Solitons in geometric flows

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Outline

- Definitions and motivation.
- Ricci solitons and clasification results.
- Yamabe solitons and classification.
- Questions.

Introduction

- We will discuss special solutions called the solitons of the Ricci flow and the Yamabe flow.
- Ricci flow equation: If (M, g_0) is a smooth Riemannian manifold then evolve the metric in time by

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \qquad g(\cdot, 0) = g_0(\cdot).$$

- Yamabe flow equation: If (M, g_0) is a smooth Riemannian manifold then evolve the metric by

$$\frac{\partial}{\partial t} g_{ij} = -R g_{ij}, \qquad g(\cdot, 0) = g_0.$$

- These two flows coincide in dimension $n = 2$.
- Motivation for studying the solitons: they often arise as finite time singularity models. In other words, if we encounter the singularity, rescale, take the blown up limit and the limiting solution is called the singularity model.

Definitions ad notation

- Solitons: Ricci (Yamabe) soliton $g(\cdot, t)$ is the solution to the Ricci (Yamabe) flow that moves by 1 parameter family of diffeomorphisms $\{\phi_t\}$ and by homotheties, that is,

$$g(\cdot, t) = \sigma(t)\phi_t^* g(\cdot, 0).$$

- Equivalently, $g(\cdot, t)$ is the Ricci (Yamabe) soliton if it solves the Ricci (Yamabe) equation and say g_0 satisfies

$$\mathrm{Ric}(g_0) + \mathcal{L}_X g_0 = \rho g_0, \qquad \text{Ricci soliton},$$

$$R g_0 + \mathcal{L}_X g_0 = \rho g_0, \qquad \text{Yamabe soliton}.$$

- When $X = \nabla f$, replace $\mathcal{L}_X g_0$ above by the $\nabla\nabla f$.
- $\rho > 0$ - shrinking Ricci (Yamabe) solitons
  $\rho = 0$ - steady Ricci (Yamabe) solitons
  $\rho < 0$ - expanding Ricci (Yamabe) solitons

Motivation

- Singularity in both flows occur when the norm of the curvature operator blows up.
- If $T < \infty$ is the singular time we say the singularity is of Type I if
$$\limsup_{t \to T} \sup_M (T - t)|\mathrm{Rm}|(\cdot, t) < \infty.$$
otherwise we say we have a Type II singularity.
- Perform a parabolic rescaling (rescale by the maximum of the curvature norm) and take the limit of the rescaled sequence - singularity model.
- Naber, Enders, Müller, Topping: There exists a rescaling around a Type I singularity of the Ricci flow so that the singularity model is the gradient shrinking Ricci soliton.

Gradient shrinking Ricci solitons in lower dimensions

- Hamilton, Ivey: The only closed gradient shrinking Ricci solitons in dimensions $n = 2, 3$ are the ones with constant positive curvature.
- Böhm, Wilking: The compact gradient shrinking Ricci solitons with positive curvature operator in any dimension have constant positive curvature.
- Hamilton-Ivey pinching estimate shows that three dimensional ancient solutions (in particular, shrinking Ricci solitons) have nonnegative sectional curvatures.
- Perelman: Every κ-noncollapsed three dimensional gradient shrinking Ricci soliton with bounded curvatures and strictly positive Ricci curvature must be compact.
- The assumptions on being κ-noncollapsed and of bounded curvatures have been removed.
- Combining the previous results we obtain: the three dimensional gradient shrinking Ricci solitons are S^3, $\mathbb{R}^3$, $S^2 \times \mathbb{R}$ and the quotients of those.

Gradient shrinking Ricci solitons in higher dimensions

- Locally conformally flat solitons have been considered by various people. Under additional assumptions on the curvature of those it was obtained that the the only locally conformally flat gradient shrinking Ricci solitons are S^n, $\mathbb{R}^n$, $S^{n-1} \times \mathbb{R}$ and the quotients of those (Ni-Wallach; Cao-Wang-Zhang; Peterson-Wylie).
- Zhang: The gradient shrinking Ricci solitons with vanishing Weyl tensor must have nonnegative curvature operator. By the result of Cao-Wang-Zhang we have the same classification as above.
- Cao, Zhu: For any fixed point $p \in M$ there is a uniform constant $c > 0$ so that

$$\frac{1}{4}(r(x) - c)^2 \le f(x) \le \frac{1}{4}(r(x + c)^2,$$

where $r(x) = \mathrm{dist}(x, p)$.

