Personal Value-at-Risk: An Overview of the Real Options in Your Life

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Government Finance

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Corporate Finance

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Personal Finance

Outline of Presentation

- ✔Introduction to Real Options in PF&I
- ✔An Optional Walk Through the Life Cycle
  - – The Trivial, The Obvious and The Complex
- ✔Case Study: Option To Take A Pension
  - – When to delay purchasing a life annuity.
  - – Modeling and valuation issues.
- ✔Summary and Conclusion

Real Options

- ✔“...Every project competes with itself delayed in time…”
- ✔Flexibility has strategic value.
- ✔Discounted Cash-Flow (DCF) analysis ignores the option value to wait.
- ✔The Net Present Value (NPV) rule assumes the project is reversible.

Real Options in PF&I:
What are the characteristics?

- ✔Real versus Financial
- ✔Corporate versus Personal
- ✔Traded versus Inseparable
- ✔Information Asymmetry
- ✔Adverse Selection
- ✔Can they be valued?

Education and Option to Expand

- ✔Undergraduate students must choose a college major early in their (academic) life.
- ✔The wrong major eliminates, or greatly reduces, certain career opportunities.
- ✔There is an option to expand that comes with choosing a flexible major, even at the expense of lost wages and more time.
- ✔I think Math Majors are the best! (Biased)

Marriage: The Option to Delay

- ✔These 'projects' are mutually exclusive.
- ✔The costs incurred, when trying to reverse or abandon the initial investments, are high.
- ✔What are the probabilities of locating a better 'project' within specified time frame?
- ✔Empirical evidence that delay is correlated with the availability of competing 'projects'.

Housing and Mortgages

- ✔The transaction costs and frictions associated with selling (or buying) a house can be substantial and time consuming.
- ✔It provides an incentive to delay purchases.
- ✔Rent vs. Buy comparison should account for all the real options in the two choices..
- ✔Floating rate 'open' mortgages contain the embedded option to 'lock in' at any time.

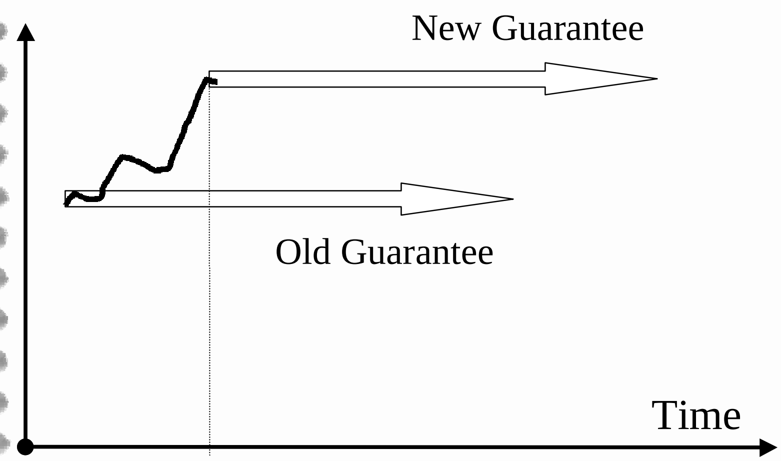
Variable Annuities

- ✔Investment gains are tax deferred. In addition, the contract contains a put option.
- ✔If you die <u>and</u> the contract is 'under water' the insurance company refunds the loss to your estate or beneficiary.
- ✔This is known as the Guaranteed Minimum Death Benefit (or GMDB), which can be a valuable option.
- ✔You pay with higher expenses (M&E fee)

VA: The Option to Lapse

- ✔The GMDB option value moves in and out of the money, depending on the performance of the sub-accounts.
- ✔If the account value increases substantially, it <u>may</u> be optimal to 'lapse' and then buy a similar contract to re-establish the basis of the guarantee. (a.k.a. 1035 exchange, IRC)
- ✔But, with DSC, when should you Lapse?

When is the optimal time to lapse?



Real Option in Life Insurance

- ✔Guaranteed insurability is an option.
- ✔Term versus whole-life policy.
- ✔Should you convert to a cash value (fixed premium) policy, or should your retain the option to convert later?
- ✔Estate taxes may decline in the future, and you tastes for bequest might change as well.

Long-Term Care Insurance

- ✔The earlier in life you buy the policy, the cheaper the periodic premiums.
- ✔This is an argument to buy at 40 vs. 60.
- ✔But:
  - – Better coverage might appear in the future.
  - – Today's policy might rise in price.
  - – The healthcare system might improve.

Pensions and RRSP Savings

- ✔Savings: qualified or outside the shelter?
- ✔Should you contribute now or later?
- ✔In some jurisdictions (U.S.) there is a penalty for early withdrawal.
- ✔The high return from matching contributions may not exceed the lost option value to delay.
- ✔Options in converting from DB to DC.

Option to Work and Retire

- ✔Once retired it is difficult to return to the labor force and continue working.
- ✔A classical economic model compares utility of income with disutility from work. Accordingly, you retire when the later is greater than the former.
- ✔However, it might be optimal to delay retirement until you are absolutely sure the option has no value.

Option to Start a Pension

- ✔The lifetime annuity is non-reversible, but can always be delayed.
- ✔The benefit is protection against longevity risk, the cost is the loss of liquidity.
- ✔When is the optimal time to exercise this option?
- ✔Detailed analysis in the Case Study

The Final Abandonment Option

- ✔SUICIDE:
- ✔A rational look at an irrational act.
- ✔The NPV of future lifetime utility must be negative, by at least the option value.

Case Study: Annuitization

- ✔At retirement, most people must decide how much of their liquid wealth should be voluntarily annuitized.
- ✔Few people choose to buy life annuities, despite the strong theoretical reasons for full annuitization.
- ✔The decision is irreversible, can always be delayed and therefore contains an option.

