Exact critical behavior for pinning model in correlated environment

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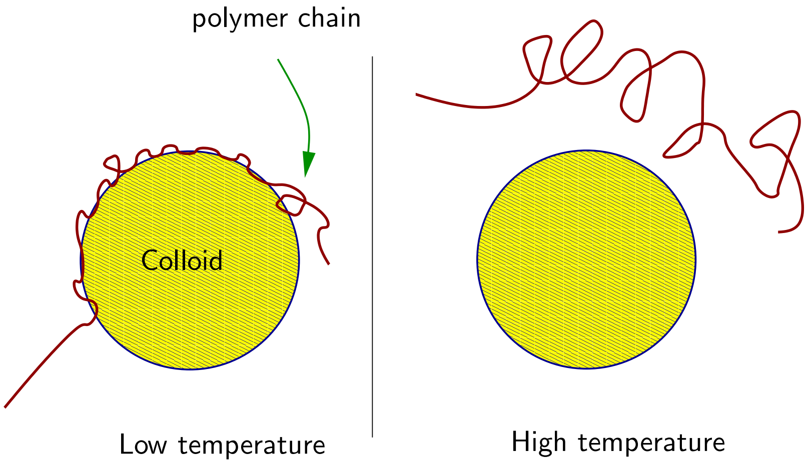
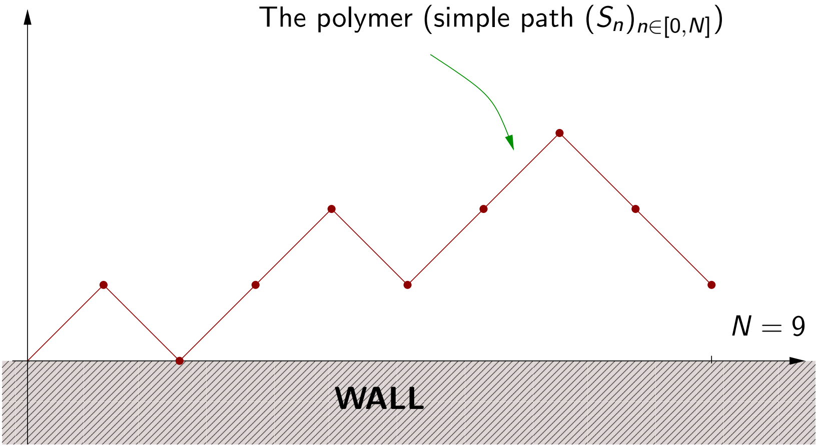
Example of Physical motivation : polymer in a solution with colloids

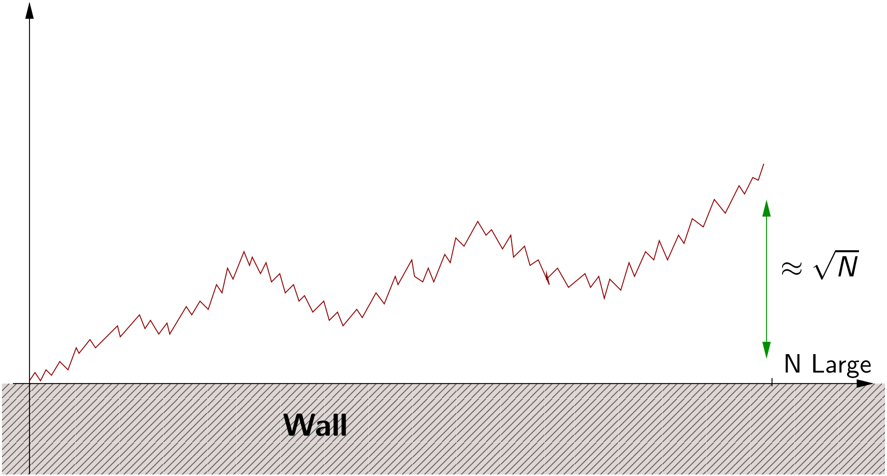


A simple modelization



The trajectory is chosen uniformly among trajectories $(S_n)_{n\in[0,N]}$ satisfying $(S_{n+1} - S_n) = \pm 1$ and $S_n \geq 0$ for all n and $S_0 = 0$.