Curvature estimates for gradient shrinking Ricci solitons

- Munteanu, S. Let M^n be a complete gradient shrinking Ricci soliton normalized such that

$$Ric + Hess_f = \frac{1}{2}g$$

Then for any $\lambda > 0$ we have $\int_M |Ric|^2 e^{-\lambda f} < \infty$.

- Munteanu, S. Assume that for some $\lambda < 1$ we have $\int_M |Rm|^2 e^{-\lambda f} < \infty$. Then the following identity holds

$$\int_M |\nabla Ric|^2 e^{-f} = \int_M |div\,(Rm)|^2 e^{-f} < \infty.$$

- Munteanu, S. The only complete shrinking gradient Ricci solitons with harmonic Weyl tensor are the quotients of $\mathbb{R}^n$, S^n and $S^{n-1} \times \mathbb{R}$.
- harmonic Weyl tensor means that $\mathrm{div}\, W = 0$.

Topological properties of gradient shrinking Ricci solitons

- A manifold is called nonparabolic if it admits a positive symmetric Green's function. Otherwise it is called parabolic. A similar definition holds for manifold ends.
- (M, g) is a Kähler-Ricci soliton if

$$R_{\alpha\bar{\beta}} + f_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}}, \qquad f_{\alpha\beta} = f_{\bar{\alpha}\bar{\beta}} = 0.$$

- Munteanu, S. Let (M, g) be a gradient shrinking Kähler-Ricci soliton as above. If u is a harmonic function with $\int_M |\nabla u|^2 < \infty$ then u has to be a constant function. As a corollary, (M, g) has at most one nonparabolic end.
- If (M, g) had at least 2 nonparabolic ends, Li and Tam have constructed the nontrivial bounded harmonic function with bounded total energy and therefore contradiction.

Steady Ricci solitons

- $R_{ij} = \nabla_i \nabla_j f$ (occur as singularity models of Type II singularities).
- Bryant: There exists unique, up to scaling, rotationally symmetric complete gradient steady Ricci soliton. It has positive sectional curvatures. The volume of geodesic balls $B_r(0)$ grow of the order $r^{\frac{n+1}{2}}$.
- H.-D. Cao, Chen Q. Let (M^n, g, f), $n \geq 3$, be a n-dimensional complete noncompact locally conformally flat gradient steady Ricci soliton with positive sectional curvature. Then (M^n, g, f) is isometric to the Bryant soliton.
- Conjecture: The only gradient three dimensional steady Ricci soliton with positive sectional curvature is the Bryant soliton.

Topology and geometry of steady Ricci solitons

- Munteanu, S. If M is a gradient steady Ricci soliton then it has at most one nonparabolic end. They managed to show that (M, g) is either connected at infinity or splits isometrically as $M = N \times \mathbb{R}$, for a compact Ricci flat manifold N, assuming that (M, g) is Kähler and certain bounds on the Ricci curvature and the volume noncollapsing.
- Munteanu,Wang: By studying the spectrum of manifolds with $\mathrm{Ric}_f = \mathrm{Ric}(g) + \nabla\nabla f \geq 0$ they have showed that steady Ricci solitons either have one end (equivalently, connected at infinity) or split isometrically as $M = N \times \mathbb{R}$, where N is a compact steady Ricci soliton.
- Munteanu, S. If (M, g) is a gradient steady Ricci soliton, there exist uniform constants $c_0, r_0 > 0$ so that for any $r > r_0$

$$\mathrm{vol}(B_p(r)) \geq c_0 r.$$

Compact Yamabe flow

- Definition: (M, g) is called a Yamabe gradient soliton if there exists a smooth potential function $f : M \to \mathbb{R}$ and a constant $\rho \in \mathbb{R}$ so that
$$(R - \rho)g_{ij} = \nabla_i \nabla_j f.$$
(by scaling assume $\rho = -1, 0, 1.$)
- Yamabe solitons are the special solutions to the Yamabe flow equation
$$\frac{\partial}{\partial t} g_{ij} = -R g_{ij}.$$
- compact Yamabe flow: Chow,B., Ye,R., Struwe,M., Schwetlick,H., etc.
- Brendle: If $3 \leq n \leq 5$ or if $n \geq 6$ (in the latter case he imposes some mild technical assumptions), then starting at any initial metric, the normalized Yamabe flow has the long time existence and converges to a metric of constant scalar curvature.