Types of Immediate Life Annuity:

- Straight Life, Joint-life, or Last survivor
- Substandard or impaired health
- Indexed or Variable

✔Pros:

- guaranteed income for life
- longevity insurance, protection

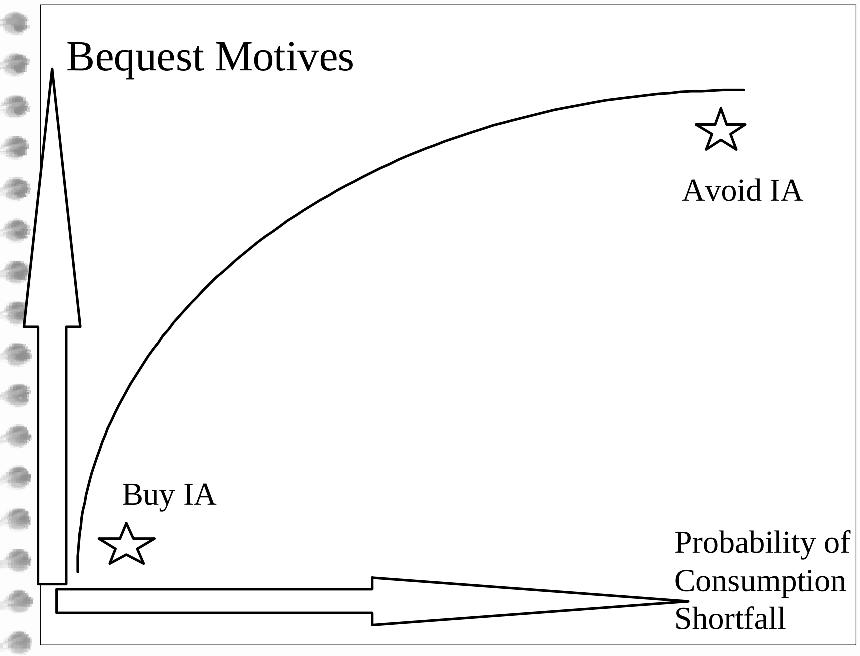
✔Cons:

- non-reversible purchase
- nothing for the estate

You Are 65 Years Old: How Long Will You Live?

Live to Age:	Female:	Male:
70	94.0%	89.2%
75	85.5%	75.1%
80	73.4%	57.6%
85	56.6%	38.2%
90	35.9%	20.1%
95	18.6%	7.5%

Source: Statistics Canada 1995/1996 Health Tables



Life Annuity Quotes: Male

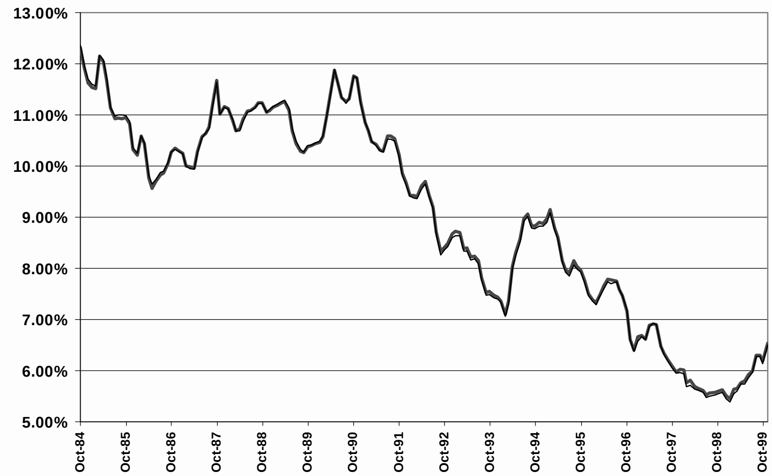
Per $100,000 Purchase (Zero Guarantee)

InsCo	60	65	70	75	80
A.	*665*	*741*	*849*	*1,002*	*1,222*
B.	*650*	*706*	*778*	*890*	*1,050*
C.	*679*	*772*	*891*	*1,042*	
D.	*675*	*752*	*864*	*1,026*	
E.	*667*	*738*	*834*	*983*	*1,194*
F.	*646*	*716*	*816*	*958*	*1,160*

Source: CANNEX, April 4, 2001

Life Annuity IRR: 1984 - 2000

Canadian: Age 65 (Source: CANNEX)



Immediate Life Annuity Pricing

$$a_x = (1 + l_x) \sum_{i=1}^{\infty} \frac{{}_i p_x}{(1 + r_i)^i}$$

"..Price is equal to actuarial present value multiplied by a load factor..."

Can you 'beat' the rate of return from a life annuity?

Fact: $$a_x(1+K)-1 \geq a_{x+1}$$

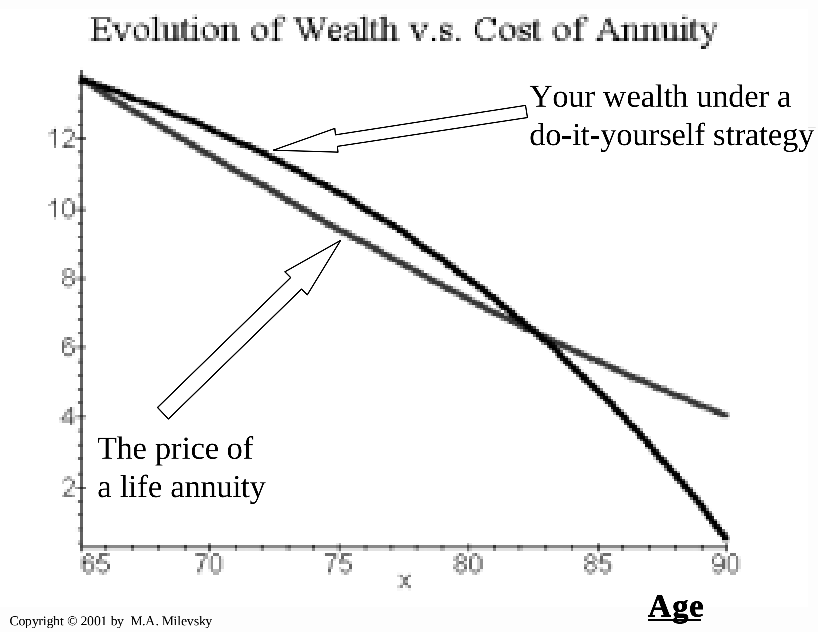
If: $$K = \frac{1}{{}_1p_x}(1+r) - \frac{l}{a_x} - 1$$

…when the actuarial load and the interest rate are constant.

Numerical Example: How To Beat a Life Annuity

Age	Death Probability	Required Return
55	2.26/1000	6.2%
65	5.76/1000	6.6%
75	16.34/1000	7.8%
85	54.05/1000	12.1%
90	95.84/1000	17.2%

Assumptions**:** R=6%, load=0%, IAM1996 Table



Model for the Option Value

$$E\left[U^{*}_{annuity}\,(W_0 + v)\right] =$$
$$E[U^{*}_{wait}\,(\tilde{W}_1)]$$

The option value is v.

