An Introduction to Quasi-Symmetric and Noncommutative Symmetric Functions.

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Affine Schubert Calculus at Fields Institute, July 7-10, 2010

Warning and Outline

This talk has nothing to do with k-Schur functions, affine Grassmanians or any other topic of this school...

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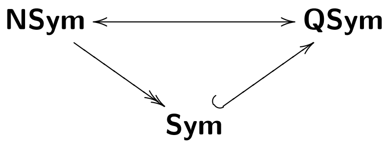
Magic Triangle



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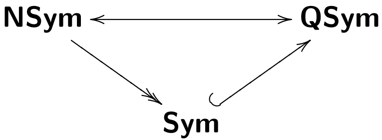

$$\textbf{Sym}: \ m_\lambda, \ h_\lambda, \ s_\lambda, \ldots$$
$$\textbf{NSym}: \ M^I, \ L^I, \ S^I, \ R^I, \ldots$$
$$\textbf{QSym}: \ M_I, \ L_I, \ldots$$

Labelling Set: Compositions

A composition is ordered set of integers: $I = (i_1, \ldots, i_n)$. The sum of all parts is denoted by $|I|$, and the number of parts – by $\ell(I)$.

$$I = (3, 1, 1, 4, 2), \ |I| = 11, \ \ell(I) = 5$$

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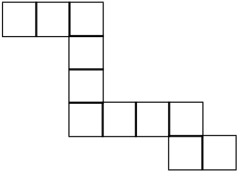
$I =$ 

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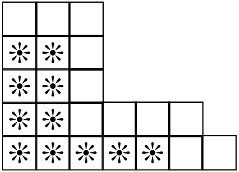
$I =$  $= \boldsymbol{\lambda}/\boldsymbol{\kappa}$

Reverse refinement order.

Let $I = (i_1, \ldots, i_n), J = (j_1, \ldots, j_k)$, $|J| = |I|$ then I is greater in the **reverse refinement order** (or, simply, **finer**) than J,

$$I \succ J$$

if every part of J can be obtained by summing some consecutive parts of I:

$$J = (i_1 + \ldots + i_{p_1}, \ldots, i_{p_{s-1}+1} + \ldots + i_{p_s}, \ldots, i_{p_{k-1}+1} + \ldots + i_n)$$

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For instance, $(3, 3, 2) = (3, 1 + 2, 2) \prec (3, 1, 2, 2)$,

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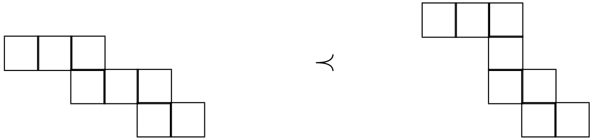

Noncommutative Operations on Compositions

For two compositions $I = (i_1, \ldots, i_{r-1}, i_r)$ and $J = (j_1, j_2, \ldots, j_s)$ one defines two operations

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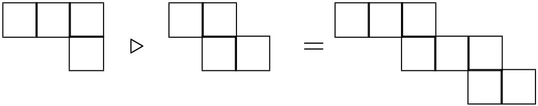

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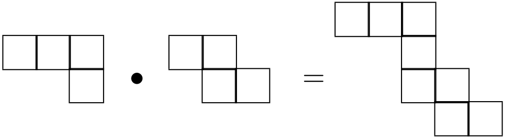
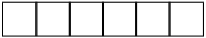


Descent Sets and Compositions.

Another way to encode a composition I of n is by a subset D of $\{1, 2, \ldots, n-1\}$. If $D = \{d_1, d_2, \ldots, d_k\}$, then

$$I = (d_1, d_2 - d_1, d_3 - d_2, \ldots, n - d_k)$$

Example: Let $n = 6$ and take a set $\{2, 3, 5\}$



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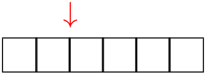

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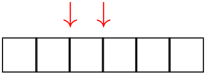

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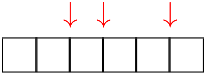

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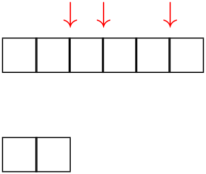

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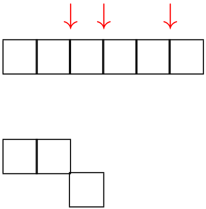

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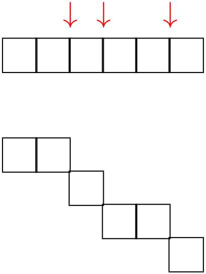

Definitions of Quasi-Symmetric Functions.

