Cyclicity one elliptic islands in the Standard family

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Workshop on Instabilities in Hamiltonian Systems
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16·06·2011

Definition (Chirikov-Taylor standard family of maps)

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $\kappa \in \mathbb{R}^+$, $\phi(x) = 2\pi^{-1}\sin(2\pi x)$; define $f_\kappa : \mathbb{T}^2 \to \mathbb{T}^2$

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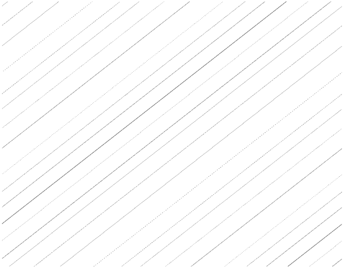
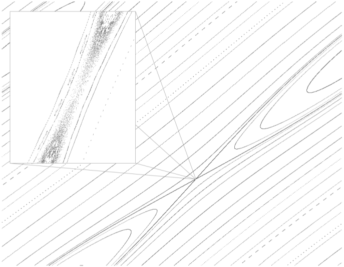


$\kappa = 0$



$0 < \kappa \ll 1$

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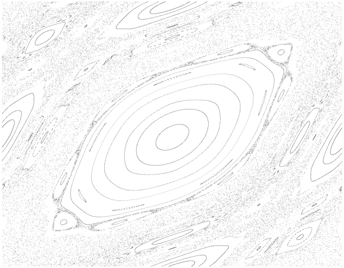
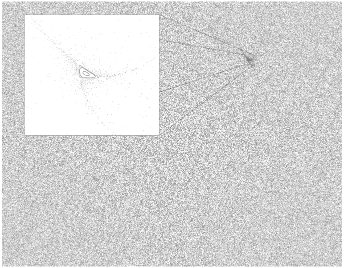


$\kappa \sim 1$



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Conjecture (Carleson·1991):

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Question (Sinai·1994)

Let $\mathcal{C}^* = \{\kappa \text{ s.t. } h_{\text{Leb}}(f_\kappa) > 0\}$; does any of the following hold?

$$\mathcal{C}^* = \mathbb{R} \setminus \{0\} \qquad \text{Leb}\, \mathcal{C}^* > 0 \qquad \mathcal{C}^* \neq \emptyset$$

By Pesin theory we know that if $h_{\text{Leb}}(f_\kappa) > 0$, there exist positive measure invariant sets on which f_κ is ergodic (*stochastic sea*)

Some related previous results

Theorem (Duarte·1994)

There exists $\kappa_0 > 0$ *and a residual set of parameters* $\mathcal{M}^D \subset [\kappa_0, \infty)$ s.t. *if* $\kappa \in \mathcal{M}^D$*, then* f_κ *has infinitely many elliptic periodic points accumulating on a locally maximal hyperbolic set that fills the torus as* $\kappa \to \infty$

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Theorem (Gorodetksi·2010)

There exists $\kappa_0 > 0$ *and a residual set of parameters* $\mathcal{M}^G \subset [\kappa_0, \infty)$ s.t. *if* $\kappa \in \mathcal{M}^G$*, then the* stochastic sea *of* f_κ *has full Hausdorff dimension*

Main results

Theorem (2011)

There exists $\kappa_0 > 0$ *and a dense set of parameters* $\mathcal{M} \subset [\kappa_0, \infty)$ *of Hausdorff dimension larger than ¼ such that if* $\kappa \in \mathcal{M}$*, then* f_κ *has infinitely many elliptic islands whose centers accumulate on a locally maximal hyperbolic set that fills the torus as* $\kappa \to \infty$

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There exists a $\kappa_0 > 0$ *such that for almost every* $\kappa \geq \kappa_0$ *the standard map* f_κ *has only finitely many cyclicity* 1 *elliptic islands*

Definitions

Basic observation

For large κ there exists a set $\mathbf{C}_\kappa \subset \mathbb{T}^2$ and cone fields $\mathscr{C}_\kappa^{\mathsf{u,s}}$ such that

$$\forall\, p \in \mathbb{T}^2 \setminus \mathbf{C}_\kappa \quad f_{\kappa *}\mathscr{C}_\kappa^{\mathsf{u}}|_p \subset \mathscr{C}_\kappa^{\mathsf{u}}|_{f_\kappa p} \quad f_\kappa^{*}\mathscr{C}_\kappa^{\mathsf{s}}|_p \subset \mathscr{C}_\kappa^{\mathsf{s}}|_{f_\kappa^{-1} p}$$

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Definition (κ-adapted slope field)

