Continuous averaging proof of the Nekhoroshev theorem

Jinxin Xue

University of Maryland at College Park

Fields Institute, Workshop on Instabilities in Hamiltonian Systems

The Nekhoroshev theorem says: Suppose we have a Hamiltonian system

$$H = H_0(I) + \varepsilon H_1(I, \theta), (I, \theta) \in \mathbb{R}^n \times \mathbb{T}^n$$

Theorem (Nekhoroshev)

When the unperturbed Hamiltonian H_0 is quasi-convex (the energy surface $H_0(I) = E$ is strictly convex) the following general estimate holds:
$\|I(t) - I(0)\| \leq C_1\varepsilon^b$ *when* $t \leq \mathcal{T}, \mathcal{T} = O(\exp(C_2/\varepsilon^a))$

$$a = b = \frac{1}{2n} \quad \text{(Lochak-Neishtadt, Pöschel)}$$

$$a = \frac{1}{2(n-1)} - \delta, \quad b = \delta(n-1), \quad 0 < \delta \leq \frac{1}{2n(n-1)}$$

$$\text{(Bounemoura, Marco)}$$

Theorem (X)

$$C_2 = \left(\frac{M_-}{M^+}\right)^{3/2} \frac{\rho_1}{8\sqrt{n}}$$

$$M_- Id \leq Hess H_0 \leq M^+ Id$$

Motivation: estimate stability time for concrete system

- Nekhoroshev theorem(Lochak's proof)

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- The proof

- Local:
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  Geometric part.
- From Local to global:
  Number theoretical part: Dirichlet's simultaneous approximation theorem.

Neishtadt's theorem on single frequency averaging

$$\begin{cases} \dot{\theta} = \omega(I) + \varepsilon f(I,\theta) \\ \dot{I} = \varepsilon g(I,\theta) \quad (I,\theta) \in \mathbb{R}^m \times \mathbb{T}^n \end{cases} \Longrightarrow \begin{cases} \dot{\psi} = \Omega(J,\varepsilon) + \varepsilon\alpha(J,\psi) \\ \dot{J} = \varepsilon\phi(J,\varepsilon) + \varepsilon\beta(J,\psi) \end{cases}$$

$$\text{If} n = 1, \quad \alpha, \beta \sim \exp(-C/\varepsilon)$$

C is determined by the complex singularity of θ (Treschev).

We need to fix a rational frequency $\omega^* \in Q^n$ and expand the Hamiltonian in the following form:

$$H = \langle I, \omega^* \rangle + G(I) + \varepsilon \bar{H} + \varepsilon \tilde{H}$$

If we do the Fourier expansion of the perturbation εH_1, then

$$\bar{H} : \langle k, \omega^* \rangle = 0, \quad \text{resonant}, \quad \langle \omega^*, \frac{\partial \bar{H}}{\partial \theta} \rangle = 0$$

$$\tilde{H} : \langle k, \omega^* \rangle \neq 0, \quad \text{nonresonant}$$

Example

Consider frequency $\omega^* = (1, 0, ..., 0) \in \mathbb{Q}^n$,

$$\bar{H} = \bar{H}(I, \theta_2, \theta_3, ..., \theta_n), \quad e^{i\langle k,\theta\rangle}, \quad k_1 = 0$$
$$\tilde{H} = \tilde{H}(I, \theta_1, \theta_2, ..., \theta_n), \quad e^{i\langle k,\theta\rangle}, \quad k_1 \neq 0$$

almost first integral

If we can kill the $\tilde{H}$ term to be exponentially small, we get an "almost first integral"

$$\langle \omega^*, I \rangle$$

In the sense that:

$$\frac{d}{dt}\langle \omega^*, I \rangle = -\varepsilon \langle \omega^*, \frac{\partial H_1(I,\theta)}{\partial \theta} \rangle = -\varepsilon \langle \omega^*, \frac{\partial \tilde{H}(I,\theta)}{\partial \theta} \rangle = O(\varepsilon)$$

Over exponentially long time.

intersection of a hyperplane with energy surface

We have two first integrals: the Hamiltonian and $\langle \omega^*, I \rangle$, we consider their intersection.

$$\{\langle \omega^*, (I(t) - I_0)\rangle = 0\} \bigcap \{H_0(I(t)) = H_0(I_0)\}$$

$$\{\text{hyperplane}\} \bigcap \{\text{convex energy surface}\}$$

analytic part

Recall:

$$H = H_0(I) + \varepsilon H_1(I, \theta)$$

Split it in the form:

$$H = \langle \omega^*, I \rangle + G(I) + \varepsilon \bar{H}(I, \theta) + \varepsilon \tilde{H}(I, \theta)$$

