Free Energy and Large Deviations for Quenched Polymers in Random Potential

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(Joint with Timo Seppäläinen and Atilla Yılmaz)

1. Model
2. Quenched LDP
3. Free Energy

Random walk

P_0 is a **Random Walk** on $\mathbb{Z}^d$ with steps in $\mathscr{R} \subset \mathbb{Z}^d$ bounded

$d \geq 1$ is arbitrary

Without loss of generality: jumps to $z \in \mathscr{R}$ are equally likely

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Examples:

- Simple random walk: $\mathscr{R} = \{\pm e_1, \ldots, \pm e_d\}$
- Directed simple random walk: $\mathscr{R} = \{e_1 \pm e_2, \ldots, e_1 \pm e_d\}$ (or $\{e_1, \ldots, e_d\}$)

Polymer in random potential

$(\Omega, \mathcal{B}, \mathbb{P}, \{T_z : z \in \mathcal{G}\})$: ergodic system
($\mathcal{G}$ is group generated by $\mathscr{R}$ and Ω is compact)

Measurable $V : \Omega \times \mathscr{R}^\ell \to \mathbb{R}$ is a **Random Potential**

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Quenched measures are

$$dQ_n^{V,\omega} = \frac{\exp\Big\{-\sum_{k=0}^{n-1} V(T_{x_k}\omega, z_{k+1,k+\ell})\Big\}}{Z_n^{V,\omega}}\, dP_0$$

$z_{i,j} = (x_i - x_{i-1}, \dots, x_j - x_{j-1})$
$Z_n^{V,\omega}$ is the normalizing constant (*partition function*)

Examples

- RWRE: $\ell = 1$ and $V(\omega, z) = -\log \pi_{0,z}(\omega)$
- Nearest-neighbor polymers or directed polymers: $\ell = 0$ $(V(\omega))$
- Stretched polymers: $\ell = 1$ and $V(\omega, z) = \Psi(\omega) - h \cdot z$

Assumptions on V

- Bounded, or
- $d = 1$ and $V \in L^1$, or
- $d \geq 2$ and $\Omega = \Omega_0^{\mathbb{Z}^d}$ and $\mathbb{P}$ is i.i.d.
  and $V(\omega, z_{1,\ell}) = \Psi(\omega_0, z_{1,\ell}) \in L^p$, $p > 2(d+1)$

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Examples: Ber, Geo, Poi, Exp, Gau, Gam, log Gam, etc

For simplicity: think of V bounded continuous

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(all have $V(\omega)$ and so RWRE is not covered)

Earlier Work: static RWRE

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Kosygina-Rezakhanlou-Varadhan '06: Diffusion in random potential

Rosenbluth '06: variational formula for rate (level 1)

Yilmaz '09: univariate level 2

R'-Seppäläinen '11: level 3

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(All require loops to be allowed. Space-Time not covered.)

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Key ingredient: **Free Energy**

$$\Lambda(g) = \lim_{n\to\infty} n^{-1} \log E_0\Big[\exp\Big\{\sum_{k=0}^{n-1} g(T_{X_k}\omega)\Big\}\Big]$$

$(g = -V)$

Point-to-Point Free Energy

$$\Lambda(g,\xi) = \lim_{n\to\infty} n^{-1} \log E_0\Big[\exp\Big\{\sum_{k=0}^{n-1} g(T_{X_k\omega})\Big\},\ X_n = [n\xi]\Big]$$

exists by subadditivity and $\Lambda(g) = \sup_\xi \Lambda(g,\xi)$

Lower Bound: Change of Measure

$$\Lambda(g) = \lim_{n\to\infty} n^{-1} \log E_0\Big[\exp\Big\{\sum_{k=0}^{n-1} g(T_{X_k}\omega)\Big\}\Big]$$
$$\geq \sup_{\mu}\{E^{\mu}[g] - H(\mu)\} = H^*(g).$$

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$$H(\mu) = \inf\{H(\mu \times q \,|\, \mu \times p) : \mu q = \mu\}$$

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$H(\mu) = \infty$ if $\mu \not\ll \mathbb{P}$ (only relevant measures)

Upper Bound: Goal

Will show that $\Lambda(g) \le K(g) \le H^*(g)$. (will define $K(g)$)

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Conclusions: $\Lambda(g) = K(g) = H^*(g)$. Two variational formulas.

And quenched LDP.

In particular: for space-time RWRE.

Class of Correctors

$F : \Omega \times \mathscr{R} \to \mathbb{R}$ such that

- $F(\omega, z) \in L^1(\mathbb{P})$ for each $z \in \mathscr{R}$ (moment)
- $\mathbb{E}[F(\omega, z)] = 0$ for each $z \in \mathscr{R}$ (mean-zero)
- $\sum_{k=0}^{n-1} F(T_{x_k}\omega, z_{k+1}) = \sum_{j=0}^{m-1} F(T_{\bar{x}_j}\omega, \bar{z}_{j+1})$ if $x_n = \bar{x}_m$ (closed-loop)

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Examples: Gradients $h(T_z\omega) - h(\omega)$ with $h \in L^1(\mathbb{P})$
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Examples: Gradients $h(T_z\omega) - h(\omega)$ with $h \in L^1(\mathbb{P})$ and their L^1-limits

Lemma: That's all!