$h : \mathbb{T}^2 \to \mathbb{RP}$ smooth such that diffeomorphisms transform κh as the slope of a vector field. In particular:

$$f_{\kappa *}h = h_1 - \frac{1}{\kappa^2 h \circ f_\kappa^{-1}} \qquad f^*_\kappa h = h_{-1} + \frac{1}{\kappa^2 [h_1 - h] \circ f_\kappa}$$

$$h_1(x,y) = \ddot\phi(x) + 2\kappa^{-1} \qquad h_{-1}(x,y) = 0$$

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Fix $\alpha \in (½, 1)$ and $\tau \in \mathbb{R}^+$, then there exist $C^{\mathsf{u,s}}$ s.t.:

$$\mathbf{C}_\kappa(\alpha, \tau) = \{p \text{ s.t. } |h_1(p)| < \tau\kappa^{\alpha-1}\}$$

$$\mathscr{C}_\kappa^{\mathsf{u}}(\alpha, \tau) = \{h \text{ s.t. } \|h - h_1\| < C^{\mathsf{u}}\kappa^{-\alpha-1}\}$$

$$\mathscr{C}_\kappa^{\mathsf{s}}(\alpha, \tau) = \{h \text{ s.t. } \|h - h_{-1}\| < C^{\mathsf{s}}\kappa^{\alpha-1}\}$$

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Let $p \in \mathbb{T}^2$ a periodic point for f_κ of least period N:

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By invariance of $\mathscr{C}^{\mathrm{u,s}}$ outside $\mathbf{C}_\kappa$:

$$p \text{ elliptic} \quad \Rightarrow \quad s(p) > 0$$

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Conjecture

There exists $\kappa_0 > 0$ such that for almost every $\kappa \geq \kappa_0$ the standard map f_κ has only finitely many elliptic islands of either bounded cyclicity or period larger than some N_s

Preliminary construction

- Fix α, τ; choose pair of critical sets $\hat{\mathbf{C}}_\kappa \subset \mathbf{C}_\kappa$:

$$\mathbf{C}_\kappa = \mathbf{C}_\kappa(\alpha, \tau) \qquad \hat{\mathbf{C}}_\kappa = \mathbf{C}_\kappa(\alpha, \tau/10)$$

Recall $\mathbf{C}_\kappa(\alpha, \tau)$ is a $\mathcal{O}\left(\tau\kappa^{\alpha-1}\right)$ strip around critical points of $\dot{\phi}$

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- Construct locally maximal hyperbolic set $\mathbf{\Lambda}_\kappa \subset \mathbb{T}^2 \setminus \hat{\mathbf{C}}_\kappa$ w/ expansion rate $> \kappa^\alpha$; Markov partition (of $\mathbf{\Lambda}_\kappa$):

$$\mathbf{\Lambda}_\kappa \ni p \mapsto \cdots a_{-2}a_{-1}a_0a_1a_2\cdots$$

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- Extend the construction to a *finite time Markov structure*:

$$\mathbb{T}^2 \setminus \mathbf{C}_\kappa \ni p \mapsto \langle \tilde{a}_{-*}\cdots\tilde{a}_{-1}\tilde{a}_0\tilde{a}_1\cdots\tilde{a}_*\rangle$$

Symbols $\tilde{a}$ belong to some extended alphabet

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$$\mathbf{C}_\kappa \ni p \mapsto \omega(p) = \langle \tilde{a}_1\cdots\tilde{a}_n\rangle$$

Note: if $p \in \mathbf{C}_k \cap f_\kappa^{-1}\mathbf{C}_k$ we set $p \mapsto \langle\rangle$

Main lemmata

$\mathcal{E}_\omega = \{\kappa$ s.t. $\exists\, p \in \mathbf{C}_\kappa,\ p$ cyclicity 1 elliptic p.p. for $f_\kappa,\ \omega(p) = \omega\}$

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- Moreover notice:

$$\sum_N \sum_{|\omega|=N} \mathrm{Leb}(\mathcal{J}_m \cap \mathcal{E}_\omega) \sim m^{-1}$$

  There are *no* cyclicity 1 elliptic p.p in a parameter set of density approaching 1 as $m \to \infty$ (i.e. Carleson conjecture holds for cyclicity 1).

