Regulators of [Higher] Chow Cycles on Calabi-Yau Varieties

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Notation

$X/_{\mathbf{C}}$ = a projective algebraic manifold, of dimension n.

$z^k(X)$ = free abelian group generated by subvarieties of codimension k in X.

Let $\mathbf{F}$ be a field, with multiplicative group $\mathbf{F}^\times$, and put $T(\mathbf{F}^\times) = \coprod_{n\geq 0} T^n(\mathbf{F}^\times)$, the tensor product of the $\mathbf{Z}$-module $\mathbf{F}^\times$. Here $T^0(\mathbf{F}^\times) := \mathbf{Z}$, $\mathbf{F}^\times = T^1(\mathbf{F}^\times)$, $a \mapsto [a]$. If $a \neq 0, 1$, set $r_a = [a] \otimes [1-a] \in T^2(\mathbf{F}^\times)$. The two-sided ideal R generated by r_a is graded, and we put:

$$K_\bullet^M \mathbf{F} = \frac{T(\mathbf{F}^\times)}{R} = \coprod_{n\geq 0} K_n^M \mathbf{F}, \quad \text{(Milnor } K\text{−theory).}$$

For example, $K_0(\mathbf{F}) = \mathbf{Z}$, $K_1(\mathbf{F}) = \mathbf{F}^\times$, and $K_2^M(\mathbf{F})$ is the abelian group generated by symbols $\{a, b\}$, subject to the Steinberg relations:

$$\{a_1 a_2, b\} = \{a_1, b\}\{a_2, b\}$$

$$\{a, 1-a\} = 1. \quad \text{for } a \neq 0, 1$$

$$\{a, b\} = \{b, a\}^{-1}$$

$$\{a, -a\} = 1$$

More generally, $K_n^M(\mathbf{F})$ is generated $\{a_1, \ldots, a_n\}$, $a_1, \ldots, a_n \in \mathbf{F}^\times$, subject to:

$$\text{(i)} \qquad (a_1, \ldots, a_n) \mapsto \{a_1, \ldots, a_n\},$$

is a multilinear function from $\mathbf{F}^\times \times \cdots \times \mathbf{F}^\times \to K_n^M(\mathbf{F})$,

$$\text{(ii)} \qquad \{a_1, \ldots, a_n\} = 0,$$

if $a_i + a_{i+1} = 1$ for some $i < n$.

Assume given a field $\mathbf{F}$ with discrete valuation $\nu : \mathbf{F}^\times \to \mathbf{Z}$, with corresponding discrete valuation ring $\{a \in \mathbf{F} \mid \nu(a) \geq 0\}$, and residue field $\mathbf{k}(\nu)$. One has maps $T : K_\bullet^M(\mathbf{F}) \to K_{\bullet-1}^M(\mathbf{k}(\nu))$. Choose $\pi \in \mathbf{F}^\times$ such that $\nu(\pi) = 1$. If we write $a = a_0\pi^i,\ b = b_0\pi^j \in K_1^M(\mathbf{F})$, then $T(a) = i \in \mathbf{Z} = K_0^M(\mathbf{k}(\nu))$ and

$$T\{a, b\} = (-1)^{ij} \frac{a^j}{b^i} \in \mathbf{k}(\nu)^\times = K_1^M(\mathbf{k}(\nu)) \quad \text{(Tame symbol)}.$$

Let $\mathcal{O}_X$ be the sheaf of regular functions on X, with sheaf of units $\mathcal{O}_X^\times$. Put

$$\mathcal{K}^M_{k,X} := \frac{(\mathcal{O}_X^\times \otimes \cdots \otimes \mathcal{O}_X^\times)}{J}, \quad \text{(Milnor sheaf)},$$

where J is the subsheaf of the tensor product generated by sections of the form:

$$\left\{\tau_1 \otimes \cdots \otimes \tau_k \;\middle|\; \tau_i + \tau_j = 1, \quad \text{for some } i \text{ and } j\right\}.$$

[For example, $\mathcal{K}^M_{1,X} = \mathcal{O}_X^\times$.]