Sublinearity

Can define a path integral $f(x,\omega) = \sum_{k=0}^{n-1} F(T_{x_k}\omega, z_{k+1})$

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Lemma: For any ξ, $n^{-1} f([n\xi],\omega) \to 0$ a.s.

Proof: Trivial for gradients. Then approximate

Upper Bound: Part 1

$$K_F(g) = \mathbb{P}\text{-ess}\sup_{\omega} \log \sum_z \frac{1}{|\mathscr{R}|} e^{g(\omega)+F(\omega,z)}$$

$$K(g) = \inf_F K_F(g) = \inf_F \mathbb{P}\text{-ess}\sup_{\omega} \log \sum_z \frac{1}{|\mathscr{R}|} e^{g(\omega)+F(\omega,z)}$$

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$$\begin{aligned} E_0\Big[\exp\Big\{g(\omega)+\cdots+g(T_{X_{n-1}}\omega)\Big\},\ X_n=[n\xi]\Big] \\ \le e^{c(\omega)n\varepsilon} E_0\Big[\exp\Big\{g(\omega)+F(\omega,Z_1)+\cdots \\ +g(T_{X_{n-1}}\omega)+F(T_{X_{n-1}}\omega,Z_n)\Big\}\Big] \\ \le e^{c(\omega)n\varepsilon} e^{nK_F(g)} \text{ (by the Markov property)} \end{aligned}$$

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So $\Lambda(g) = \sup_\xi \Lambda(g,\xi) \le K(g)$

Upper Bound: Part 2

$$H^*(g) = \sup_{\mu \ll \mathbb{P}} \Big\{ E^\mu[g] - \inf\{H(\mu \times q \,|\, \mu \times p) : \mu q = \mu\} \Big\}$$

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Solution: Approximate with finite $\mathcal{B}_k$

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$$H^*(g) \geq \sup_{\frac{d\mu}{d\mathbb{P}}\ \mathcal{B}_k\text{-meas}} \inf_h E^\mu\Big[\log \sum_z \frac{1}{|\mathscr{R}|} e^{g(\omega)+h(T_z\omega)-h(\omega)}\Big]$$

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Uniform integrability gives a limit $F(\omega, z)$ that is a corrector

$H^*(g) \geq K(g)$ as desired

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Mean-zero gives
$\mathbb{E}[(h_k(T_z\omega) - h_k(\omega))^-] = \mathbb{E}[(h_k(T_z\omega) - h_k(\omega))^+] \leq C$

So: can throw away the bad part! (Note that it is nonnegative)

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Solution: The resulting $F(\omega, z)$ has $\mathbb{E}[F(\omega, z)] = c(z) \geq 0$.
So redefine as $F(\omega, z) - c(z)$

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Closed-loop for $F(\omega, z)$ implies same for $c(z)$

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Inequality goes the right way because $c(z) \geq 0$

$$
\begin{aligned}
H^*(g) &\geq \mathbb{P}\text{-ess}\sup_{\omega} \log \sum_{z} \frac{1}{|\mathscr{R}|} e^{g(\omega)+F(\omega,z)} \\
&\geq \mathbb{P}\text{-ess}\sup_{\omega} \log \textstyle\sum_{z} \frac{1}{|\mathscr{R}|} e^{g(\omega)+F(\omega,z)-c(z)} \\
&\geq K(g)
\end{aligned}
$$

Result

$$\Lambda(g) = \lim_{n\to\infty} n^{-1} \log E_0\Big[\exp\Big\{\sum_{k=0}^{n-1} g(T_{X_k}\omega)\Big\}\Big]$$

$$= \sup_{\mu}\{E^{\mu}[g] - H(\mu)\}$$

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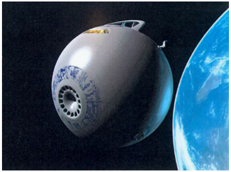

The IID: Infinite Improbability Drive

(The Hitchhiker's Guide to the Galaxy)

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We need $\mathbb{E}[|V|^p] < \infty$ for $p \geq 1$ to apply ergodic arguments.

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Lemma

Let $(Y_i)_{i\in\mathbb{Z}^d}$ be nonnegative and ergodic. Assume $E[Y^p] < \infty$ for p "large enough." Fix $z \in \mathbb{Z}^d$. Then,

$$\lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} \, n^{-1} \sup_{|i|\leq n} |Y_{i+z} + \cdots + Y_{i+\varepsilon nz}| = 0.$$

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Question 1: does it work under any large enough but finite p and mere ergodicity?

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We need $\mathbb{E}[|V|^p] < \infty$ for $p \geq 1$ to apply ergodic arguments.

The only place where one needs p to be large enough is:

Lemma

Let $(Y_i)_{i\in\mathbb{Z}^d}$ be nonnegative and ergodic. Assume $E[Y^p] < \infty$ for p "large enough." Fix $z \in \mathbb{Z}^d$. Then,

$$\lim_{\varepsilon\to 0} \varlimsup_{n\to\infty} n^{-1} \sup_{|i|\leq n} |Y_{i+z} + \cdots + Y_{i+\varepsilon nz}| = 0.$$

Trivial if variables are bounded. Works by the SLLN if $d = 1$.
Works by Borel-Cantelli if variables are i.i.d. and $p > 2(d+1)$.

Question 2: does it work under i.i.d. and only $p > d$?

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Question 3: what about just ergodicity and $p > d$?