Entropy in Measurable Dynamics

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Two systems (G, X_1, μ_1) and (G, X_2, μ_2) are *isomorphic* if there exists a measure-space isomorphism $\phi : X_1 \to X_2$ with $\phi(gx) = g\phi(x)$ for a.e. $x \in X_1$ and for all $g \in G$.

Main Problem: Classify systems up to isomorphism.

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- G acts on K^G by shifting. $(gx)(f) = x(g^{-1}f)$ for all $x \in K^G, g, f \in G$.
- (G, K^G, κ^G) is the *Bernoulli shift* over G with *base space* (K, κ).

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von Neumann's question: Is the full 2-shift over $\mathbb{Z}$ isomorphic to the full 3-shift over $\mathbb{Z}$?

Ideas from Information Theory

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$I(t)$ for $0 \leq t \leq 1$ should satisfy:

1. $I(t) \geq 0$.
2. $I(t)$ is continuous.
3. $I(ts) = I(t) + I(s)$.

So $I(t) = -\log_b(t)$ for some $b > 1$.

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Let $T : X \to X$ be measure-preserving. The *entropy rate* of ϕ w.r.t T is:

$$h(T, \phi) = \lim_{n \to \infty} \frac{1}{2n+1} H\Big(\bigvee_{i=-n}^{n} \phi \circ T^i\Big).$$

Coding

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ϕ is a generator if Φ is an isomorphism from (G, X, μ) to $(G, A^G, \Phi_*\mu)$.

Kolmogorov's entropy

Theorem (Kolmogorov, 1958)

Let $T : X \to X$ be an automorphism of (X, μ). If ϕ and ψ are finite-entropy generators for $(\mathbb{Z}, X, \mu) = (\langle T \rangle, X, \mu)$ then $h(T, \phi) = h(T, \psi)$.

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Theorem (Sinai, 1959)

If ϕ is any finite-entropy observable then $h(T, \phi) \leq h(\mathbb{Z}, X, \mu)$. Hence we may define the entropy of $(\mathbb{Z}, X, \mu)$ to be $\sup_\phi h(T, \phi)$.

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Theorem (Kolmogorov, 1958)

If $(\mathbb{Z}, K^{\mathbb{Z}}, \kappa^{\mathbb{Z}})$ is isomorphic to $(\mathbb{Z}, L^{\mathbb{Z}}, \lambda^{\mathbb{Z}})$ then $H(K, \kappa) = H(L, \lambda)$. So the full 2-shift is not isomorphic to the full 3-shift.

Questions

- Does the converse hold?

- What if $\mathbb{Z}$ is replaced with some other group G?

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Definition

A group G is *Ornstein* if whenever (K, κ), (L, λ) are two standard probability spaces with $H(\kappa) = H(\lambda)$ then (G, K^G, κ^G) is isomorphic to (G, L^G, λ^G).

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- If G contains an Ornstein subgroup H then G is Ornstein [Stepin, 1975].
- Is every countably infinite group Ornstein?

Classification

Theorem (Ornstein, 1970)

Bernoulli shifts over $\mathbb{Z}$ are completely classified by their entropy.

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Theorem

If G is infinite and amenable then Bernoulli shifts over G are completely classified by their entropy (which equals their base measure entropy).

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What if G is nonamenable?

Factor maps

Definition

Let (G, X, μ), (G, Y, ν) be two systems and $\phi : X \to Y$ a measurable map with $\phi_* \mu = \nu$, $\phi(gx) = g\phi(x)$ for a.e. $x \in X$ and all $g \in G$. Then ϕ is a *factor map* from (G, X, μ) to (G, Y, ν).

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$\Longrightarrow$ the full 2-shift over G cannot factor onto the full 4-shift over G.

The Ornstein-Weiss Example

Theorem (Ornstein-Weiss, 1987)

If $\mathbb{F} = \langle a, b \rangle$ is the rank 2 free group then the full 2-shift over $\mathbb{F}$ factors onto the full 4-shift over $\mathbb{F}$.

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More Counterexamples

Theorem (Karen Ball, 2005)

If G has infinitely many ends then the 2-shift over G factors onto every Bernoulli shift over G.

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Theorem

If G contains a nonabelian free subgroup then every nontrivial Bernoulli shift over G factors onto every other Bernoulli shift over G.

New Results

Theorem

If G is a sofic group (e.g., a linear group) then Kolmogorov's direction holds. I.e., if (G, K^G, κ^G) is isomorphic to (G, L^G, λ^G) then $H(K, \kappa) = H(L, \lambda)$.

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The idea: For $n > 0$, count the number of sequences $(a_1, a_2, \ldots, a_n)$ with elements $a_i \in A$ that approximate the above sequence.

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$\phi^W_* \mu$ is a measure on A^W that encodes the local statistics .

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Let u be the uniform measure on $\{1, \ldots, n\}$. $\psi^W_* u$ is a measure on A^W that encodes the local statistics of the sequence $(\psi(1), \ldots, \psi(n)) \in A^n$.