Introduce the Gersten-Milnor complex:

$$\mathcal{K}^M_{k,X} \to K^M_k(\mathbf{C}(X)) \to \bigoplus_{\operatorname{cd}_X Z=1} K^M_{k-1}(\mathbf{C}(Z)) \to \cdots$$

$$\to \bigoplus_{\operatorname{cd}_X Z=k-2} K^M_2(\mathbf{C}(Z)) \to \bigoplus_{\operatorname{cd}_X Z=k-1} K^M_1(\mathbf{C}(Z))$$

$$\to \bigoplus_{\operatorname{cd}_X Z=k} K^M_0(\mathbf{C}(Z)) \to 0$$

We have

$$K_0^M(\mathbf{C}(Z)) = \mathbf{Z}, \qquad K_1^M(\mathbf{C}(Z)) = \mathbf{C}(Z)^\times,$$

$$K_2^M(\mathbf{C}(Z)) = \big\{\text{symbols } \{f,g\} \ / \ \text{Steinberg relations}\big\}$$

The last three terms of this complex then are:

$$\bigoplus_{\mathrm{cd}_X Z = k-2} K_2^M(\mathbf{C}(Z)) \overset{T}{\to} \bigoplus_{\mathrm{cd}_X Z = k-1} \mathbf{C}(Z)^\times \overset{\mathrm{div}}{\to} \bigoplus_{\mathrm{cd}_X Z = k} \mathbf{Z} \to 0$$

where div is the divisor map of zeros minus poles of a rational function, and T is the Tame symbol map:

$$T : \bigoplus_{\mathrm{codim}_X Z = k-2} K_2^M(\mathbf{C}(Z)) \to \bigoplus_{\mathrm{codim}_X D = k-1} K_1^M(\mathbf{C}(D))$$

defined as follows. First $K_2^M(\mathbf{C}(Z))$ is generated by symbols $\{f,g\}$, $f,\ g \in \mathbf{C}(Z)^\times$.

For $f,\ g \in \mathbf{C}(Z)^\times$,

$$T\big(\{f,g\}\big) = \sum_D (-1)^{\nu_D(f)\nu_D(g)} \left(\frac{f^{\nu_D(g)}}{g^{\nu_D(f)}} \right)_D$$

Let

$$\underline{\mathcal{K}}_{k,X}^M = \ \mathrm{Image}\Big(\mathcal{K}_{k,X}^M \to K_k(\mathbf{C}(X)) \Big).$$

Definition. For $0 \le m \le 2$,

$$CH^k(X,m) = H_{\mathrm{Zar}}^{k-m}(X, \underline{\mathcal{K}}_{k,X}^M).$$

Bloch's Higher Chow Groups

Using a norm argument, the above definition is a special case of Bloch's higher Chow groups, defined as follows:

Let $W/_{\mathbf{C}}$ a quasi-projective variety. Recall $z^k(W) =$ free abelian group generated by subvarieties of codimension k in W.

$$\Delta^m = \mathrm{Spec}\left\{\frac{\mathbf{C}[t_0,\dots,t_m]}{(1-\sum_{j=0}^m t_j)}\right\} \simeq \mathbf{C}^m.$$

$z^k(X,m) = \big\{\xi \in z^k(X\times\Delta^m) \bigm| \xi$ meets all faces [defined by setting subsets of $\{t_0,\dots,t_m\}$ to 0] properly$\big\}$. [Note: $z^k(X,0) = z^k(X)$.]

$\partial_j : z^k(X,m) \to z^k(X,m-1)$, restriction to j-th face.

$\delta = \sum_{j=0}^m (-1)^j \partial_j : z^k(X,m) \to z^k(X,m-1), \quad \delta^2 = 0.$

$CH^\bullet(X,\bullet) =$ homology of $\big\{z^\bullet(X,\bullet),\delta\big\}$.

We put $CH^k(X) := CH^k(X,0)$.

$CH^k(X,m)$ **versus** $H^{k-m}_{\mathrm{Zar}}(X,\mathcal{K}^M_{k,X})$

There is a diagram of the following sort:

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
z^k(X,2) & \overset{\text{Norm}}{\to} & \Gamma\big(\bigoplus_{\mathrm{cd}_X Z=k-2} K_2^M(\mathbf{C}(Z))\big) \\
\downarrow & & \downarrow \\
z^k(X,1) & \underset{\overset{\text{Graph}}{\leftarrow}}{\overset{\text{Norm}}{\to}} & \Gamma\big(\bigoplus_{\mathrm{cd}_X Z=k-1} K_1^M(\mathbf{C}(Z))\big) \\
\downarrow & & \downarrow \\
z^k(X) & = & \Gamma\big(\bigoplus_{\mathrm{cd}_X Z=k} K_0^M(\mathbf{C}(Z))\big)
\end{array}
$$

This leads to:

$$H^{k-m}_{\mathrm{Zar}}(X,\mathcal{K}^M_{k,X}) \simeq CH^k(X,m), \quad \text{for } 0 \le m \le 2.$$

Connections with [higher] K-theory

The Riemann-Roch theorem furnishes isomorphisms:

$$\mathrm{ch} : K_m(X)\otimes\mathbf{Q}\tilde{\to}CH^\bullet(X,m;\mathbf{Q}), \begin{cases} \text{Grothendieck} & [m=0] \\ \text{Bloch} & [m\geq 1] \end{cases}$$

where ch is the Chern character map.