Maximize Utility of Consumption

$$U(w) = \frac{w^{(1-\beta)}}{1-\beta}$$

$$({}_t p_x) = \exp\{ -\int_x^{x+t} \lambda(s)\,ds \}$$

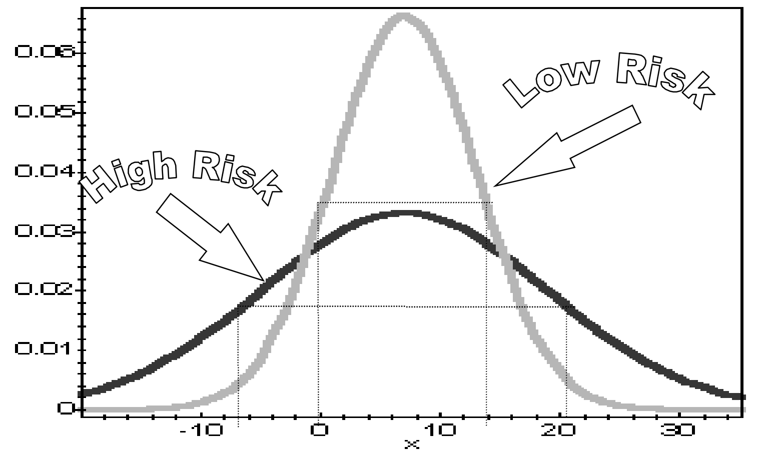
$$dW_{x+t} = \left(\delta W_{x+t} - \frac{1}{a_x} \right) dt$$

Value of option to wait one year.

Age (Male)	β = 3	β = 2
60	2.33%	4.92%
65	2.04%	4.70%
70	1.52%	4.25%
75	0.71%	3.58%
80	Negative	2.44%
85	Negative	0.67%

Assumptions: PowerUtility, Lognormal Assets, (μ=12%, σ=20%, r=6%), IAM2000 Table

What asset class should you select, while you are waiting to annuitize?



Conclusion: Life Annuity

- ✔ The decision is irreversible.
- ✔ Do not buy it too early, since you are losing the option to buy it later.
- ✔ Do-it-yourself and-then-switch gives the best odds of success.
- ✔ Later in life (80+), the return from life annuities are very high.

Real Options in PF&I: Lessons Beyond the Numbers

- ✔Look around you: options are everywhere.
- ✔A precise value is impossible to obtain.
- ✔Don't give for free: rationality is increasing.
- ✔Ask yourself: What does it cost to reverse?
- ✔The Knowledge is the Option:
  - – I will learn more…by tomorrow.

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The Real Option to Delay Annuitization: It's not Now-or-Never

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Abstract

The academic literature continues to puzzle over the extremely low levels of voluntary annuitization amongst the elderly. This phenomenon is inconsistent with a strict interpretation of the Modigliani life-cycle hypothesis, as originally developed by Yaari (1965). Although many plausible explanations have been suggested to reconcile theory and practice, none seem to contain any rigorous normative advice on when exactly one *should* annuitize.

In contrast to previous thinking – and motivated by the financial derivative literature – this paper focuses attention on the Real Option embedded in the decision to annuitize. Indeed, a fixed life annuity is a project with a positive net-present value (NPV) when compared to maintaining liquid wealth in the risk free asset. However, this project should nevertheless be *deferred*, since the option to wait has value. This result is driven by the complete irreversibility of the life annuity purchase and the higher risk-adjusted returns from alternative asset classes in the early stages of retirement. Practically speaking, I estimate that the Real Option to Defer Annuitization (RODA) is quite valuable until the mid-80s, at which point fixed life annuities become the optimal asset class. Moreover, individuals with higher risk tolerance and greater health asymmetry are endowed with an even larger option value to wait.

"...Every project competes with itself delayed in time..."
Steven Ross, FMA Keynote Lecture, 1995

1 Introduction, Motivation and Objectives

Much academic literature has documented – and continues to puzzle over – the extremely low levels of voluntary annuitization exhibited amongst elderly retirees. Strictly speaking, this phenomenon is inconsistent with results of a standard Modigliani life-cycle model of savings and consumption, as described by Yaari (1965). In a life-cycle model with no bequest motives, Yaari (1965) demonstrated that all consumers hold actuarial notes as opposed to liquid assets. This implies that when given the chance, retirees should convert their liquid assets to life annuities which provides longevity insurance and protection against outliving ones money. The rationale behind Yaari's result is that returns from actuarial notes (life annuities) dominate all other assets because the 'living' inherit the assets and returns of the 'dead'. Moreover, at older ages, the higher probability of dying increases the relative return, conditional on survival, from actuarial notes.

For example, there is a 20% chance that a 95 year old female will die in the next year[1]. If five such females enter into a one-year life annuity agreement, by investing \$100 each in a pool yielding 5%, the funds will grow to \$525 by year end. Of the starting five, four are expected to survive, leaving \$525/4=\$131.24 per survivor. This is a net (expected) return of 31.25%. This far exceeds the risk free return of 5% (or perhaps any risky return), because the annuitants have seceded control of assets in the event of death. Although technically this agreement is a classical tontine (or pure endowment contract), and not a common life annuity, the underlying idea is exactly the same. By pooling mortality risk and ceding bequests, everyone gains.

Nevertheless, despite the highly appealing arguments in favour of annuitization, there is little evidence that retirees are voluntarily embracing this arrangement. As Modigliani (1986), Friedman and Warshawsky (1990), Mirer (1994), Poterba and Wise (1996), Brown (1999) and many others have pointed out, very few people consciously choose to annuitize their marketable wealth. In the comprehensive Health and Retirement Survey (HRS), conducted in the U.S, only 1.57 percent of the HRS respondents reported annuity income. Likewise, only 8.0 percent of respondents with a defined contribution pension plan selected an annuity payout. The U.S.-based society of actuaries and LIMRA, as reported in Sondergeld (1997), conducted a study that shows only 0.3% of variable annuity (VA) contracts were annuitized during the 1992-1994 period.