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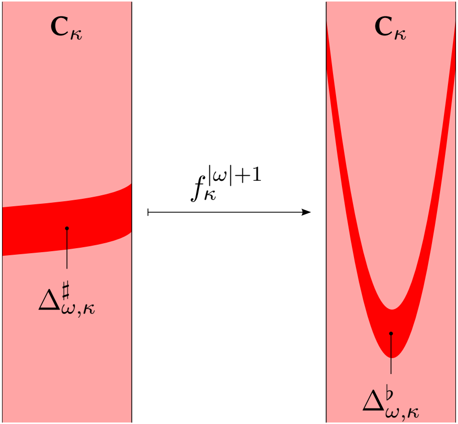
- Construct a Cantor set in $\mathcal{B}$ by induction using the lemma fixing the middle symbols according to the enumeration $\square$

Define $\Delta^\sharp_{\omega,\kappa} = \{p \in \mathbf{C}_\kappa \text{ s.t. } \omega(p) = \omega\}$; $\Delta^\flat_{\omega,\kappa} = f_\kappa^{|\omega|+1}\Delta^\sharp_{\omega,\kappa}$



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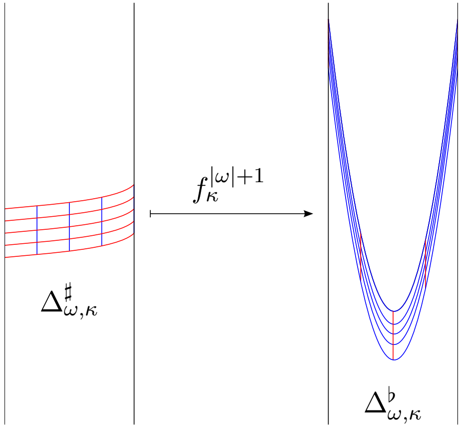
Adapted coordinates on $\Delta^\sharp_{\omega,\kappa}$ and $\Delta^\flat_{\omega,\kappa}$



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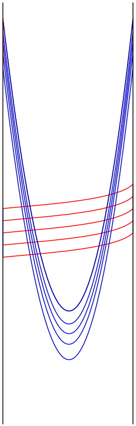

Leaf functions: $\ell^{\sharp}_{\omega}[\zeta;\kappa](\xi)$ and $\ell^{\flat}_{\omega}[\eta;\kappa](\xi)$

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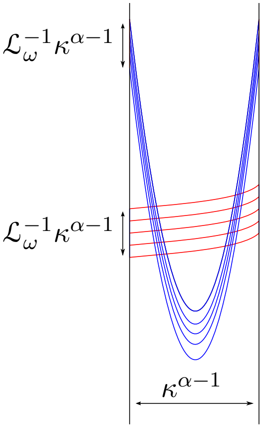

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Geometrical bounds:

$$\mathcal{L}_\omega \sim \sup_{p\in\Delta^\sharp_{\omega,\kappa}} \frac{\mathrm{d}x_{|\omega|+1}}{\mathrm{d}y_0} \geq \kappa^{|\omega|\alpha}$$

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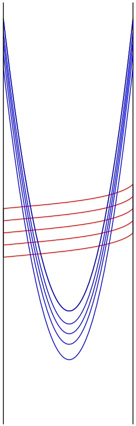
Adapted coordinates on $\Delta^{\sharp}_{\omega,\kappa}$ and $\Delta^{\flat}_{\omega,\kappa}$



Leaf functions: $\ell^{\sharp}_{\omega}[\zeta;\kappa](\xi)$ and $\ell^{\flat}_{\omega}[\eta;\kappa](\xi)$

Geometrical bounds:

$$\mathcal{L}_\omega \sim \sup_{p\in\Delta^{\sharp}_{\omega,\kappa}} \frac{\mathrm{d}x_{|\omega|+1}}{\mathrm{d}y_0} \geq \kappa^{|\omega|\alpha}$$

Dependence on κ:

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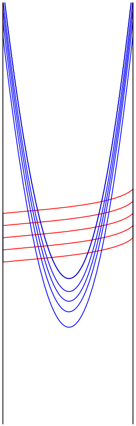

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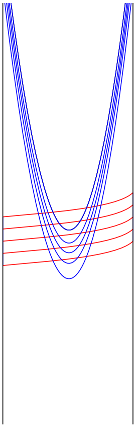

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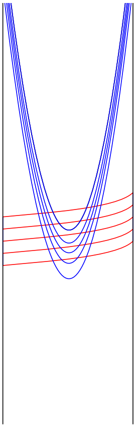

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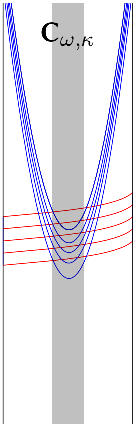

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ω-adapted $\mathbf{C}_{\omega,\kappa}$ of thickness $\sim \kappa^{-1}\mathcal{L}_\omega^{-1}$

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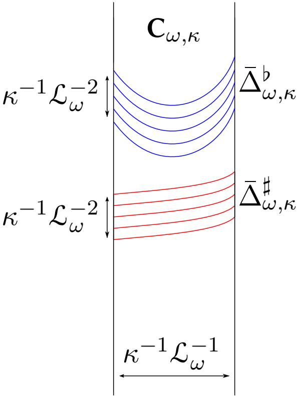