Complete Yamabe flow

- complete Yamabe flow is not well understood.
- Type of singularities: If $T < \infty$ is a singular time, which is the time when the norm of the Riemannian curvature blows up, then if

$$\limsup_{t \to T} [(T - t) \sup_M |\mathrm{Rm}|(\cdot, t)] < \infty, \quad \text{Type I singularity.}$$

  Otherwise we have a Type II singularity.
- Yamabe flow is conformal, that is, if e.g., we are on $\mathbb{R}^n$ and we write $g(\cdot, t) = u(\cdot, t)^{\frac{4}{n+2}} dx^2$ then $u(\cdot, t)$ evolves by the fast diffusion equation

$$\frac{\partial}{\partial t} u = \frac{(n-1)}{m} \Delta_{\mathbb{R}^n} u^m.$$

- Barenblatt solutions: $B_k(x, t) = \left(\frac{C^*(T-t)}{k(T-t)^{2\gamma} + |x|^2} \right)^{\frac{1}{1-m}}$, where $m = \frac{n-2}{n+2}$, $\beta = \frac{n}{n-2-nm}$ and $\gamma = -\frac{\beta}{n}$.

Complete Yamabe flow

- Assumption: The initial condition u_0 is trapped in between two Barenblatt solutions, i.e.

$$\left(\frac{C^* T}{k_1 + |x|^2}\right)^{\frac{1}{1-m}} \leq u_0(x) \leq \left(\frac{C^* T}{k_2 + |x|^2}\right)^{\frac{1}{1-m}},$$

  for some constants $k_1 > k_2 > 0$.
- Daskalopoulos, S. Let u solve the fast diffusion equation as above, for $\frac{N-4}{N-2} < m < \frac{N-2}{N}$, with initial value u_0 satisfying the assumption. Then, the rescaled solution converges, as $\tau \to \infty$, uniformly on $\mathbb{R}^N$, and also in $L^1(\mathbb{R}^N)$, to the rescaled Barenblatt solution $\tilde{B}_{k_0}$, for some $k_0 > 0$ which turns out to be the Yamabe shrinker.
- Daskalopoulos, S. The previous theorem about the asymptotic singular profile is valid even for ranges $0 < m \leq \frac{n-4}{n-2}$, $n \geq 4$ if we assume, in addition, that for some k_0, the difference $u_0 - B_{k_0} \in L^1(\mathbb{R}^n)$.

Complete Yamabe flow

- The previous two results show that complete non-compact solutions to the Yamabe flow develop a finite time singularity of Type I, and after re-scaling the metric converges to the Barenblatt solution.
- Daskalopoulos, S. There exists a class of solutions u of the fast diffusion equation with initial data $u_0 = \left(\frac{C^* T}{|x|^2}\right)^{\frac{1}{1-m}} (1 + o(1))$ as $|x| \to \infty$ with the following properties:

Complete Yamabe flow

- The vanishing time T^* of u satisfies $T^* > T$.
- The solution u satisfies as $|x| \to \infty$, the growth conditions

$$u(x,t) \geq \left(\frac{C^*\,(T-t)}{1+|x|^2} \right)^{\frac{2}{1-m}}, \qquad \text{on } 0 < t < T$$

and

$$u(x,t) \leq \frac{C(t)}{|x|^{\frac{m}{N-2}}}, \qquad \text{on } T < t < T^*.$$

In particular, u becomes integrable on $t > T$.

- At time T there is a singularity. We conjecture it is the Type II singularity.

Singularity model - rigidity theorem

- Daskalopoulos, S. Let $g(x,t)$ be a complete eternal solution to the locally conformally flat Yamabe flow on a simply connected manifold M, with uniformly bounded sectional curvature and strictly positive Ricci curvature. If the scalar curvature R assumes its maximum at an interior space-time point P_0, then $g(x,t)$ is necessarily a Yamabe gradient steady soliton.
- Singularity models of Type II singularities are eternal solutions that live on $(-\infty, \infty)$.

Yamabe solitons

- Daskalopolous, S. If (M, g, f) is a compact gradient Yamabe soliton, not necessarily locally conformally flat, then g is the metric of constant scalar curvature.
- interested in complete noncompact locally conformally flat Yamabe gradient solitons with positive sectional curvature.
- Carron, Herzlich: Every locally conformally flat complete noncompact manifold with nonnegative Ricci curvature is either globally conformally flat to plane or isometric to a flat manifold or locally isometric to a cylinder.
- we will first provide the classification of rotationally symmetric Yamabe solitons, which are globally conformally flat.
- Dsakalopoulos, S. All locally conformally flat complete Yamabe solitons with positive sectional curvature have to be rotationally symmetric.