Entropic repulsion



Taking into account polymer/interface interaction

To each trajectory, one associate S a potential energy

$$H_N(S) = \#\{ \text{ contacts of } S \text{ with the wall } \} = \#\{n : S_n = 0\}. \qquad (1)$$

Given h, one denotes by $\mu_N^{(h)}$ the probability law for which each path has probability proportionnal to $\exp(hH_N)$

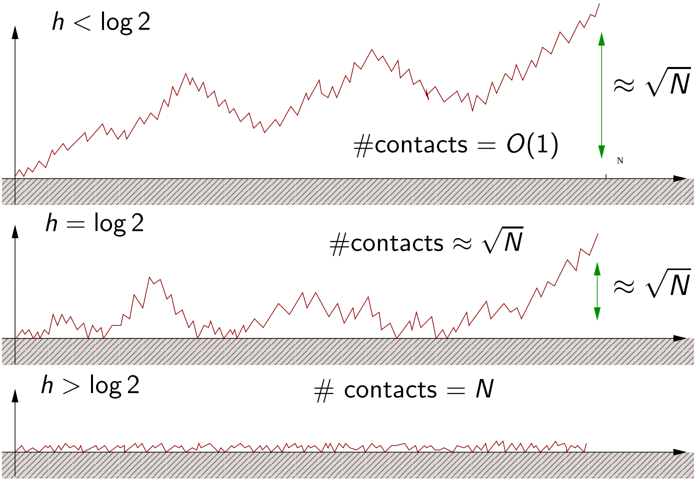
$$\mu_N^{(h)}(S) = \frac{1}{Z_N(h)} \exp(hH_N(S)). \qquad (2)$$

where

$$Z_N(h) := \sum_{\text{all possible paths } S} \exp(hH_N(S)). \qquad (3)$$

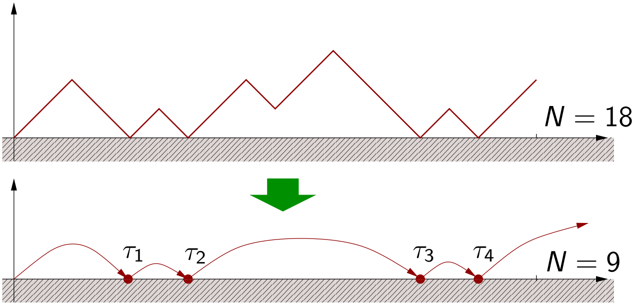
It is **partition fonction** of the system.

Phase transition phenomenon



More general frame work

Observation The introduction of an interaction with the wall does not change the law of excusions out of the wall, but only their length. For S we change only the law of the return times to 0. One can focus on the study of these return times



For simple random walk, return times take only even values. We divide them by 2 so that τ_1 can have any integer value with positive probability.

Renewal Process

Let be $\tau = (\tau_n)_{n\in\mathbb{N}}$ be a sequence of random variable (law **P**) satisfying

- $\tau_0 = 0$, **P**-p.s.
- $(\tau_n - \tau_{n-1})_{n\in\mathbb{N}}$ is an i.i.d. sequence > 0.
- $P(\tau_1 = n) = K(n) \sim \frac{cste.}{n^{1+\alpha}}$ for some $\alpha > 0$ ($\alpha = 1/2$=Simple Random Walk) one consider $\alpha \neq 1$.

One can alternatively consider τ as a set, $\tau = \{\tau_n \ ; \ n \in \mathbb{N}\}$.

Transient renewal

One can have $K(\infty) = \mathbf{P}(\tau_1 = \infty) > 0$.In this case, the renewal stops after the first jump of infinite size, one talk about transient renewal (see preceding example). In the other case, the renewal is said to be recurrent.