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Theorem

$$h(T, \phi) = \inf_{W \subset \mathbb{Z}} \inf_{\epsilon > 0} \lim_{n \to \infty} \frac{1}{n} \log \left| \{ \psi : \{1, \dots, n\} \to A \ : \ d_W(\phi, \psi) < \epsilon \} \right|.$$

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For $W \subset G$, let $\mathcal{G}(W) \subset \{1, \ldots, m\}$ be the set of all p such that

$$\begin{aligned} \sigma(fg)p &= \sigma(f)\sigma(g)p \ \ \forall f, g \in W \text{ with } fg \in W, \\ \sigma(f)p &\neq \sigma(g)p \ \ \Leftarrow f \neq g \in W. \end{aligned}$$

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σ is a *(W, ϵ)-approximation* to G if $|\mathcal{G}(W)| \geq (1 - \epsilon)m$.

Sofic Groups

A sequence $\Sigma = \{\sigma_i\}_{i=1}^{\infty}$ of maps $\sigma_i : G \to \mathrm{Sym}(m_i)$ is a *sofic approximation* if σ_i is an (W_i, ϵ_i)-approximation with $\epsilon_i \to 0$ and $W_i \to G$ (i.e., $\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} W_i = G$).

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- Is every countable group sofic?

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$\phi : X \to A$ be a measurable map into a finite set.

Entropy for Sofic Groups

Let (G, X, μ) be a system,

$\Sigma = \{\sigma_i\}$ be a sofic approximation to G where $\sigma_i : G \to \mathrm{Sym}(m_i)$,

$\phi : X \to A$ be a measurable map into a finite set.

The idea: Count the number of observables $\psi : \{1, \ldots, m_i\} \to A$ so that $(G, [m_i], u_i, \psi)$ approximates (G, X, μ, ϕ).

Approximating

If $W \subset G$ is finite, let $\phi^W \colon X \to A^W$ be the map $\phi^W(x) := \Big(\phi(wx)\Big)_{w \in W}$.

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Let $d_W(\phi, \psi)$ be the l^1-distance between $\phi^W_* \mu$ and $\psi^W_* u$.

Entropy for sofic groups

$$h(\Sigma,\phi) := \inf_{W\subset G}\inf_{\epsilon>0}\limsup_{i\to\infty}\frac{\log\left|\{\psi:\{1,\dots,m_i\}\to A\ :\ d_W(\phi,\psi)\le\epsilon\}\right|}{m_i}.$$

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If ϕ_1 and ϕ_2 are generating then $h(\Sigma, \phi_1) = h(\Sigma, \phi_2)$. So let $h(\Sigma, G, X, \mu)$ be this common number.

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ϕ and ψ are *combinatorially equivalent* if there exists finite subsets $K, L \subset G$ such that ϕ^K refines ψ and ψ^L refines ϕ.

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Proposition

If ϕ is a simple splitting of ψ then $h(\Sigma, \phi) = h(\Sigma, \psi)$.

Applications: von Neumann algebras

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Major problem: classify these algebras up to isomorphism in terms of the group/action data.

Theorem (Connes, 1976)

If G is infinite and amenable and the action $G \curvearrowright (X, \mu)$ is free and ergodic then $L^\infty(X, \mu) \rtimes G$ is hyperfinite. In particular, all such algebras are isomorphic.

Rigidity

Definition

(G_1, X_1, μ_1) and (G_2, X_2, μ_2) are *von Neumann equivalent (vNE) if* $L^\infty(X_1, \mu_1) \rtimes G_1 \cong L^\infty(X_2, \mu_2) \rtimes G_2$.

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Theorem (Popa, 2006)

If G is an ICC property T group then any two von Neumann equivalent Bernoulli shifts over G are isomorphic.

Corollary

If, in addition, G is sofic and Ornstein then Bernoulli shifts over G are classified up to vNE by base measure entropy. E.g., this occurs when $G = PSL_n(\mathbb{Z})$ for $n > 2$.

Applications: orbit equivalence

Definition

(G_1, X_1, μ_1) is orbit equivalent (OE) to (G_2, X_2, μ_2) if there exists a measure-space isomorphism $\phi : X_1 \to X_2$ such that $\phi(G_1 x) = G_2\phi(x)$ for a.e. $x \in X_1$.

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Theorem (Dye 1959, Connes-Feldman-Weiss 1981)

If G_1 and G_2 are amenable and infinite and their respective actions are ergodic and free then (G_1, X_1, μ_1) is OE to (G_2, X_2, μ_2).

OE rigidity

Theorem (Kida, 2008)

Let G be the mapping class group of a genus g surface with n holes. Assume $3g + n - 4 > 0$ *and* $(g, n) \notin \{(1, 2), (2, 0)\}$*. If* (G, X, μ) *is free and ergodic then it is strongly orbitally rigid. I.e., if* (G_2, X_2, μ_2) *is free, ergodic and OE to* (G, X, μ) *then it is isomorphic to* (G, X, μ)*.*

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If G is as above then Bernoulli shifts over G are classified up to OE by base measure entropy.