PDE formulation of Yamabe solitons

- Proposition: Let $g_{ij} = u^{\frac{4}{n+2}}\, dx^2$ be a rotationally symmetric Yamabe gradient soliton $(R - \rho)g_{ij} = \nabla_i \nabla_j f$. Then, u is a smooth solution of the elliptic equation

$$\frac{n-1}{m}\,\Delta u^m + \beta\, x \cdot \nabla u + \gamma\, u = 0, \qquad \text{on } \mathbb{R}^n \tag{1}$$

where $\beta \geq 0$ and

$$\gamma = \frac{2\beta + \rho}{1 - m}, \qquad m = \frac{n-2}{n+2}.$$

In addition, any smooth solution of the above elliptic equation with β and γ as above defines a gradient Yamabe soliton.

Classification of rotationally symmetric Yamabe solitons

- Proposition: Let $m = \frac{n-2}{n+2}$. The elliptic equation admits non-trivial radially symmetric smooth solutions if and only if $\beta \geq 0$. More precisely, we have:
- Yamabe shrinkers $\rho = 1$: For any $\beta > 0$ and $\gamma = \frac{2\beta+1}{1-m}$, there exists an one parameter family u_λ, $\lambda > 0$, of smooth cigar solutions with $u_\lambda(x) = O(|x|^{-\frac{2}{1-m}})$, as $|x| \to \infty$. In the case $\gamma = \beta n$ the solutions are given in the closed form

$$u_\lambda(x) = \left(\frac{C_n}{\lambda^2 + |x|^2}\right)^{\frac{1}{1-m}}, \qquad C_n = (n-2)(n-1),$$

known as the Barenblatt solutions. When $\beta = 0$ and $\gamma = \frac{1}{1-m}$ we have the explicit solutions (spheres) of fast-decay rate

$$u_\lambda(x) = \left(\frac{C_n\,\lambda}{\lambda^2 + |x|^2}\right)^{\frac{2}{1-m}}, \qquad C_n = (4n(n-1))^{\frac{1}{2}}.$$

Classification of rotationally symmetric Yamabe solitons

- Yamabe expanders $\rho = -1$: For any $\beta > 0$ and $\gamma = \frac{2\beta - 1}{1-m} > -\frac{1}{1-m}$, there exists an one parameter family u_λ, $\lambda > 0$ of smooth solutions .
- Yamabe steady solitons $\rho = 0$: For any $\beta > 0$ and $\gamma = \frac{2\beta}{1-m} > 0$, there exists an one parameter family $u = u_\lambda$, $\lambda > 0$, of smooth solutions with $u = O((\frac{\log |x|}{|x|^2})^{\frac{1}{1-m}})$, as $|x| \to \infty$. We will refer to them as logarithmic cigars. If $\beta = 0$ and therefore $\gamma = 0$, then u is a constant, defining the euclidean metric on $\mathbb{R}^n$.
- In all of the above cases the solution u_λ is uniquely determined by its value at the origin.

Positive sectional curvature

- The logarithmic cigars and the Yamabe expanders found in the previous Proposition have strictly positive sectional curvatures as long as $\gamma > 0$. The Yamabe shrinkers have strictly positive sectional curvatures as long as $\beta > \frac{1}{n-2}$.
- PDE equation $\frac{n-1}{m}\Delta u^m + \beta x \cdot \nabla u + \gamma u = 0$ implies $R(0) = \gamma$. We show the shrinkers have scalar curvature bigger than $\rho = 1$ as long as $\beta > \frac{1}{n-2}$.
- It turns out the nonnegativity of sectional curvature K_0 which is the curvature of the 2-planes perpendicular to the spheres $\{x\} \times S^{n-1}$ is equivalent to the scalar curvature R being decreasing in distance r from the origin.
- The last follows from : R can not attain local minimum. We argue this using

$$(n-1)\Delta R + \beta x \cdot \nabla R v + R(R-\rho)v = 0,$$

where v is the conformal factor in cylindrical coordinates.

Rotational symmetry of Yamabe solitons

- Dsakalopoulos, S. All locally conformally flat complete Yamabe solitons with positive sectional curvature have to be rotationally symmetric.
- inspired by the proof of H.D.-Cao, Chen,Q. in the case of locally conformally flat complete steady Ricci solitons.
- f is the Yamabe soliton potential function. Let Σ_c be the level surface of f, that is

$$\Sigma_c = \{x \in M \ : \ f(x) = c\}.$$

- if c is the regular value, we can express the metric g as

$$g = \frac{1}{G(f,\theta)} df^2 + h_{ab}(f,\theta) d\theta^a \, d\theta^b,$$

where $G(f,\theta) = |\nabla f|^2$ and $\theta_2, \ldots \theta_n$ are the intrinsic coordinates for Σ_c.