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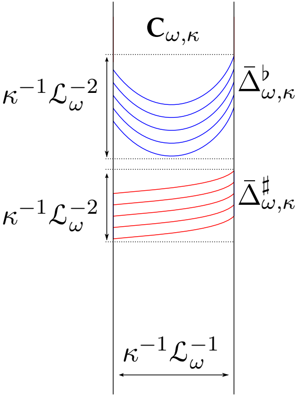

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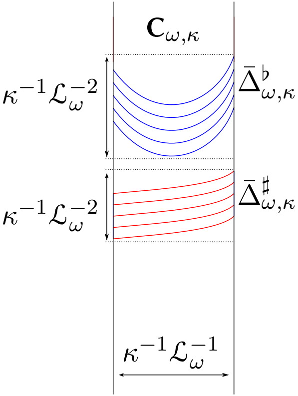

If $p \in \Delta_{\omega,\kappa}$ cyclicity 1 elliptic p.p.:

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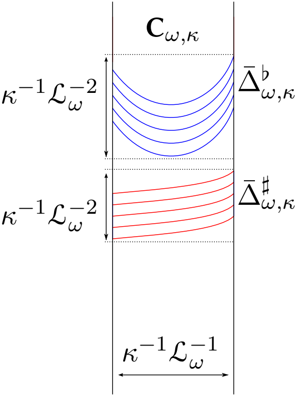

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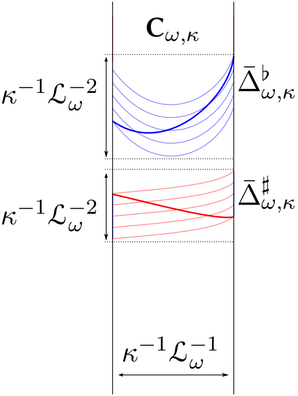

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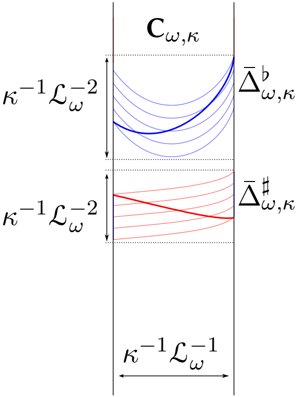

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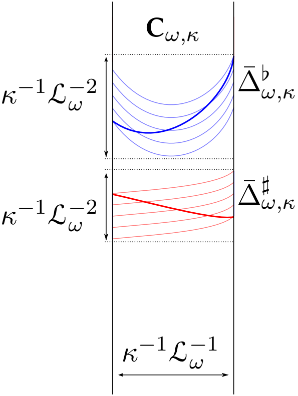

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$+$ at most 1 elliptic p.p. with given ω

Density argument

Observation

$\Delta^{\sharp}_{\omega,\kappa}$ and $\Delta^{\sharp}_{\omega',\kappa}$ are close if the *first* symbols of ω, ω' agree
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Definition (Bicylinder of rank r)

$$\Omega = \{\omega \text{ with prescribed first and last } r \text{ symbols}\}$$

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$$\omega = \langle \underbrace{a_1 a_2 \cdots a_r}_{\Omega^\sharp} * * * \cdots * * * \underbrace{a_{n-r} a_{n-r+1} \cdots a_{n-2} a_{n-1}}_{\Omega^\flat} \rangle$$

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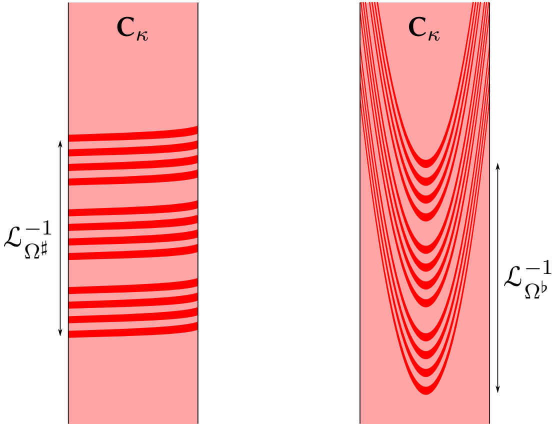

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- Arbitrary choice of central $\log N$ symbols in $\omega(p)$

Final remarks

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- The construction gives plenty of information on the parameter space. In particular it seems it can be used to study higher cyclicity orbits *without* a geometrical understanding of multiple passages through the critical set.
- The construction provides estimates on the size of elliptic islands; such estimates seem to be sharp for cyclicity 1.