For every composition $I = (i_1, \ldots, i_k)$, the **quasi-symmetric monomial** is defined

$$M_I = \sum_{s_1 < \ldots < s_k} x_{s_1}^{i_1} \ldots x_{s_k}^{i_k}$$

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and **quasi-symmetric fundamental**

$$L_I = \sum_{J \succeq I} M_J$$

Examples of Quasi-Symmetric Functions

Monomials:

$$M_{12}(x_1, x_2, x_3) = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2$$
$$M_{21}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3$$

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$$L_{13} = M_{13} + M_{1^2 2} + M_{121} + M_{1^4}$$

Expansion of Schur Functions in Quasi-Symmetric Fundamental.

Consider a standard (skew-)tableau. A **descent** of SYT T is an integer i such that $i+1$ appears in a row of T above i. The descent set of T, $Des(T)$ – is the set of all descents of T.
Example: (desents are marked in bold)

1	4		2	4		2	3		3	4		1	3	
	2	**3**		**1**	**3**		**1**	4		1	**2**		**2**	4

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$$s_{\lambda/\mu} = \sum_{T:\ \text{SYT of shape } \lambda/\mu} L_{Des(T)}$$

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Example continues:

$$s_{32/1} = L_{3,1} + L_{1,2,1} + L_{1,3} + 2L_{2,2}$$

Backsteps.

The **backsteps** of a permutation $w = (w_1, w_2, \ldots, w_n) \in S_n$ are $BS(w) = \{i \mid i+1 \text{ is to the left of } i \text{ in } w\}$.
Denote the reading word (left to right, top to bottom) of T – $w(T)$. Then

$$Des(T) = BS(w(T))$$

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So, equivalently, one can look at the reading words of these tableaux: $142\mathbf{3}, 24\mathbf{13}, 23\mathbf{1}4, 341\mathbf{2}, 13\mathbf{2}4$ and record their backsteps.

$$s_{\lambda/\mu} = \sum_{T:\ \text{SYT of shape } \lambda/\mu} L_{BS(w(T))}$$

Classical Symmetric Functions as Determinats.

Recall that in **Sym** there is a number of identities expressing one type of function (elementary, complete, Schur) as a determinant of other (power sums, complete, etc.). For instance,

$$e_n = \frac{1}{n!}\begin{vmatrix} p_1 & 1 & \dots & 0 & 0 \\ p_2 & p_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{n-1} & \dots & \dots & p_1 & n-1 \\ p_n & \dots & \dots & p_2 & p_1 \end{vmatrix}$$

Quasi-Determinants.

Consider an almost-triangular matrix with noncommutative entries a_{ij} and commutative off-diagonal entries b_j. Its quasideterminant (with respect to the bottom left element) is a sum of all weighted paths starting at the bottom row, ending at the first column, taking northward until encoutering commutative off-diagonal entry and then turning east.

$$\begin{vmatrix} a_{11} & b_1 & 0 \\ a_{21} & a_{22} & b_2 \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix} = a_{31} - \frac{a_{32}\, a_{11}}{b_1} - \frac{a_{33}\, a_{21}}{b_2} + \frac{a_{33}\, a_{22}\, a_{11}}{b_1\, b_2}$$

Noncommutative Elementary and Homogeneous Symmetric Functions.

Define **elementary symmetric** functions Λ_n:

$$\Lambda_n = \frac{(-1)^{n-1}}{n} \begin{vmatrix} \Psi_1 & 1 & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{n-1} & \Psi_{n-2} & \dots & \dots & n-1 \\ \boxed{\Psi_n} & \Psi_{n-1} & \dots & \dots & \Psi_1 \end{vmatrix}$$

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and **complete symmetric** functions S_n:

$$S_n = \frac{1}{n} \begin{vmatrix} \Psi_1 & -(n-1) & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{n-1} & \Psi_{n-2} & \dots & \dots & -1 \\ \boxed{\Psi_n} & \Psi_{n-1} & \dots & \dots & \Psi_1 \end{vmatrix}$$

Noncommutative Monomials.