In the face of poor empirical evidence, various theories have been proposed to salvage this aspect of the life-cycle hypothesis, and to justify the low demand for longevity insurance. For example, in one of the earlier papers on this puzzle, Kotlikoff and Spivak (1981) argued that family risk pooling may be preferred to public annuity markets, especially given the presence of adverse selection and transaction costs. Indeed, a married couple functions as a mini annuity market, as explained by Brown and Poterba (1998). Friedman and Warshawsky (1990) showed that average yields on

[1] Source: Statistics Canada Population Standard Life Tables: 1990-1992

individual life annuities, during the late 1970's and early 1980's, were lower than plausible alternative investments. The reduced yield was largely attributed to actuarial loads and profits, which have declined over time, according to recent work by Mitchell, Poterba, Warshawsky and Brown (MPWB, 1999). In a different vein, Kotlikoff and Summers (1981) argued that intergenerational transfers accounted for the vast majority of U.S. savings and therefore bequest motives 'solve' the puzzle. This view is echoed by Bernheim (1991) and Hurd (1989). In other words, individuals do not annuitize wealth simply because they want to bequeath assets. Recall that a generic life annuity flow will terminate at death. Bernheim (1991) further argues that large pre-existing annuities in the form of Social Security and government pensions, might serve as an additional deterrent to voluntary annuitization. In a distinct line of reasoning, Yagi and Nishigaki (1993) argue that the actual design of annuities impede full annuitization. One can not obtain a life annuity that provides arbitrary payments contingent on survival, which would be dictated by a Yaari (1965) model. They must be either fixed (in nominal or real terms) or variable (linked to an index). This constraint forces consumers to hold both marketable wealth and annuities. In related research, Milevsky (1998) computed the probability of success from mimicking the consumption from a fixed immediate annuity and investing the balance. His argument was that there is a 'high probability' that equities will outperform fixed immediate annuities, until late in life. However, he did not address the maximization of utility, nor the option value embedded in the decision to annuitize.

In summary, many explanations exist for why people *do not* annuitize further wealth. Although these justifications have explanatory power, they fail to provide advice on optimal product design as well as normative strategies for the elderly. Furthermore, they can not account for the casual observation that most people shun life annuities, simply because they want to 'maintain control' of their assets.

In this paper we intend to pursue a slightly different approach to the issue, and attempt another solution to the puzzle. Specifically – and motivated by the financial option pricing paradigm – we would like to focus attention on the Real Option[2] embedded in the decision to annuitize. Heuristically, due to the irreversibility of annuitization, the decision to purchase a life annuity is akin to exercising an American-style mortality-contingent claim. It is only optimal to do so when the remaining time value of the option becomes worthless. Options derive their value from the volatility of the underlying state variables. Therefore, if one accounts for future mortality and investment uncertainty, the embedded option provides an incentive to delay annuitization until the option value has been eliminated. The option is *real* in the sense that it is not directly separable or tradeable.

Indeed, as Yaari (1965), and many others, have illustrated, the availability of a (fair) life annuity relaxes the budget constraint which then induces greater consumption and utility. Therefore, all

[2]We are using the term Real Option in the personal, as opposed to corporate finance, sense. Strictly speaking, this differs from the classical use of the term in the literature. Our Real Option exists because the irreversibility of the decision to annuitize. Once purchased, it can not be sold in a secondary market. We refer the interested reader to the work by Berk (1999), Amram and Kulatilaka (1999), Trigeorgis (1996), Ross (1995), Ingersoll and Ross (1992) and Hubbard (1994) for additional information about Real Options.

else being equal, the consumer annuitizes wealth as soon as he or she is given the (fair) opportunity to do so. However, these classical arguments are predicated on the existence of only one financial asset, 'off' which the annuities (actuarial notes) are priced. This framework *de facto* assumes that the budget constraint will not improve over time. However, in practice, a risky asset is an alternative to the risk free investment, and by taking a chance in the risky asset, the future budget constraint may improve. In other words, it might be worth waiting, since tomorrows budget constraint may allow for a larger annuity flow and greater utility. In the meantime, of course, the individual is assumed to withdraw consumption from liquid wealth, so as to mimic the life annuity[3]. In fact, when the volatility in our model is set equal to zero, the option to delay has no value, which corresponds with the Yarri (1965) solution. Likewise, uncertainty about future interest rates, mortality, insurance loads and product design all add value to the option to delay. Stated differently, *our main argument is that retirees should refrain from annuitizing today because they may get an even better 'deal' tomorrow.*

It is important to note that our argument is more than just a play on the equity risk premium; namely that sufficiently risk-tolerant consumers should invest in the risky asset. Rather, we are arguing that any multiperiod framework that ignores the irreversibility of the life annuity purchase, is not capturing the option value in waiting. It is most likely that this is the reason why Richard's (1975) merging of Merton (1971) and Yaari (1965) also yields a full annuitization result. The irreversibility has been ignored. Likewise, Brugiavini (1993), examined the optimal time to annuitize, and concludes that it should be *early* in the life cycle. However, her model is driven by adverse selection considerations, and abstracts somewhat from the multiple sources of uncertainty that might induce people to wait.

In related research, Kapur and Orszag (1999) introduced immediate annuities into a Merton (1971) framework by assuming that the risk-free rate is augmented by a mortality bonus that is proportional to the instantaneous hazard rate. Their model (a) does not include any variable immediate annuities, and (b) they assume a tontine structure which implies instantaneous recontracting. In their set-up, the optimal time to completely annuitize becomes the point at which the mortality credits exceed the risk premium, which is quite late in life.

In a similar vein, Blake, Cairns and Dowd (2000) conducted extensive computer simulations to determine the annuity and pension drawdown policy which provides the highest level of (exponential) utility. However, they did not examine the implications of annuitizing at different ages, as it pertains to the option value of waiting.

Therefore, to price this option to wait, we proposed a methodology very similar to the wealth equivalent metric of MPWB (1999). We define the value of the Real Option to Defer Annuitization (RODA) as the percentage increase in wealth that would *substitute* for the ability to defer. We answer the question: How much would the consumer require in compensation for loosing the opportunity to wait? This number is clearly preference dependent, which means the option value

[3]In the spirit of the "buy term and invest the difference" battle cry, we call this strategy: consume term and invest the difference.

is <u>not</u> priced in the arbitrage-free framework of Black and Scholes (1973). But this valuation methodology is primarily due to the lack of secondary market for this Real Option. Furthermore, our option value may actually be negative, in which case we argue that the consumer is better off annuitizing right now, since waiting can only destroy wealth.

The tools we employ are standard in the asset pricing and actuarial literature, which requires very little in the way of new mathematics. From a practical and empirical perspective, we estimate that the Real Option to Defer Annuitization (RODA) remains quite valuable until the mid-80s. Moreover, individuals with higher risk tolerance and greater health asymmetry are endowed with an even larger option value to wait. Finally, we argue that the availability of (low cost) variable immediate annuities reduces the option value to wait and should increase annuitization arrangements in the future.