Polymer measure as modification of the renewal measure

For fixed h and N, one defines $\mathbf{P}_N^{(h)}$ a probability law on τ with its Radon-Nykodim derivative w.r.t. the reference measure $\mathbf{P}$

$$\frac{\mathrm{d}\mathbf{P}_N^{(h)}}{\mathrm{d}\mathbf{P}}(\tau) := \frac{1}{Z_N^{(h)}} \exp(h \sum_{n=1}^{N} \mathbf{1}_{n\in\tau}). \tag{4}$$

where

$$Z_N^{(h)} := \mathbf{E}\left[\exp(h \sum_{n=1}^{N} \mathbf{1}_{n\in\tau})\right]. \tag{5}$$

One wants to study the typical behavior of τ under the probability law $\mathbf{P}_N^{(h)}$ when N tends to infinity.

Free energy and localization

One defines the polymer free-energy by

$$\mathrm{F}(h) := \lim_{N\to\infty} \frac{1}{N} \log Z_N^{(h)}. \tag{6}$$

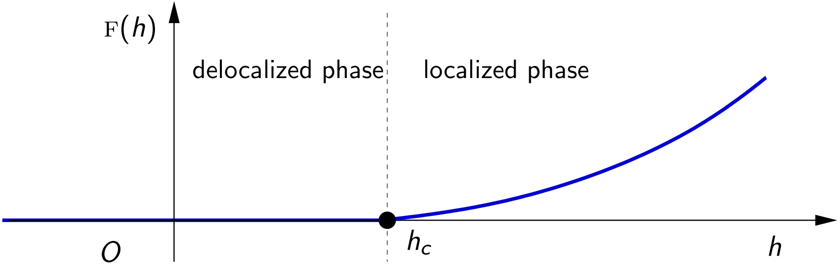
It is a limit of convex non-decreasing function.

$$\begin{aligned}\frac{\partial}{\partial h} \log Z_N^{(h)} &= \frac{1}{Z_N^{(h)}} \frac{\partial}{\partial h} \mathbf{E}\left[\exp(h \sum_{n=1}^{N} \mathbf{1}_{n\in\tau})\right] \\ &= \frac{1}{Z_N^{(h)}} \mathbf{E}\left[\sum_{n=1}^{N} \mathbf{1}_{n\in\tau} \exp(h \sum_{n=1}^{N} \mathbf{1}_{n\in\tau})\right] = \mathbf{E}_N^{(h)}\left[|\tau \cap [0,N]|\right].\end{aligned} \tag{7}$$

Convexity allows to interchange derivation and limit. One obtains

$$\mathrm{F}'(h) = \lim_{N\mapsto\infty} \mathbf{E}_N^{(h)}\left[\frac{|\tau \cap [0,N]|}{N}\right]. \tag{8}$$

Free energy



One has

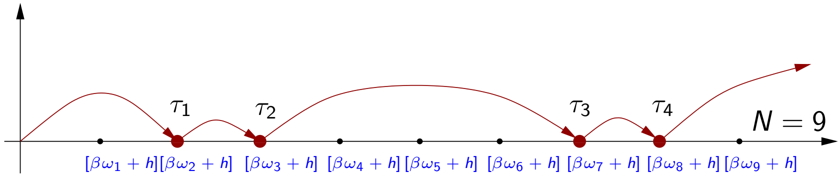
$$h_c = -\log \mathbf{P}(\tau_1 < \infty)$$
$$\mathrm{F}(h) \overset{h \to h_c+}{\sim} cste.(h - h_c)^{1 \vee \frac{1}{\alpha}} \tag{9}$$

$1 \vee \frac{1}{\alpha}$ is the critical exponent for the phase transition in h. h_c is the critical point. In particular, if the original renewal is recurrent, the polymer is pinned for every $h > 0$.

Inhomogeneous model

Let $(\omega_n)_{n\in\mathbb{N}}$, be a sequence (deterministic or random) of real numbers (with some regularity).

One consider a model where the energy gain given by a return to zero is depends on the location where the polymer returns. One fixes two parameters $\beta > 0$ and $h \in \mathbb{R}$,



The energy of the above trajectory is $H_N(\tau) := \beta(\omega_2 + \omega_3 + \omega_7 + \omega_8) + 4h$.