One has cycle class maps (regulators)

$$\mathrm{cl}_{k,m} : CH^k(X,m) \to \widetilde{H}^{2k-m}(X,\mathbf{A}(k)),$$

where $\widetilde{H}^{2k-m}(X,\mathbf{A}(k))$ is a suitable cohomology theory.

[Here $\mathbf{A}\subset\mathbf{R}$ is a subring, and $\mathbf{A}(k)=\mathbf{A}(2\pi\sqrt{-1})^k$.]

Examples of $H^{k-m}_{\mathrm{Zar}}(X, \mathcal{K}^M_{k,X})$ and corresponding regulators

(I) The Case $m = 0$

$$\begin{array}{ccc}
\bigoplus_{\mathrm{codim}_X Z = k-1} K_1^M(\mathbf{C}(Z)) & \to & \bigoplus_{\mathrm{codim}_X Z = k} K_0^M(\mathbf{C}(Z)) \\
\| \wr & & \| \wr \\
\bigoplus_{\mathrm{codim}_X Z = k-1} \mathbf{C}(Z)^\times & \overset{\mathrm{div}}{\to} & \bigoplus_{\mathrm{codim}_X Z = k} \mathbf{Z}
\end{array}$$

$$\Rightarrow H^k_{\mathrm{Zar}}(X, \mathcal{K}^M_{k,X}) \simeq \frac{\Gamma\big(\bigoplus_{\mathrm{codim}_X Z = k} \mathbf{Z}\big) \;= z^k(X)}{\mathrm{div}\big(\Gamma(\bigoplus_{\mathrm{codim}_X Z = k-1} \mathbf{C}(Z)^\times)\big)}$$

$$= \left\{\frac{\text{Free abelian group of codim k subvarieties in } X}{\text{Divisors of rational functions on codim } \mathrm{k}-1 \text{ Z in } X}\right\}$$

$$=: CH^k(X)$$

The fundamental class map:

$$\mathrm{cl}_k : CH^k(X) \to H^{2k}_{\mathrm{DR}}(X, \mathbf{C}) \simeq H^{2n-2k}_{\mathrm{DR}}(X, \mathbf{C})^\vee.$$

Let $V \subset X$ be a subvariety of codimension k in X, $\{w\} \in H^{2n-2k}_{\mathrm{DR}}(X, \mathbf{C})$. Define $\mathrm{cl}_k(V)(w) = \frac{1}{(2\pi\sqrt{-1})^k} \int_{V^*} w$, and extend to $CH^k(X)$ by linearity. [Note that $\dim_{\mathbf{R}} V = 2n - 2k$, and where $V^* = V \backslash V_{\mathrm{sing}}$.] Evidently, cl_k is induced by a corresponding map

$$\mathrm{cl}_k : CH^k(X) \to Hg^k(X) := H^{2k}(X, \mathbf{Z}(k)) \cap H^{k,k}(X).$$

The Abel-Jacobi map:

Set $CH^k_{\mathrm{hom}}(X) = \ker(\mathrm{cl}_k : CH^k(X) \to H^{2k}(X, \mathbf{Z}))$. We construct a second cycle class map:

$$\Phi_k : CH^k_{\mathrm{hom}}(X) \to J^k(X),$$

where

$$J^k(X) = \frac{H^{2k-1}(X, \mathbf{C})}{F^k H^{2k-1}(X, \mathbf{C}) + H^{2k-1}(X, \mathbf{Z})}$$
$$\simeq \frac{F^{n-k+1} H^{2n-2k+1}(X, \mathbf{C})^\vee}{H_{2n-2k+1}(X, \mathbf{Z})},$$

is a compact complex torus, called the Griffiths jacobian.

<u>Prescription for</u> Φ_k: Let $\xi \in CH^k_{\mathrm{hom}}(X)$. Then $\xi = \partial\zeta$ bounds a $2n - 2k + 1$ real dimensional chain ζ in X. Let $\{w\} \in F^{n-k+1} H^{2n-2k+1}(X, \mathbf{C})$. Define:

$$\Phi_k(\xi)(\{w\}) = \int_\zeta w \quad \text{(modulo periods)}.$$

Remarks

(i) Presumably, one should expect $k+1$ (higher) cycle class maps on $CH^k(X)$, each defined on the kernel of the previous one, which would "explain" the Chow group as a concrete realization of a conjectural motivic filtration.