Rotational symmetry for Yamabe solitons

- Goal: $G = G(f)$, $h_{ab} = h_{ab}(f)$ and (Σ_c, h_{ab}) is a space form with constant positive curvature. This would imply

$$g = \psi^2(f)df^2 + \phi^2(f)g_{S^{n-1}}.$$

- Identities on Yamabe solitons:

$$\nabla G = 2R\nabla f, \qquad (n-1)\nabla R = \mathrm{Ric}(\nabla f, \cdot).$$

- we show that the Ricci tensor of our soliton metric g has at most 2 distinct eigenvalues.
- we use the Harnack expression for the Yamabe flow, introduced by Chow, which is

$$Z(g, X) = (n-1)\Delta R + \langle \nabla R, X\rangle + \frac{1}{2(n-1)} R_{ij}X_iX_j + R^2.$$

The eigenvalues of Ricci tensor

- At any point $p \in \Sigma_c$, the Ricci tensor of g has either a unique eigenvalue λ, or it has two distinct eigenvalues λ and μ, of multiplicity 1 and $n-1$ respectively. In either case, $e_1 = \frac{\nabla f}{|\nabla f|}$ is an eigenvector with eigenvalue λ. Moreover, for any orthonormal basis $e_2, \dots e_n$ tangent to the level surface Σ_c at p, we have
- $\mathrm{Ric}(e_1, e_1) = \lambda$
- $\mathrm{Ric}(e_1, e_b) = R_{1b} = 0, \quad b = 2, \dots n$
- $\mathrm{Ric}(e_a, e_b) = R_{aa}\delta_{ab}, \;\; a, b = 2, \dots, n,$
- where either $R_{11} = \dots R_{nn} = \lambda$ or $R_{11} = \lambda$ and $R_{22} = \dots = R_{nn} = \mu$.

Eigenvalues of Ricci tensor

- $\mathrm{Ric} > 0$ so choose a vector field X to satisfy

$$\nabla_i R + \frac{1}{n-1} R_{ij} X_j = 0.$$

- Z evolves by

$$\Box Z = RZ + A_{ij} X_i X_j + g^{kl} R_{ij} (R g_{ik} - \nabla_i X_k)(R g_{jl} - \nabla_j X_l).$$

- in local coordinates $\{x_i\}$ where $g_{ij} = \delta_{ij}$ and the Ricci tensor is diagonal with eigenvalues $\{\lambda_1, \ldots \lambda_n\}$ we have
-
$$A_{ij} = \begin{pmatrix} \nu_1 & & \\ & \ddots & \\ & & \nu_n \end{pmatrix}.$$

where

$$\nu_i = \frac{1}{2(n-1)(n-2)} \sum_{k,l \neq i, k > l} (\lambda_k - \lambda_l)^2.$$

- Lemma: Let c be a regular value of f and $\Sigma_c = \{f = c\}$. Then,
- the function $G = |\nabla f|^2$ and the scalar curvature R are constant on Σ_c, that is, they are functions of f only.
- the mean curvature H of Σ_c is constant.
- the sectional curvature of the induced metric on Σ_c is constant.
- proof: let $\{e_1, e_2, \dots e_n\}$ be an orthonormal frame with $e_1 = \frac{\nabla f}{|\nabla f|}$ and $e_2, \dots e_n$ tangent to Σ_c.
- $\nabla G = 2R\nabla f \Rightarrow \nabla_a G = 0$,

$$(n-1)\nabla R = \mathrm{Ric}(\nabla f, \cdot) \Rightarrow (n-1)\nabla_a R = \mathrm{Ric}(\nabla f, e_a) = R_{1a} = 0.$$

for $a \in \{2, \dots n\}$.

Questions

- What is the classification of Yamabe solitons if we drop the assumption on being locally conformally flat?
- Is there an analogue of Perelman's W functional or the reduced volume functional for the Yamabe flow which will have a consequence that every finite time singularity model of a Type I singularity is a Yamabe shrinker?
- Examples of Type II singularities in the complete Yamabe flow.
- Classification of gradient shrinking Ricci solitons and the geometric properties of those.