One defines (as in the homogeneous case) the polymer measure $\mathbf{P}_N^{(\omega,\beta,h)}$ by defining its Radon-Nykodim derivative w.r.t. $\mathbf{P}$:

$$\frac{\mathrm{d}\mathbf{P}_{N,h}^{\omega,\beta}}{\mathrm{d}\mathbf{P}}(\tau) := \frac{1}{Z_{N,h}^{\omega,\beta}} \exp\left(\sum_{n=1}^{N} (\beta\omega_n + h)\mathbf{1}_{n\in\tau}\right). \tag{10}$$

where

$$Z_{N,h}^{\omega,\beta} := \mathbf{E}\left[\exp\left(\sum_{n=1}^{N} (\beta\omega_n + h)\mathbf{1}_{n\in\tau}\right)\right]. \tag{11}$$

is the partition function.

Examples of environment to be considered

- ω is a periodic deterministic sequence [Bolthausen, Caravenna, Giacomin, Zambotti...].
- ω is a (typical fixed realization of a) sequence of IID random variables [Alexander, Birkner, Den Hollander, Derrida, Giacomin, Toninelli, Zygouras...].
- ω is a (typical fixed realization of a) sequence of correlated random variables with correlation vanishing with the distance [the work presented today].

Proposition

In all the cases we consider in this talk one can define the free-energy :

$$\mathrm{F}(\beta,h) := \lim_{N\to\infty} \frac{1}{N} \log Z_{N,h}^{\omega,\beta} = \lim_{N\to\infty} \frac{1}{N} \mathbb{E}\left[\log Z_{N,h}^{\omega,\beta,h}\right], \tag{12}$$

($\mathbb{P}$ denotes the law of the environment when it is random). which is also non-negative, non-decreasing and convex in h.

As in the homogeneous case, the positivity of the free-energy allows to decide whether the polymer is localized or not. One can define.

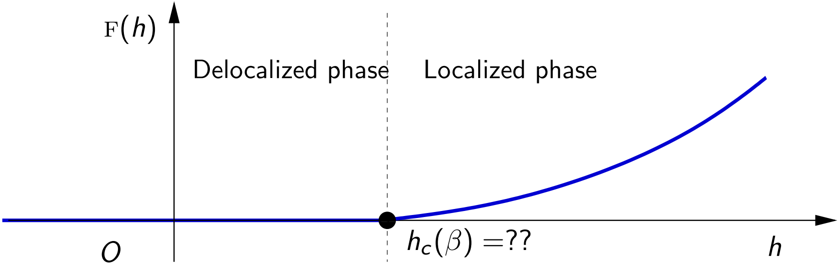
$$h_c(\beta) := \inf\{h : \mathrm{F}(\beta,h) > 0\} \in [-\infty, \infty). \tag{13}$$

where the transition phase in h occurs. One has

$$\begin{aligned} \mathbf{E}_{N,h}^{\omega,\beta)} \left[|\tau \cap [0,N]|\right] &= o(N) \text{ if } \mathrm{F}(\beta,h) = 0, \\ \mathbf{E}_{N,h}^{\omega,\beta} \left[|\tau \cap [0,N]|\right] &= \frac{\partial}{\partial h}\left[\mathrm{F}(\beta,h)\right] L(1+O(1)) \text{ if } \mathrm{F}(\beta,h) > 0. \end{aligned} \tag{14}$$

Questions about the disordered model

- Can one compute the value of $h_c(\beta)$?
- Has one $\mathrm{F}(\beta, h) \sim (h - h_c)_+^{\nu}$ for some $\nu > 0$.
- If yes, do we have $\nu = 1 \vee \alpha^{-1}$ like in the homogeneous case.
- What can one say on the behavior of the trajectories in the delocalized phase and at the critical point.



Case of periodic environment

Theorem (Bolthausen Giacomin '05, Caravenna Giacomin Zambotti '07)

When ω is a deterministic periodic sequence

- $h_c(\beta)$ *can be computed exactly, and is related to the main eigenvalue of some matrix.*
- *One has* $\mathrm{F}(\beta, h) \sim cste.(h - h_c)_+^{\max(1,\alpha^{-1})}$ *near the critical point.*
- *Number of contact is* $O(1)$ *if* $h < h_c$, $N^{\min(\alpha,1)}$, *for* $h = h_c$.