(ii) Both maps ($\mathrm{cl}_k,\ \Phi_k$) can be combined to give $\mathrm{cl}_{k,0} : CH^k(X) = CH^k(X,0) \to H^{2k}_{\mathcal{D}}(X,\mathbf{Z}(k)) := \mathbf{H}^{2k}(\mathbf{Z}(k) \to \Omega_X^{\bullet<k})$, where the Deligne cohomology

$$H^i_{\mathcal{D}}(X,\mathbf{A}(j)) := \mathbf{H}^i(\mathbf{A}(j) \to \Omega_X^{\bullet<j}),$$

with commutative diagram:

$$\begin{array}{ccccccccc}
0 \to & CH^k_{\mathrm{hom}}(X) & \to CH^k(X) \to & \frac{CH^k(X)}{CH^k_{\mathrm{hom}}(X)} & \to 0 \\
 & \Phi_k \downarrow & \mathrm{cl}_{k,0} \downarrow & \mathrm{cl}_k \downarrow & \\
0 \to & J^k(X) & \to H^{2k}_{\mathcal{D}}(X,\mathbf{Z}(k)) \to & Hg^k(X) & \to 0
\end{array}$$

We use the fact that for any given subring $\mathbf{A} \subset \mathbf{R}$, the s.e.s.:

$$0 \to \Omega_X^{\bullet<j}[-1] \to \big(\mathbf{A}(j) \to \Omega_X^{\bullet<j}\big) \to \mathbf{A}(j) \to 0,$$

yields (by applying $\mathbf{H}^\bullet(-)$, and Hodge theory) the s.e.s.:

$$0 \to \frac{H^{i-1}(X,\mathbf{C})}{H^{i-1}(X,\mathbf{A}(j)) + F^j H^{i-1}(X,\mathbf{C})}$$

$$\to H^i_{\mathcal{D}}(X,\mathbf{Z}(j)) \to H^i(X,\mathbf{A}(j)) \cap F^j H^i(X,\mathbf{C}) \to 0.$$

Now put $\mathbf{A} = \mathbf{Z}$, and $(i,j) = (2k,k)$. The map $\mathrm{cl}_{k,0}$ can be defined using a local version of the above s.e.s. together with weak purity.

(II) Case $m = 1$

Recall the Tame symbol map

$$T : \bigoplus_{\mathrm{codim}_X Z = k-2} K_2^M(\mathbf{C}(Z)) \to \bigoplus_{\mathrm{codim}_X D = k-1} K_1^M(\mathbf{C}(D))$$

is defined as follows. First $K_2^M(\mathbf{C}(Z))$ is generated by symbols $\{f, g\}$, $f,\ g \in \mathbf{C}(Z)^\times$.

For $f,\ g \in \mathbf{C}(Z)^\times$,

$$T\big(\{f,g\}\big) = \sum_D (-1)^{\nu_D(f)\nu_D(g)} \left(\frac{f^{\nu_D(g)}}{g^{\nu_D(f)}}\right)_D$$

Thus $CH^k(X,1) :=$

$$H^{k-1}_{\mathrm{Zar}}(X, \mathcal{K}^M_{k,X}) \simeq \left\{\frac{\sum_j f_j \otimes D_j \mid \sum_j \mathrm{div}(f_j) = 0}{\mathrm{Im}(\mathrm{Tame})}\right\}$$

There is a s.e.s.:

$$0 \to \underbrace{\frac{H^{2k-2}(X,\mathbf{C})}{F^k H^{2k-2}(X,\mathbf{C}) + H^{2k-2}(X,\mathbf{Z}(k))}}_{\frac{F^{n-k+1}H^{2n-2k+2}(X,\mathbf{C})^\vee}{H_{2n-2k+2}(X,\mathbf{Z}(n-k))}} \to H^{2k-2}_{\mathcal{D}}(X,\mathbf{Z}(k))$$

$$\to \underbrace{H^{2k-1}(X,\mathbf{Z}(k)) \cap F^k H^{2k-1}(X,\mathbf{C})}_{H^{2k-1}(X,\mathbf{Z}(k))_{\mathrm{tor}}} \to 0.$$