Although this paper addresses voluntary annuitization using fixed immediate annuities, the same arguments would also apply in the retirement decision of 'taking a lump sum' versus participating in an employer pension plan. If a similar life-contingent benefit stream can be acquired in the future - whether in the open market or within the pension plan – the option to wait has some value.

The remainder of this paper is organized as follows. In section 2 we provide a simple 3-period model that illustrates our main argument. Section 3 does the same in multi-period continuous-time model, and derives some numerical estimates for the option value. Section 4 concludes the paper.

2 Simple 3-period Model for Life Annuities.

2.1 Classical Derivation.

We now illustrate the 'option value' of deferring annuitization with a simple 3-period example. Our problem starts at time zero with a consumer who has an initial endowment of w. All consumption takes place at the end of the period, and the probability of dying during the three periods is: $q_0 < q_1 < q_2$. If the individual is fortunate to survive to the end of third period, she consumes and immediately dies. For simplicity, we assume that both the consumer and the insurance company, are aware of, and agree on, these probabilities of death. Also, for simplicity, we assume the consumer's subjective rate of time preference is set equal to the risk free rate and we ignore income taxes[4]. Later on we shall discuss the implications of relaxing both these assumptions and the effect on the option value.

We define c_1, c_2, c_3 to be the consumption that takes place at the end of the period. The variable R denotes the (risk-free) interest rate 'off' which the annuities are priced.

[4]See Brown (1999) et. al. as well as Charupat and Milevsky (2001) for more information about the tax treatment and possible tax-advantage of fixed immediate annuities.

The optimization problem is:

$$\max_{\{c_1,c_2,c_3\}} E[U_3|w] = \frac{(1-q_0)u(c_1)}{1+R} + \frac{(1-q_0)(1-q_1)u(c_2)}{(1+R)^2} + \frac{(1-q_0)(1-q_1)(1-q_2)u(c_3)}{(1+R)^3}, \qquad (1)$$

$$\textbf{s.t.} \quad w = \frac{(1-q_0)c_1}{1+R} + \frac{(1-q_0)(1-q_1)c_2}{(1+R)^2} + \frac{(1-q_0)(1-q_1)(1-q_2)c_3}{(1+R)^3}, \qquad (2)$$

where $u(c)$ is a twice differentiable utility function that is positive, increasing and strictly concave. Specifically, we will assume a functional form that exhibits constant relative risk aversion so that $u(c) = c^{(1-\beta)}/(1-\beta)$ and $-cu''(c)/u'(c) = \beta$, which is the coefficient of relative risk aversion (RRA). In the event that $\beta = 1$, the function collapses to $u(c) = \ln(c)$, which is consistent with the limit. Also, a utility of bequest is ignored in our paper, since this can only increase the value of not annuitizing. The annuity contract 'appears' in equation (2), by virtue of the (expected) mortality-adjusted discounting of consumption. All else being equal, higher values of q_i increase the consumption attainable in the annuity market. The same initial w can be used to finance a higher consumption stream. Likewise, setting all $q_i = 0$ in equation (2), tightens the budget constraint and reduces the feasible consumption set. This is akin to solving the problem without annuity markets.

The Lagrangian becomes:

$$\begin{aligned}\max_{\{c_1,c_2,c_3,\lambda\}} \quad L_3 = {} & \frac{(1-q_0)u(c_1)}{1+R} + \frac{(1-q_0)(1-q_1)u(c_2)}{(1+R)^2} + \frac{(1-q_0)(1-q_1)(1-q_2)u(c_3)}{(1+R)^3} \\ & + \lambda\left(w - \frac{(1-q_0)c_1}{1+R} - \frac{(1-q_0)(1-q_1)c_2}{(1+R)^2} - \frac{(1-q_0)(1-q_1)(1-q_2)c_3}{(1+R)^3}\right) \qquad (3)\end{aligned}$$

The first order condition is:

$$\frac{\partial L_3}{\partial c_i} = 0, \quad i = 1, 2, 3, \qquad \text{and} \qquad \frac{\partial L_3}{\partial \lambda} = 0 \qquad (4)$$

This leads to an optimal (constant) consumption of:

$$c_i^* = c^* = \frac{w}{a_3}, \quad i = 1, 2, 3, \qquad \text{and} \qquad E[U_3^*|w] = u\left(\frac{w}{a_3}\right) a_3, \qquad (5)$$

where a_3 is the initial price of a \$1 life annuity – paid over three periods contingent on survival – and defined equal to:

$$a_3 = \frac{(1-q_0)}{1+R} + \frac{(1-q_0)(1-q_1)}{(1+R)^2} + \frac{(1-q_0)(1-q_1)(1-q_2)}{(1+R)^3} \qquad (6)$$

This is the classical annuity result, originally derived by Yaari (1965), which states that all liquid wealth is annuitized – held in the form of actuarial notes – and consumption is constant across all (living) periods. As mentioned earlier, in the absence of annuity markets, the budget constraint, in equation (2), is *tightened* to equate present value of consumption and initial wealth, and the optimal consumption decreases in proportion to the probability of survival.

The constant consumption result is predicated on (i) the time preference being set to the risk free rate, and (ii) symmetric mortality beliefs. If these numbers are different, the optimal consumption stream might not be constant. In some cases, it might even induce holdings of non-annuitized assets. We ignore these cases for now – and refer the interested reader to Yagi and Nishigaki (1993) for this line of reasoning – since our main point is that *even* in the simple case, one should not annuitize.

2.1.1 Numerical Example

For example, when $w = 1$, $q_0 = 0.10$, $q_1 = 0.25$, $q_2 = 0.60$, $R = 0.10$ and $\beta = 1.5$, we have that $u(c) = -2/\sqrt{c}$, and:

$$a_3 = 1.5789, \qquad c^* = \frac{1}{a_3} = 0.63336, \qquad E[U_3^*|1] = u(\frac{1}{a_3})a_3 = -3.9679$$

and when $\beta = 1$, (log utility) consumption remains the same, since all asset are annuitized, but $\ln(1/a_3)a_3 = -0.7211$ *utiles*. (The negative utility is obviously not a problem since the function is defined up-to any constant.)