Case of IID environment

Set $\lambda(\omega) := \log \mathbb{E}[\exp(\beta\omega_1)] < \infty$. Then one has by Jensen's inequality that

$$
\begin{aligned}
\mathrm{F}(\beta, h) &= \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\log Z_{N,h}^{\omega,\beta}\right] \\
&\leq \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left[Z_{N,h}^{\omega,\beta}\right] = \lim_{N\to\infty} \frac{1}{N} \log \mathbf{E}\mathbb{E}\left[e^{\sum_{n=1}^{N}(\beta\omega_n + h)\mathbf{1}_{n\in\tau}}\right] = \mathrm{F}(0, h+\lambda(\beta)
\end{aligned}
$$

(

as

$$
\mathbb{E}\left[Z_{N,h}^{\omega,\beta}\right] = Z_{N,h+\lambda(\beta)}. \tag{16}
$$

Therefore one has

$$
h_c(\beta) \geq h_c(0) - \lambda(\beta). \tag{17}
$$

Whether this inequality is sharp or not may depend on the value of α and β.

Case of IID environment, irrelevant disorder

Theorem (Alexander '08, Toninelli '08, L '10)

When β is small and $\alpha < 1/2$ then

$$h_c(\beta) = h_c(0) - \lambda(\beta) \tag{18}$$

and $\mathrm{F}(\beta, h) \sim cste.(h - h_c)_+^{\alpha^{-1}}$, as is the homogeneous case. Moreover when $h = h_c(\beta)$

$$\mathbf{E}_{N,h}^{\omega,\beta}\left[\tau \cap [0,N]\right] \sim cste.N^{\alpha}. \tag{19}$$

And the number of contacts is typically $O(1)$ in when $h < h_c(\beta)$.

Case of IID environment, irrelevant disorder

Theorem (Giacomin Toninelli '06, Derrida Giacomin L Toninelli '09, Alexander Zygouras '09, Giacomin L Toninelli '10, L '10)

For any β and $\alpha \geq 1/2$ then

$$h_c(\beta) > h_c(0) - \lambda(\beta) \tag{20}$$

and $\mathrm{F}(\beta, h) \leq cste.(h - h_c)_+^2$ (so that the critical behavior has to be different) Moreover when $h = h_c(\beta)$

$$\mathbf{E}_{N,h}^{\omega,\beta}\left[\tau \cap [0,N]\right] \leq .N^{1/2+o(1)}. \tag{21}$$

The number of contacts is $O(\log N)$ in when $h < h_c(\beta)$.

Our model of pinning in correlated environment [Proposed by Giacomin]

Let $\hat{\tau}$ be (a fixed typical realization of) a recurrent renewal process satisfying

$$\mathbb{P}\left[\hat{\tau}_1 = n\right] = \hat{K}(n) \sim \frac{\hat{c}_k}{n^{1+\widetilde{\alpha}}}. \tag{22}$$

with $\widetilde{\alpha} > 1$ (finite mean). Let X_n be sequence of IID random variables in $\{-1, 0\}$. One sets

$$\omega_i = X_n \quad \forall i \in (\hat{\tau}_{n-1}, \hat{\tau}_n]. \tag{23}$$

The sequence of ω_i are correlated and one has

$$\mathrm{cov}(\omega_n \omega_{n+k}) \approx k^{1-\widetilde{\alpha}}. \tag{24}$$

What to expect in this case ?

- Contrary to the IID case, we don't have clear prediction from physics litterature (Weinrib Halperin '83 give general prediction for disorded system with quenched disorder).
- One has trivially $\mathrm{F}(0, h-\beta) \leq \mathrm{F}(\beta, h) \leq \mathrm{F}(0, h)$.
- Annealed bound do not give anything better as

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left[Z_{N,h}^{\beta,\omega}\right] = \mathrm{F}(0, h). \tag{25}$$

Results

Theorem (Berger L '11)

One has that for every $\beta > 0$ there exists a constant $C(\beta)$ such that for all $h \in [0,1]$

$$(C(\beta))^{-1} |\log h|^{1-\widetilde{\alpha}} (\mathrm{F}(0,h))^{\widetilde{\alpha}} \le \mathrm{F}(\beta,h) \le C(\beta) |\log h|^{1-\widetilde{\alpha}} (\mathrm{F}(0,h))^{\widetilde{\alpha}}. \tag{26}$$

And in particular $h_c(\beta) = 0$. As a consequence the number of contacts is typically $O(1)$ in when $h < h_c(\beta)$.