Free Groups: a special case

Let $\mathbb{F} = \langle s_1, \ldots, s_r \rangle$. Let $\mathbb{F}$ act on (X, μ).

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Theorem

If ϕ_1 and ϕ_2 are generating then $f(\phi_1) = f(\phi_2)$. So we may define $f(\mathbb{F}, X, \mu) = f(\phi_1)$. Moreover, $f(\mathbb{F}, K^{\mathbb{F}}, \kappa^{\mathbb{F}}) = H(K, \kappa)$.

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For each $n \geq 1$, let $\sigma_n : \mathbb{F} = \langle s_1, \ldots, s_r \rangle \to \mathrm{Sym}(n)$ be chosen uniformly at random.

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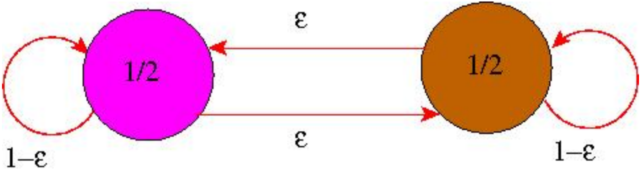
$h_*(\phi) = f(\phi)$.

A Markov chain example



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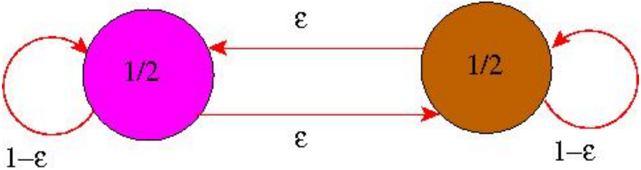
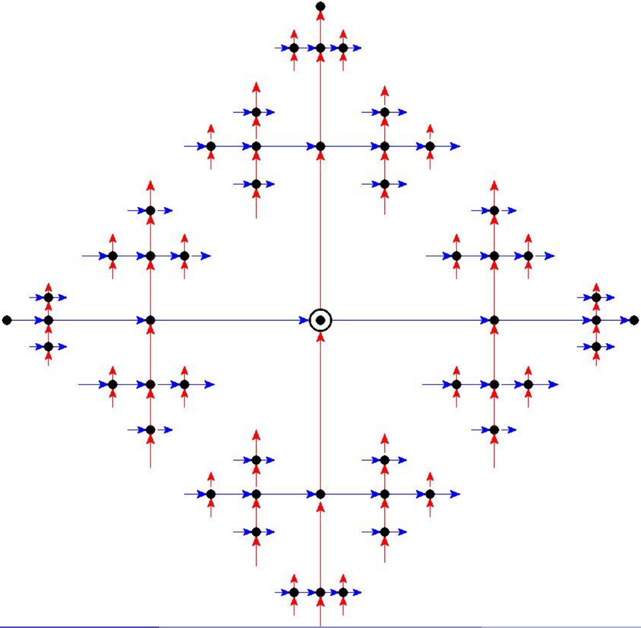
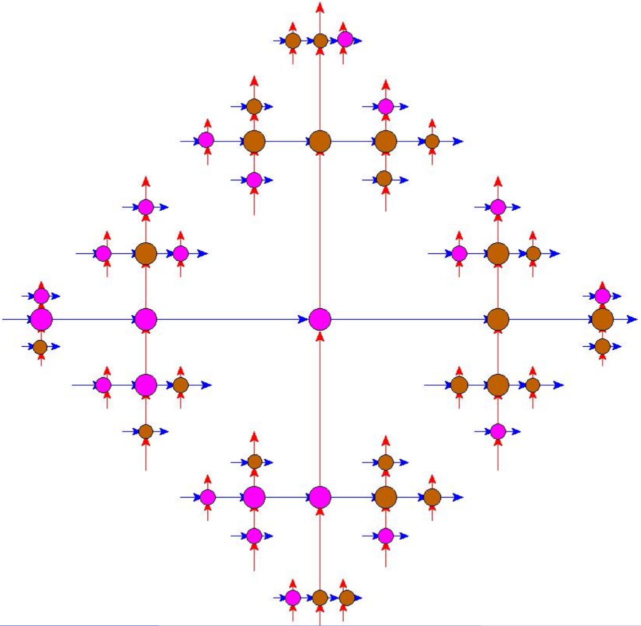


The Cayley graph



The Ising model



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$$F(\mu_\epsilon, \phi) = h_*\big(\mathbb{F}, \{magenta, brown\}^{\mathbb{F}}, \mu_\epsilon\big).$$

Systems of algebraic origin

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If $\mathbb{F}$ acts by automorphisms on $\mathcal{G}$ with closed normal subgroup $\mathcal{N}$ then $f(\mathbb{F}, \mathcal{G}, Haar_{\mathcal{G}}) = f(\mathbb{F}, \mathcal{N}, Haar_{\mathcal{N}}) + f(\mathbb{F}, \mathcal{G}/\mathcal{N}, Haar_{\mathcal{G}/\mathcal{N}})$.

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Let $\mathcal{G} = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{F}}$. Let $\mathcal{N} = \{\mathbf{0}, \mathbf{1}\}$. By Ornstein-Weiss' example,

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