Goal: Use the continuous averaging, kill $\tilde{H}$ to

$$\exp\left(-\frac{2\pi\rho_1}{M^+\mathcal{R}T}\right)$$

$\mathcal{R}$: size of working region. $\quad |I| \leq \mathcal{R}$

From local to global:

Dirichlet theorem for simultaneous approximation:
For any $\alpha \in R^n, Q \in \mathbb{R}^1, \; and \; Q > 1$.
There exists an integer q, $1 \leq q < Q$, s.t.

$$\|q\alpha - \mathbb{Z}^n\|_\infty \leq Q^{-1/n}$$

Derivation of continuous averaging

$$\mathcal{L}_F H = \{H, F\}$$

$$H_\delta = \{H, F\}$$

Change of variables vs. evolution of Hamiltonian

The Hilbert transform

Fourier expansion

$$H_1(I,\theta) = \sum_k H^k(I) e^{i\langle k,\theta\rangle}$$

Then Define:

$$F(I,\theta) = i \sum_k \sigma_k H^k(I) e^{i\langle k,\theta\rangle}$$

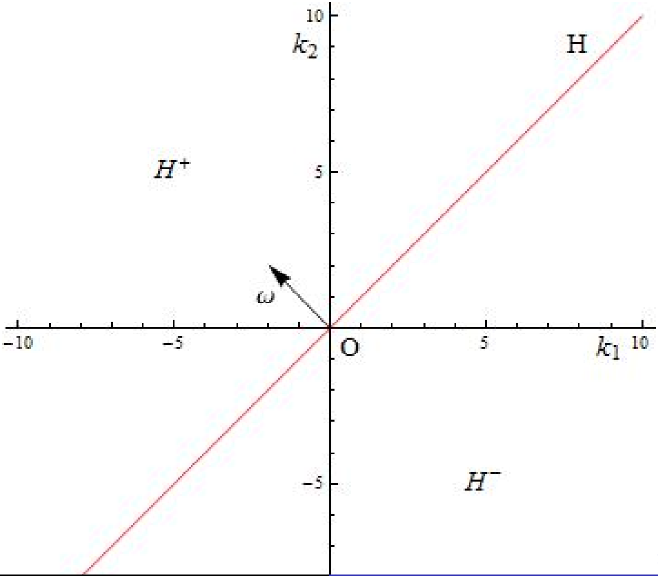
$$\sigma_k = sign(\langle k, \omega^* \rangle)$$

Example:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

Figure



comparison with the iterative Lie method

$$H(I,\theta) = H_0(I) + \varepsilon H_1(I,\theta)$$

$$\frac{dH}{dt} = \mathcal{L}_{\varepsilon F} H = \{H, \varepsilon F\}$$

$$e^{\mathcal{L}_{\varepsilon F}} H = H + \{H, \varepsilon F\} + \frac{1}{2}\{\{H, \varepsilon F\}, \varepsilon F\} + \ldots$$

$$= H_0 + \varepsilon H_1 + \varepsilon\{H_0, F\} + O(\varepsilon^2)$$

cohomological equation,

$$H_1 + \{H_0, F\} = 0$$

Fourier expansion gives:

$$H_1(I, \theta) = \sum_{k \in \mathbb{Z}} H^k(I) e^{ik\theta}$$
$$F(I, \theta) = \sum_{k \in \mathbb{Z}} F^k(I) e^{ik\theta}$$

In fact, we are only able to solve

$$H_1 - H^0 + \{H_0, F\} = 0$$

Fourier coefficients:

$$H^k(I) + i\langle k, \omega\rangle F^k = 0, \;\; k \neq 0$$
$$F^k = i\frac{H^k(I)}{\langle k, \omega\rangle} \;\; \omega := \frac{\partial H_0}{\partial I}$$

Continuous averaging for Nekhoroshev

$$H_\delta = -\{F, H\}$$

$$H_\delta = -\{F, \langle \omega^*, I\rangle\} - \{F, G\} - \{F, \varepsilon \bar{H}\} - \{F, \varepsilon \tilde{H}\}$$

$$\Longrightarrow$$

$$\begin{cases} \bar{H}_\delta = -\overline{\{\xi \tilde{H}, \tilde{H}\}} \\ \tilde{H}_\delta = -\{\xi \tilde{H}, \langle \omega^*, I\rangle\} - \{\xi \tilde{H}, G\} - \{\xi \tilde{H}, \varepsilon \bar{H}\} - \widetilde{\{\xi \tilde{H}, \varepsilon \tilde{H}\}} \end{cases}$$

