- Gaussian matrices (Candes, Tao), $m \simeq n/\log(2N/n)$
- Matrices with rows selected randomly from the Fourier matrix (Candes & Tao, Rudelson & Vershynin), $m \simeq n/\log^4(N)$
- Matrices with independent subgaussian isotropic rows (Mendelson, Pajor, Tomczak-Jaegermann), $m \simeq n/\log(2N/n)$
- Matrices with independent log-concave isotropic columns (LPTA), $m \simeq n/\log^2(2N/n)$

# Neighbourly polytopes

## Theorem (LPTA)

*Assume that $X_i'$s are $\psi_r$. Let $\theta \in (0, 1/4)$ and assume that $N \leq \exp(c\theta^C n^c)$ and $m \log^{2/r}\left(\frac{2N}{\theta m}\right) \leq \theta^2 n$. Then, with probability at least $1 - \exp(-c\theta^C n^c)$*

$$\delta_m\Big(\frac{1}{\sqrt{n}}\Gamma\Big) \leq \theta.$$

## Corollary (LPTA)

*Let $X_1, \ldots, X_N$ be random vectors drawn from an isotropic $\psi_r$ ($r \in [1, 2]$) convex body in $\mathbb{R}^n$. Then, for $N \leq \exp(cn^c)$, with probability at least $1 - \exp(-cn^c)$, the polytope $K(\Gamma)$ (resp. $K_+(\Gamma)$) is m-symmetric-neighbourly (resp. m-neighbourly) with*

$$m = \lfloor c\frac{n}{\log^{2/r}(CN/n)}\rfloor.$$

# Method of proof

We use the same approximation techniques as for the KLS problem to bound

$$B_m = \sup_{|\mathrm{supp}\ z| \le m,\ |z|=1} \left| \left| \sum_{i \le N} z_i X_i \right|^2 - \sum_{i \le N} z_i^2 |X_i|^2 \right|^{1/2}$$

**Theorem (B. Klartag)**

$$\mathbb{P}\left( \max_{i \le N} \left| \frac{|X_i|^2}{n} - 1 \right| \ge \varepsilon \right) \le C \exp(-c\varepsilon^C n^c).$$

Thus

$$\delta_n(n^{-1/2}\Gamma) \le n^{-1} B_m^2 + \varepsilon$$

with overwhelming probability.

# Smallest singular value

## Definition

*For an* $n \times n$ *matrix* $\Gamma$ *let* $s_1(\Gamma) \geq s_2(\Gamma) \geq \ldots \geq s_n(\Gamma)$ *be the singular values of* $\Gamma$*, i.e. eigenvalues of* $\sqrt{\Gamma\Gamma^T}$*. In particular*

$$s_1(\Gamma) = \|A\|, \; s_n(\Gamma) = \inf_{x \in S^{n-1}} |\Gamma x| = \frac{1}{\|A^{-1}\|}$$

## Theorem (Edelman, Szarek)

*Let* $\Gamma$ *be an* $n \times n$ *random matrix with independent* $\mathcal{N}(0,1)$ *entries. Let* $s_n$ *denote the smallest singular values of* $\Gamma$*. Then, for every* $\varepsilon > 0$*,*

$$\mathbb{P}(s_n(\Gamma) \leq \varepsilon n^{-1/2}) \leq C\varepsilon,$$

*where* $C$ *is a universal constant.*

## Theorem (Rudelson, Vershynin)

*Let* $\Gamma$ *be a random matrix with independent entries* $X_{ij}$*, satisfying* $\mathbb{E}X_{ij} = 0$, $\mathbb{E}X_{ij}^2 = 1$, $\|X_{ij}\|_{\psi_2} \le B$*. Then for any* $\varepsilon \in (0,1)$*,*

$$\mathbb{P}(s_n(\Gamma) \le \varepsilon n^{-1/2}) \le C\varepsilon + c^n,$$

*where* $C > 0, c \in (0,1)$ *depend only on B.*

## Theorem (Guédon, Litvak, Pajor, Tomczak-Jaegermann, R.A.)

*Let* $\Gamma$ *be an* $n \times n$ *random matrix with independent isotropic log-concave rows. Then, for any* $\varepsilon \in (0,1)$*,*

$$\mathbb{P}(s_n(\Gamma) \le \varepsilon n^{-1/2}) \le C\varepsilon + C\exp(-cn^c)$$

*and*

$$\mathbb{P}(s_n(\Gamma) \le \varepsilon n^{-1/2}) \le C\varepsilon^{n/(n+2)} \log^C(2/\varepsilon).$$

## Corollary

*For any* $\delta \in (0,1)$ *there exists* $C_\delta$ *such that for any* $n$ *and* $\varepsilon \in (0,1)$*,*

$$\mathbb{P}(s_n(\Gamma) \leq \varepsilon n^{-1/2}) \leq C_\delta \varepsilon^{1-\delta}.$$

## Definition

For an $n \times n$ matrix $\Gamma$ define the **condition number** $\kappa(\Gamma)$ as

$$\kappa(\Gamma) = \|\Gamma\| \cdot \|\Gamma^{-1}\| = \frac{s_1(\Gamma)}{s_n(\Gamma)}.$$

## Corollary

*If* $\Gamma$ *has independent isotropic log-concave columns, then for any* $\delta > 0$*,* $t > 0$*,*

$$\mathbb{P}(\kappa(\Gamma) \geq nt) \leq \frac{C_\delta}{t^{1-\delta}}.$$

## Definition

A random vector $X = (X_1, \ldots, X_N)$ is unconditional if its distribution is the same as that of $(\varepsilon_1 X_1, \ldots, \varepsilon_N X_N)$ for any choice of signs $\varepsilon_1, \ldots, \varepsilon_N \in \{-1, 1\}$.

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## Theorem (LPTA 2010)

*Let $A$ be an $n \times N$ matrix with independent log-concave isotropic* **unconditional** *rows. Let $\theta \in (0, 1)$ and assume that $m \log^2\left(\frac{2N}{m}\right) \leq \theta^2 n$. Then, with high probability,*

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**Tool:** A comparison principle for norms of unconditional log-concave vectors by Rafał Latała.

# Asymptotic spectral distribution, singular values

## Definition

The empirical spectral distribution of a random $n \times n$ matrix $A$ is the random measure defined as

$$\nu = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$ and $\delta_x$ is the Dirac mass at $x$.

## Theorem (Marchenko-Pastur 1967)

*Let $A_n = (X_{ij})_{i \le N_n, j \le n}$, where $X_{ij}$ are i.i.d. mean zero variance one random variables. If $N_n/n \to y \in (0, \infty)$ then the empirical spectral distribution of $\frac{1}{n} A_n A_n^T$ converges almost surely to a non-random measure depending only on $y$ (the Marchenko-Pastur distribution with parameter $y$).*

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## Theorem (Pajor, Pastur 2007)

*Let $A_n$ be an $N_n \times n$ random matrix with independent log-concave isotropic rows. If $N_n/n \to y \in (0, \infty)$, then the empirical spectral distribution of $\frac{1}{n} A_n A_n^T$ converges almost surely to the Marchenko-Pastur law with parameter $y$.*

# Asymptotic spectral distribution, eigenvalues

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*Let $A_n$ be an $n \times n$ matrix with i.i.d. mean zero, variance one entries. Then the empirical spectral distribution of $\frac{1}{\sqrt{n}}A_n$ converges almost surely to the uniform measure on the unit disc.*

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Thank you