Define **noncommutative monomial symmetric function** corresponding to a composition $I = (i_1, \dots, i_n)$ as a quasideterminant of an n by n matrix:

$$M^I = \frac{(-1)^{n-1}}{n} \begin{vmatrix} \Psi_{i_n} & 1 & 0 & \dots & 0 & 0 \\ \Psi_{i_{n-1}+i_n} & \Psi_{i_{n-1}} & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{i_2+\dots+i_n} & \dots & \dots & \dots & \Psi_{i_2} & n-1 \\ \boxed{\Psi_{i_1+\dots+i_n}} & \dots & \dots & \dots & \Psi_{i_1+i_2} & \Psi_{i_1} \end{vmatrix}$$

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where $n = \ell(I)$. In particular

$$M^{1^n} = \Lambda_n$$

where Λ_n is an elementary symmetric function.

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where $n = \ell(I)$. If one were to allow power sums to commute, say $\chi(\Psi_k) = p_k, \forall k$, i.e. projecting from **NSym** to **Sym**, then

$$m_{\boldsymbol{\lambda}} = \sum_{I:\ \mathfrak{S}(I)=\boldsymbol{\lambda}} \chi(M^I)$$

Noncommutative Fundamental and Ribbon Schur Functions.

Define **noncommutative fundamental** symmetric functions mimicing the definition in **QSym**

$$L^I = \sum_{J \succeq I} M^J$$

and **ribbon Schur functions** by Jacobi-Trudi formula using quasi-determinants:

$$R^I = (-1)^{\ell(I)-1} \begin{vmatrix} S_{i_n} & 1 & 0 & \dots & \dots \\ S_{i_n+i_{n-1}} & S_{i_{n-1}} & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ S_{i_n+\ldots+i_2} & S_{i_{n-1}+\ldots+i_2} & \dots & S_{i_2} & 1 \\ \boxed{S_{i_n+\ldots+i_1}} & S_{i_{n-1}+\ldots+i_1} & \dots & \dots & S_{i_1} \end{vmatrix}$$

Genocchi Backsteps.

The **G-backsteps** of a permutation $w = (w_1, w_2, \ldots, w_n) \in S_n$ are **positions** of $GBS(w) = \{i \mid i+1 \text{ is to the left of } i \text{ in } w\}$ minus 1.

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The **G-backsteps** of a permutation $w = (w_1, w_2, \ldots, w_n) \in S_n$ are **positions** of $GBS(w) = \{i \mid i+1 \text{ is to the left of } i \text{ in } w\}$ minus 1.

Example

$\begin{array}{ccc} 1 & 4 & \\ & 2 & \mathbf{3} \end{array}$ $\quad \begin{array}{ccc} 2 & 4 & \\ & \mathbf{1} & \mathbf{3} \end{array}$ $\quad \begin{array}{ccc} 2 & 3 & \\ & \mathbf{1} & 4 \end{array}$ $\quad \begin{array}{ccc} 3 & 4 & \\ & 1 & \mathbf{2} \end{array}$ $\quad \begin{array}{ccc} 1 & 3 & \\ & \mathbf{2} & 4 \end{array}$

$$GBS(142\mathbf{3}) = \{3\}$$
$$GBS(24\mathbf{13}) = \{2,3\}$$
$$GBS(23\mathbf{1}4) = \{2\}$$
$$GBS(341\mathbf{2}) = \{3\}$$
$$GBS(13\mathbf{2}4) = \{2\}$$

Expansion of Ribbon Schur in Noncommutative Fundamental.

$$R^I = \sum_{T\colon \text{ SYT of shape } I} L^{GBS(w((T))}$$

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$$\begin{array}{|c|c|c} \hline 1 & 4 & \\ \hline & 2 & \mathbf{3} \\ \hline \end{array} \qquad \begin{array}{|c|c|c} \hline 2 & 4 & \\ \hline & \mathbf{1} & \mathbf{3} \\ \hline \end{array} \qquad \begin{array}{|c|c|c} \hline 2 & 3 & \\ \hline & \mathbf{1} & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|c} \hline 3 & 4 & \\ \hline & 1 & \mathbf{2} \\ \hline \end{array} \qquad \begin{array}{|c|c|c} \hline 1 & 3 & \\ \hline & \mathbf{2} & 4 \\ \hline \end{array}$$

$$R^{2,2} = 2L^{3,1} + L^{2,1,1} + 2L^{2,2}$$