Use the antisymmetricity of the Poisson bracket, we obtain the following.

$$\bar{H}_\delta = -2i\varepsilon\overline{\{H^+, H^-\}}$$
$$H_\delta^+ = -i\{H^+, H_0\} - i\varepsilon\{H^+, \bar{H}\} - 2i\varepsilon\widetilde{\{H^+, H^-\}}^+$$
$$H_\delta^- = i\{H^-, H_0\} + i\varepsilon\{H^-, \bar{H}\} - 2i\varepsilon\widetilde{\{H^+, H^-\}}^-$$

Linearization

$$\bar{H}_\delta = 0$$

$$H_\delta^+ = -i\{H^+, \langle\omega^*, I\rangle + G\} = i\langle\omega^*, \frac{\partial H^+}{\partial\theta}\rangle - i\frac{\partial G}{\partial I}\frac{\partial H^+}{\partial\theta}$$

$$H_\delta^- = i\{H^-, \langle\omega^*, I\rangle + G\} = -i\langle\omega^*, \frac{\partial H^-}{\partial\theta}\rangle + i\frac{\partial G}{\partial I}\frac{\partial H^-}{\partial\theta}$$

Property of the rational frequency

ω^* is a rational vector $\frac{1}{q}(p_1, p_2, ..., p_n), \quad p_i, q \in \mathbb{Z}$.

So the period T of this vector is:

$$2\pi/T = \frac{1}{q} g.c.d.(p_1, p_2, ..., p_n)$$

Those k's with $\langle \omega^*, k \rangle \neq 0$, give us

$$|\langle k, \omega^* \rangle| = \frac{1}{q}|k.(p_1, p_2, ..., p_n)| \geq \frac{2\pi}{T}$$

decay of Fourier coefficients, I

1. For $\langle k, \omega^* \rangle > 0$, $H^k e^{i\langle k,\theta\rangle}$,

$$H_\delta^+ = i\langle \omega^*, \frac{\partial H^+}{\partial \theta} \rangle$$

$$H_\delta^k e^{i\langle k,\theta\rangle} = i\langle \omega^*, \frac{\partial}{\partial \theta}(H^k e^{i\langle k,\theta\rangle}) \rangle = -|\langle k, \omega^* \rangle| H^k e^{i\langle k,\theta\rangle}$$

$$\Longrightarrow H^k(\delta) = e^{-|\langle k,\omega^* \rangle|\delta} H^k(0) \leq e^{-\frac{2\pi\delta}{T}} |H^k(0)|$$

decay of Fourier coefficients, II

2. For $\langle k, \omega^* \rangle < 0$, $H^k e^{i\langle k,\theta\rangle}$

$$H_\delta^- = -i\langle \omega^*, \frac{\partial H^-}{\partial \theta}\rangle$$

$$H_\delta^k e^{i\langle k,\theta\rangle} = -i\langle \omega^*, \frac{\partial}{\partial \theta}(H^k e^{i\langle k,\theta\rangle})\rangle = -|\langle k, \omega^*\rangle| H^k e^{i\langle k,\theta\rangle}$$

$$\Longrightarrow H^k(\delta) = e^{-|\langle k,\omega^*\rangle|\delta} H^k(0) \le e^{-\frac{2\pi\delta}{T}} |H^k(0)|$$

imaginary flow

$$H_\delta^+ = -i\frac{\partial G}{\partial I}\frac{\partial H^+}{\partial \theta}$$

The imaginary flow,

$$\frac{d\theta}{d\delta} = i\frac{dG}{dI}, \quad \theta(\delta) = \theta(0) + iG'\delta$$

Characteristic method,

$$\Longrightarrow \frac{dH^+}{d\delta} = 0$$

imaginary flow: continued

$$H^k e^{i\langle k,\theta\rangle} \simeq e^{-|k|\rho}.e^{i\langle k,\theta(0)+iG'\delta\rangle}$$
$$= e^{-|k|\rho-\langle k,G'\delta\rangle}.e^{i\langle k,\theta(0)\rangle}$$

$$|k|\rho > |\langle k, G'\delta\rangle|$$

Speed of the imaginary flow has upper bound

$$|\frac{dG}{dI}| \le M^+\mathcal{R}$$

$$\Longrightarrow \delta < \frac{\rho}{M^+\mathcal{R}}$$

The Stopping time

estimate of constant

$$M^{+}\mathcal{R}\delta \leq \rho$$

$$e^{-\dfrac{2\pi\delta}{T}}$$

$$\Longrightarrow e^{-\dfrac{2\pi\rho}{M^{+}\mathcal{R}T}}$$

THANK YOU!