Theorem (Berger L '11)

for every $\beta > 0$ there exists a constant $C(\beta)$ such that for $h = 0$, one has on has for almost every ω,

$$\lim_{K\to\infty} \limsup_{N\to\infty} \mathbf{P}^{\omega,\beta}_{N,0}\left[\tau \cap [0,N] \geq K\right] = 0. \tag{27}$$

Lowerbound on the free-energy, finite volume criterion

From subadditivity properties, if one sets

$$Z_{N,h}^{\beta,\omega,\mathrm{pin}} := \mathbf{E}\left[e^{\sum_{n=1}^{N}(\beta\omega_n+h)\mathbf{1}_{n\in\tau}}\mathbf{1}_{N\in\tau}\right]. \tag{28}$$

One has

$$\mathrm{F}(\beta,h) = \lim_{N\to\infty}\frac{1}{N\mathbb{E}\left[\hat{\tau}_1\right]}\mathbb{E}\log Z_{\hat{\tau}_N,h}^{\omega,\beta,\mathrm{pin}} = \sup_{N\in\mathbb{N}}\frac{1}{N\mathbb{E}\left[\hat{\tau}_1\right]}\mathbb{E}\left[\log Z_{\hat{\tau}_N,h}^{\omega,\beta,\mathrm{pin}}\right]. \tag{29}$$

Therefore it is sufficient to bound

$$\frac{1}{N}\mathbb{E}\log Z_{\hat{\tau}_N,h}^{\omega,\beta,\mathrm{pin}} \tag{30}$$

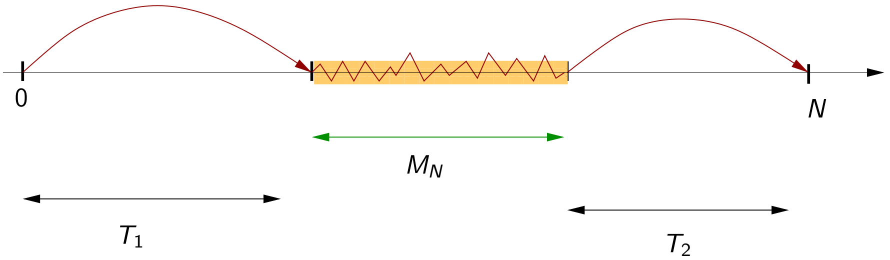
for some given value of N.

Longuest stretch strategy

The strategy to is consider only trajectories that touch the longuest stretch of $+$. Set

$$M_N := \max\{\hat{\tau}_i - \hat{\tau}_{i-1} \mid X_i = 0,\ i \le N\}. \tag{31}$$

Note that M_N is typically of order $N^{\widetilde{\alpha}^{-1}}$.



One has

$$Z^{\omega,\beta}_{\hat{\tau}_N,h} \ge e^{2h-2\beta} Z_{M_N,h} K(T_1) K(T_2) \tag{32}$$

Longest stretch strategy,end

$$\log Z^{\omega,\beta}_{N,h} \geq 2(h-\beta) \log Z^{\mathrm{pin}}_{M_N,h} \log K(T_1) + \log K(T_2) \tag{33}$$

M_N is of order $N^{\widetilde{\alpha}^{-1}}$, T_1 and T_2 are of order N. We also have

$$Z^{\mathrm{pin}}_{n,h} \geq cste.n^{-1} e^{n\mathrm{F}(h)}. \tag{34}$$

so that

$$\mathbb{E} \log Z^{\omega,\beta}_{\hat{\tau},h} \geq C_1 h^{\max(1,\alpha^{-1})} N^{\widetilde{\alpha}^{-1}} - C_2 \log N. \tag{35}$$

Hence setting $N = C_3 h^{\max(1,\alpha^{-1})} |\log h|^{\widetilde{\alpha}}$

$$\mathrm{F}(\beta,h) \geq \frac{1}{\mathbb{E}[\hat{\tau}_1] N} \mathbb{E} \log Z^{\omega,\beta}_{\hat{\tau},h} \geq cste.h^{\widetilde{\alpha}\max(1,\alpha^{-1})} |\log h|^{1-\widetilde{\alpha}}. \tag{36}$$