2.2 Consume and Defer.

Our main idea is to allow the individual to consume c^*, at the end of the period, and then reconsider annuitization at that time. In the meantime, the assets are invested and subjected to the risky return. The risky return can fall in one of two states. We use the superscript u to denote the 'up' state of nature, and d to denote the 'down' state of nature. There is a probability p of a good return X_u, and $(1-p)$ of a bad return X_d. Therefore, by waiting, next period's optimization problem will be one of two cases.

In the event the liquid assets earned the 'high' return, the optimization problem will be:

$$\max_{\{c_{u2},c_{u3}\}} \quad E[U_{u2}|wX_u - c^*] = \frac{(1-q_1)u(c_{u2})}{1+R} + \frac{(1-q_1)(1-q_2)u(c_{u3})}{(1+R)^2}, \tag{7}$$

$$\textbf{s.t.} \quad wX_u - c^* = \frac{(1-q_1)c_{u2}}{(1+R)} + \frac{(1-q_1)(1-q_2)c_{u3}}{(1+R)^2}, \tag{8}$$

In the event of a 'bad' return, the 2nd period optimization problem becomes:

$$\max_{\{c_{d2},c_{d3}\}} \quad E[U_{d2}|wX_d - c^*] = \frac{(1-q_1)u(c_{d2})}{1+R} + \frac{(1-q_1)(1-q_2)u(c_{d3})}{(1+R)^2}, \tag{9}$$

$$\textbf{s.t.} \quad wX_d - c^* = \frac{(1-q_1)c_{d2}}{(1+R)} + \frac{(1-q_1)(1-q_2)c_{d3}}{(1+R)^2}, \tag{10}$$

As before, the optimal consumption is constant, and equal to:

$$c_d^* = \frac{wX_d - c^*}{a_2} \qquad \text{and} \qquad E^*[U_{d2}|wX_d - c^*] = u\left(\frac{wX_d - c^*}{a_2}\right)a_2, \tag{11}$$

$$c_u^* = \frac{wX_u - c^*}{a_2} \qquad \text{and} \qquad E^*[U_{u2}|wX_u - c^*] = u\left(\frac{wX_u - c^*}{a_2}\right)a_2, \tag{12}$$

where the 2-period annuity factor is:

$$a_2 = \frac{(1-q_1)}{1+R} + \frac{(1-q_1)(1-q_2)}{(1+R)^2} = a_3\left(\frac{1+R}{1-q_0}\right) - 1 \tag{13}$$

We now arrive at our main expression.

$$E^*[U_{\text{wait}}|w] = \frac{1-q_0}{1+R}\left(pu\left(\frac{wX_u - c^*}{a_2}\right)a_2 + (1-p)u\left(\frac{wX_d - c^*}{a_2}\right)a_2 + u(c^*)\right). \tag{14}$$

The utility of deferral captures the gains from 'taking a chance' on next period's budget constraint. Specifically, the utility of deferral weighs next periods utility of consumption by the probability of either return-state $\{u, d\}$ occurring and the probability of survival, and then discounts for time. Hence, as long as:

$$E^*[U_{\text{wait}}|w] > E^*\left[U_3|w\right], \tag{15}$$

one is better-off waiting. Finally, as discussed in the introduction, the value of the option to defer one period is defined equal to the quantity v that equates both utilities.

$$E^*[U_{\text{wait}}|w] = E^*\left[U_3|w+v\right] \tag{16}$$

We provide the intuition with the help of a numerical example. We use the same parameters as in the previous example, namely: $w = 1$, $q_0 = 0.10$, $q_1 = 0.25$, $q_2 = 0.60$, $R = 0.10$ and $u(c) = -2/\sqrt{c}$. In this case, $c^* = 0.6333$ and $E[U_3^*|1] = -3.9679$. If the individual is faced with a one-time decision, the optimal consumption is 0.6333 units per period, and the maximum utility is: -3.9679. Now, assume the individual can defer the decision by investing w in an asset with a stochastic return with two possible outcomes: X_u and X_d. Specifically, let $p = 0.70$ denote the probability that the non-annuitized investment factor will be: $X_u = 1.45$ (which is a 45% return), and $1 - p = 0.30$ is the probability that the non-annuitized investment factor will be: $X_d = 1.00$ (which is a 0% return). The expected investment return is therefore: 31.50%.

In the event X_u occurs, the investor has 1.45 units at the end of the first period, from which she consumes $c^* = 0.6333$, to mimic the annuity. This leaves her with 0.8166 for the 2nd period budget constraint. Likewise, if X_d occurs, the investor has 1.00 units at the end of the first period, from which she consumes $c^* = 0.6333$, leaving her with only 0.3666 for the 2nd period budget constraint. Assuming she will annuitize at the end of the first period, her discounted expected utility from the decision to defer, is:

$$E^*[U_{\text{wait}}|w] = -3.9193 > -3.9679 = E^*\left[U_3|w\right]$$

Furthermore, if we give the individual $v = 0.02491$, at time zero, she would be indifferent between annuitized immediately and deferring for one period. We conclude that the value of the option to defer one period is worth 2.49% of initial wealth.

A few technical comments are in order.

- For the deferral to make financial sense, the stochastic return from the investable asset must exceed the mortality-adjusted risk free rate in at least one state of nature. In our 3-period, 2-states of nature context, X_u must be greater that $(1+R)/(1-q_0)$, otherwise $E^*[U_{\text{wait}}|w]$ will never exceed $E^*\left[U_3|w\right]$, regardless of how high p is, or how low q_0 is.

- One does not require abnormally high investment returns in order to justify 'waiting'. In fact, the entire analysis could have been conducted with a stochastic interest rate R, instead of a stochastic investment return. (Or both, for that matter.) The key insight is that waiting might change the budget constraint in the consumers favour. The budget constraint might change on the left-hand side, which is an increase (or decrease) in initial wealth, or on the right-hand side, with an increase (or decrease) in the interest rate 'off' which the annuity is priced. As long as the risk-adjusted odds of a favorable change in the budget constraint are high enough, the option to wait has value. This insight is quite important since *any* possible change in the future price of the annuity provides an option value. This would include any changes in design, liquidity or pricing that might improve tomorrow's budget constraint.

- When $\beta = 1$, which is log-utility, the 'value' of the one period option is 4.26%, which is higher than the case of $\beta = 1.5$. As one would expect, the lower the level of risk aversion (β) the higher is the (utility-adjusted) incentive to take some financial risk and defer the decision to annuitize. This increases the value of the option. The same is true in the other direction. A higher aversion to risk decreases the value of the option. For a high enough value – which in our case is $\beta = 2.1732$ – the individual should not defer annuitization since the risk is too high.

- Although we have not addressed this issue in our formal analysis, if the consumer has a less favorable view of her own mortality, the option to defer is even *more* valuable. Specifically, if q_0^b which is used in the budget constraint to price the annuity, is *lower* than the subjective q_0^s used in the objective function, the maximum utility will be reduced at time zero, which increases the value of v that equates equation (16). We refer the interested reader to Hurd and McGarry (1997) for a discussion and experiments involving 'subjective' versus 'objective' assessments of survival probabilities. This might go a long way towards explaining why individuals who believe themselves to be less healthy than average are more likely to avoid annuities, despite having no declared bequest motive. In the classical Yaari (1965) framework, subjective survival rates do not play a role in the optimal policy. In our context, the individual might be speculating on next periods budget constraint, in the (risk-adjusted) hope it will improve.

- Our annuities $\{a_3, a_2\}$ are priced in a profitless environment in which loads and commissions are set to zero. Indeed, the MPWR (1999) study finds values-per-premium dollar in the 0.75 to 0.93 region depending on the relevant mortality table, yield curve, sex and age. In our context, this would imply another incentive to defer, since X_u is more likely to exceed the mortality-adjusted risk free rate. This would hold true as long as the proportional insurance loads do not *increase* as a function of age.[5]

[5] Table 3, on page 1308 of MPWR (1999) seems to indicate that loads *decrease* from age 55 to 75, when annuitant tables are benchmarked against the corporate yield curve. This, once again, provides an incentive to defer.

- Finally, although we christen v the option value, we must be carefully in referring to it as the value of the option to defer (and consume) for one period. In theory, the individual might also defer for two periods, and then annuitize. To be absolutely precise, we should think of v as a lower bound on the option value, since one might consider deferring for many periods. We shall return to this issue later in our analysis.

Having considered the basic intuition in a simple 3-period example, we now more to a continuous-time model in which some realistic estimates are developed for the option value.

3 Continuous-time Model of Option to Wait

Analogous to the objective function (1) in discrete time, we seek to maximize discounted lifetime utility described by:

$$EU_x = \int_0^\infty \frac{{}_tp_x e^{-rt}}{1-\beta} (c_{x+t})^{1-\beta}\, dt, \tag{17}$$

where the conditional probability of survival is defined by the hazard rate (or force of mortality) λ_s, so that:

$$({}_tp_x) = e^{-\int_0^t \lambda_{x+s} ds} \tag{18}$$

It represents the probability that an individual at age (x) survives to age $(x+t)$, when subjected to the continous-time hazard rate λ_s.

The continous-time budget constraint, in the presence of life annuities, is similar to equation (2), where:

$$1 = \int_0^\infty ({}_tp_x) e^{-rt} dt \tag{19}$$

For simplicity, we initially standardize wealth to one unit of consumption, so the left hand side of equation (19) is the special case of $w = 1$.

Marginal utility is equalized across all periods for the optimal consumption plan, which implies that:

$$c^*_{x+t} = c^*_x = \frac{1}{a_x} \qquad \forall t \geq 0, \tag{20}$$

and the utility from annuitizing at time zero, is:

$$EU^*_x = \frac{a_x}{1-\beta} \left(\frac{1}{a_x} \right)^{1-\beta}. \tag{21}$$

We re-iterate that the optimality of complete annuitized consumption is a direct consequence of (i) symmetric mortality beliefs, (ii) risk free time preferences, (iii) constant relative risk aversion utility, and (iv) no bequest motives. These assumptions are not critical to the option-to-defer argument that follows, but does simplify the exposition. In fact, changing any of these four assumptions might actually increase the option value.

Now, continuing with our strategy, if the individual consumes-and-defers in continuous-time by investing at a (fixed) rate of δ, the wealth obeys the Ordinary Differential Equation:

$$dW_{x+t} = \left(\delta W_{x+t} - \frac{1}{a_x} \right) dt, \tag{22}$$

The intuition for the ODE in equation (22), is that all non-annuitized wealth will grow at a rate of δ, while $a_x^{-1}dt$ will be withdrawn to mimic the consumption stream provided by the life annuity. The solution to the ODE is:

$$W_{x+t} = \left(1 - \frac{1}{\delta a_x}\right) e^{\delta t} + \frac{1}{\delta a_x} \qquad \forall t < t^*, \tag{23}$$

where $x + t^*$ is the age-of-ruin, where t satisfies:

$$t^* = \begin{cases} \frac{-\ln[1-\delta a_x]}{\delta} & \text{if} \quad \delta < \frac{1}{a_x} \\ \infty & \text{if} \quad \delta \geq \frac{1}{a_x} \end{cases} \tag{24}$$

As of yet, we have not *randomized* the risky investment return δ. First, we will compute the attainable consumption assuming liquid assets earn a fixed rate δ, then we will randomize the return to derive the expected discounted utility.

The affordable consumption at age $x + t$, assuming liquid assets have earned a continuously compounded rate of δ during the time period t, is:

$$c_{x+t} = \frac{W_{x+t}}{a_{x+t}} = \left(\frac{1}{a_{x+t}} - \frac{1}{\delta a_x a_{x+t}}\right) e^{\delta t} + \frac{1}{\delta a_x a_{x+t}}, \tag{25}$$

while the utility from the remaining lifetime consumption at the time is:

$$EU^*_{x+t} = \frac{a_{x+t}}{1-\beta} (c_{x+t})^{1-\beta} = \frac{a_{x+t}}{1-\beta} \left(\left(\frac{1}{a_{x+t}} - \frac{1}{\delta a_x a_{x+t}}\right) e^{\delta t} + \frac{1}{\delta a_x a_{x+t}}\right)^{1-\beta} \tag{26}$$

Therefore, analogous to the derivation in the 3-period model, the utility from waiting is equal to, and denoted by:

$$EU_x(\text{wait}) = ({}_t p_x) e^{-rt} \left[EU^*_{x+t} + \frac{1}{1-\beta} \left(\frac{1}{a_x}\right)^{1-\beta} \right]. \tag{27}$$