Lower bound

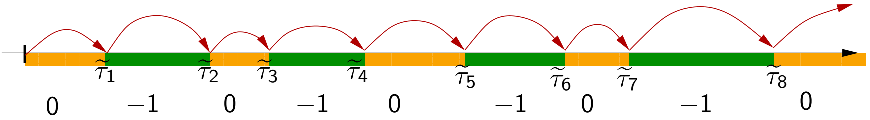
- One has to bound the contribution of all trajectories (more difficult).
- One uses a coarse graining argument to reduce the problem to some finite volume estimate.
- Proof is much easier if one accepts to drop a log factor.

Lower bound, simplified proof $\alpha > 1$

Alternative definition for ω. Consider $\widetilde{\tau}$ a renewal process satisfying

$$\widetilde{K}(n) = \widetilde{\mathbf{P}}(\widetilde{\tau}_1 = n) \sim \frac{\widetilde{c}_k}{n^{1+\widetilde{\alpha}}}. \tag{37}$$

And given a realization of $\widetilde{\tau}$ construct ω as follows



- $\omega_i = 0$ if $i \in (\widetilde{\tau}_{2n}, \widetilde{\tau}_{2n+1}]$.
- $\omega_i = -1$ if $i \in (\widetilde{\tau}_{2n+1}, \widetilde{\tau}_{2n+2}]$.

Coarse graining

Set $T_i = \widetilde{\tau}_{2i}$ and $T_i - T_{i-1} = \xi_i$.

Proposition

One have

$$Z^{\omega,\beta}_{T_N,h}] \le \prod_{i=1}^{N} \left(\left(\max_{x \in (T_{i-1},T_i]} Z^{\omega,\beta}_{[x,T_i]} \right) \vee 1 \right). \tag{38}$$

where

$$Z^{\omega,\beta}_{[x,T_i]} := e^{\beta\omega_x + h} Z^{\theta^x\omega,\beta}_{T_i - x,h}. \tag{39}$$

One has

$$Z^{\omega,\beta}_{[x,T_i]} \leq \exp(h\xi_i)\mathbf{E}\left[e^{-\beta \mathbf{1}_{x-T_i\in\tau}}\right] \leq \exp(h\xi_i). \tag{40}$$

(recall $\xi_i = T_i - T_{i-1}$) The renewal is positive reccurent, so that $\mathbf{P}(n \in \tau)$ is uniformly bounded away from zero.
Therefore the exist $C > 0$ such that if $\xi_i \leq Ch^{-1}$

$$Z^{\omega,\beta}_{[x,T_i]} \leq 1 \quad \forall x \in (T_{i-1}, T_i], \tag{41}$$

$$\log Z^{\omega,\beta}_{T_N,h} \le h \sum_{i=1}^{N} \xi_i \mathbf{1}_{\xi_i \ge C h^{-1}}. \tag{42}$$

Therefore

$$\mathrm{F}(\beta, h) = \lim_{N\to\infty} \frac{1}{N\widetilde{\mathbf{E}}[\xi_1]} \log Z^{\omega,\beta}_{T_N,h} \le \frac{h\widetilde{\mathbf{E}}\left[\xi_1 \mathbf{1}_{\xi_1 \ge h^{-1}}\right]}{\widetilde{\mathbf{E}}[\xi_1]} \le C' h^{\widetilde{\alpha}}. \tag{43}$$

Idea to prove 0(1) contacts at criticality

Prove that among trajectories that have a contact (a large) with the defect line

- those who have more than $(\log a)^2$ where $\omega = -1$ have low probabilities (energetic reasons).
- then prove that touching less than $(\log a)^2$ point where $\omega = -1$ has a small probability (entropic reasons).

Conclusion

- We found a model for which disorder modifies the phase transition and for which we can describe the modification in a precise way.
- It gives an example of case where behavior of trajectories ot the critical point (the number of contact point), is not the one you would expect by looking at the free energy exponent.