As long as $EU_x(\text{wait}) > EU^*_x$, then deferring is preferred. We now randomize $\widetilde{\delta}$, using a measure $\mathbf{P}(\delta)$, so that the expectation $EU_x(\text{wait})$ must be taken with respect to the return uncertainty over the period t. In this case we obtain:

$$\begin{aligned} EU_x(\text{wait}) = ({}_t p_x) e^{-rt} & \left[\int_{-\infty}^{\infty} \frac{a_{x+t}}{1-\beta} \left(\left(\frac{1}{a_{x+t}} - \frac{1}{\delta a_x a_{x+t}}\right) e^{\delta t} + \frac{1}{\delta a_x a_{x+t}}\right)^{1-\beta} d\mathbf{P}(\delta) \right. \\ & \left. + \frac{1}{1-\beta} \left(\frac{1}{a_x}\right)^{1-\beta} \right]. \end{aligned} \tag{28}$$

We have integrated EU^*_x over the distribution for the random variable $\widetilde{\delta}$. Naturally, depending on the distribution (and support) of the measure $\mathbf{P}$ we will obtain different values for $EU_x(\text{wait})$. The more optimistic (bullish) the investor, the higher is the subjective value of the option to wait. If we define V_{x+t} as the value of this option, which compensates for the ability to wait, we must solve

$$EU_x(\text{wait}) = \frac{a_x}{1-\beta} \left(\frac{1 + V_{x+t}}{a_x}\right)^{1-\beta}, \tag{29}$$

to isolate V_{x+t}, where $EU_x(\text{wait})$ is taken from equation (28). Finally, cancelling the constant $(1-\beta)$, we get:

$$V_{x+t} = a_x \left(\frac{({}_t p_x) e^{-rt}}{a_x} \left(\eta + \left(\frac{1}{a_x} \right)^{1-\beta} \right) \right)^{1/(1-\beta)} - 1, \tag{30}$$

where:

$$\eta = \int_{-\infty}^{\infty} a_{x+t} \left(\left(\frac{1}{a_{x+t}} - \frac{1}{\delta a_x a_{x+t}} \right) e^{\delta t} + \frac{1}{\delta a_x a_{x+t}} \right)^{1-\beta} d\mathbf{P}(\delta) \tag{31}$$

3.1 Numerical Estimates

We use a Gompertz approximation to mortality, which is common in the actuarial annuity literature, as explained by Carriere (1994) and Frees, Carriere and Valdez (1996). This model has also been used in the economics literature when pricing insurance. See Mullin and Philipson (1997), for example. In this paper, we calibrate the parameters to the Annuity 2000 Mortality Table with projection scale G. The force of mortality is therefore $\lambda_x = \exp((x-m)/b)/b$, where m is a modal value, and b is a scale parameter. For males we fit parameters $(88.18, 10.5)$ and for females we have $(92.63, 8.78)$. The random variable $\widetilde{\delta}$ is assumed to be normally distributed with a mean value of 12% and a standard deviation of 20%. This is roughly in line with Chicago's Ibbotson Associates (2000) numbers which are widely used by practitioners when simulating long-term investment returns. Note that a continuous-time normality assumption for $\widetilde{\delta}$ implies a lognormal distribution for annual returns. We display option values for two different levels of risk aversion, $\beta = 2$, and $\beta = 3$. Finally, we compute the option value assuming the decision is deferred for exactly one year, $t = 1$. This choice may seem arbitrary at first, especially since most individuals might be interested in deferring for longer than one year. However, *a priori* there is no reason for choosing any specific t value, and one can always add the one year values to obtain a rough estimate of the value of a multi-period wait. More importantly, if the option value is negative for a deferral of one year, it is optimal to annuitize immediately. The following table provides some estimates for a variety of initial ages.

For example, a 60 year old female who is moderately risk averse ($\beta = 2$) and is contemplating annuitizing \$100 of liquid assets, is endowed with a real option worth \$5.19. If she chooses to purchase a fixed immediate annuity at age 60 – and thus forgo the opportunity to wait – she is essentially 'giving up' an option that is worth *at least* \$5.19. We emphasize, once again, that this option value is a lower bound on the true value of waiting since we are only looking at one year.

On the other hand, a 90 year old female, with the same level of risk aversion, obtains no value from waiting. Intuitively, the implied (mortality adjusted) return from the life annuity exceeds the return from the risky asset class. In fact, waiting destroy value since the option value is negative.

4 Conclusion

In this paper we argue that premature annuitization destroys the *option* value contained in the irreversible decision to annuitize *later*. Specifically, by consuming term and investing the difference

Age	$\beta = 3$ Female (Male)	$\beta = 2$ Female (Male)
60	2.64 (2.33)%	5.19 (4.92)%
65	2.50 (2.04)%	5.10 (4.70)%
70	2.21 (1.52)%	4.88 (4.25)%
75	1.71 (0.71)%	4.45 (3.58)%
80	0.07 (Negative)%	3.68 (2.44)%
85	Negative	2.28 (0.67)%
90	Negative	Negative

Table 1: *The Option value of waiting one more year to annuitize, as a function or risk aversion, for Females (Males). We assume the funds are invested in an asset that is expected to earn a 0.12 return, with a standard devation of 0.20. The risk free rate is 0.06. The mortality table is IAM2000 with Scale G.*

(self annuitization), the retiree stands to gain from the possibility of a relative improvement in the future budget constraint. In our simple model, the stochastic investment returns from alternative asset classes will likely induce a relaxation in the budget constraint. Risk averse agents should weigh the costs and benefits of annuitizing now versus later, in the face of a stochastic future budget constraint. In practice, a deterioration in health status, an increase in interest rates, better liquidity features, preferential tax changes, or a reduction in actuarial loads will all serve to increase the future annuity payout, if the retiree waits and is sufficiently risk tolerant.

These results are in the spirit of Kapur and Orszag (1999), where they examine the 'best' time to annuitize in the context of optimal discounted utility of consumption. Similar to their results, we find that most individuals in their 60s and 70s should hold a substantial portion of their wealth in non-annuitized assets since the option value to wait is quite large.

We must emphasize, however, that the availability of variable immediate annuities (VIAs) will reduce the value of the option to defer since the alternative asset class yields the exact same pre-mortality return. Nevertheless, even with VIAs, the presence of asymmetric mortality information, as well as loads and other fees, creates an option to wait that has *some* value. The question is "how much?" We believe that this framework develops a parsimonious valuation methodology for computing the magnitude of the option value as well as the optimal time to exercise this timing option.

In sum. Analogous to the literature in the corporate finance arena, any irreversible personal financial decision should only be undertaken when the option value to wait – and do it tomorrow – has no value.

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