We can provide a geometric description of $\mathrm{cl}_{k,1}$ if we work with *null homologous* higher Chow cycles, viz., the kernel of $CH^k(X,1) \to H^{2k-1}(X,\mathbf{Z}(k)) \cap F^kH^{2k-1}(X,\mathbf{C}) = H^{2k-1}(X,\mathbf{Z}(k))_{\mathrm{tor}}$. Assume given a higher Chow cycle $\xi = \sum_{i=1}^N Z_i \otimes f_i$ representing a class in $CH^k(X,1)$. Then via a proper modification, we can view $f_i : Z_i \to \mathbf{P}^1$, and consider the $2n-2k+1$-chain $\gamma_i = f_i^{-1}([0,\infty])$. Then $\sum_{i=1}^N \mathrm{div}\, f_i = 0$ implies that $\gamma := \sum_{i=1}^N \gamma_i$ defines a $2n-2k+1$-cycle. Such a cycle is assumed zero, since the Poincaré dual lies in $H^{2k-1}(X,\mathbf{Z}(k)) \cap F^kH^{2k-1}(X,\mathbf{C})$. Therefore $\partial\zeta = \gamma$ for some $2n-2k+2$-chain ζ. For $\omega \in F^{n-k+1}H^{2n-2k+2}(X,\mathbf{C})$, the current defining $\mathrm{cl}_{k,1}$ is given by:

$$\begin{aligned}\mathrm{cl}_{k,1}(\xi)(\omega) \ &= \ \frac{1}{(2\pi\sqrt{-1})^{n-k+1}}\Bigg[\sum_{i=1}^N \int_{Z_i\backslash\gamma_i} \omega \log f_i \\ &\qquad + \ 2\pi\sqrt{-1}\int_\zeta \omega\Bigg],\end{aligned}$$

where we choose a branch of the log function defined on $\mathbf{C}\backslash[0,\infty)$.

Remarks. (i) γ has the following cohomological interpretation: Suppose that Z is smooth of dimension m, and $f : Z \to \mathbf{P}^1$ is a dominant morphism. Let $\eta \in E^{2m-1}_{Z,d-\text{closed}}$. Then

$$\int_Z \frac{df}{f} \wedge \eta = (2\pi\sqrt{-1}) \int_{f^{-1}[0,\infty]} \eta.$$

Hence for $\xi = \sum_{i=1}^N Z_i \otimes f_i$, $\sum_i \operatorname{div}(f_i) = 0$, $\eta \in E^{2n-2k+1}_{X,d-\text{closed}}$, and $\gamma = \sum_i f_i^{-1}[0,\infty]$:

$$\sum_i \int_{Z_i} \frac{df_i}{f_i} \wedge \eta = (2\pi\sqrt{-1}) \sum_i \int_{f_i^{-1}[0,\infty]} \eta$$
$$= (2\pi\sqrt{-1}) \int_\gamma \eta.$$

But the current

$$\eta \mapsto \sum_i \int_{Z_i} \frac{df_i}{f_i} \wedge \eta,$$

represents the image of $\mathrm{cl}_{k,1}(\xi)$ in $H^{2k-1}(X)$. This follows from a diagram of the sort ($Z := \bigcup_i Z_i$, $\Sigma_Z = \bigcup_i |\mathrm{div}_{Z_i}(f_i)| \cup Z_{\mathrm{sing}}$, $U_Z = Z \backslash \Sigma_Z$, $U_X = X \backslash \Sigma_Z$):

$$\begin{array}{ccccc} & & H^{2k-1}_{\mathcal{D}}(X, \mathbf{A}(k)) & \to & H^{2k-1}(X, \mathbf{A}(k)) \\ & & \downarrow & & \downarrow \\ H^1_{\mathcal{D}}(U_Z, \mathbf{A}(1)) & \to & H^{2k-1}_{\mathcal{D}}(U_X, \mathbf{A}(k)) & \to & H^{2k-1}(U_X, \mathbf{A}(k)) \end{array}$$

(ii) If $w \in H^{n,n-2k+2}(X)$, and $k \geq 2$, we get the transcendental regulator formula:

$$\mathrm{cl}_{k,1}(\xi)(\omega) = \frac{1}{(2\pi\sqrt{-1})^{n-k}} \int_\zeta w.$$

This defines a map

$$CH^k(X,1) \to \frac{H^{0,2k-2}(X)}{H^{2k-2}(X,\mathbf{Z}(k))} \simeq \frac{H^{n,n-2k+2}(X)^\vee}{H_{2n-2k+2}(X,\mathbf{Z}(n-k))}$$

(iii) Let $\pi_{k-1} : \mathbf{C} = \mathbf{R}(k) \oplus \mathbf{R}(k-1) \to \mathbf{R}(k-1)$ be the projection. The real regulator

$$r_{k,1} : CH^k(X,1) \to H_{\mathcal{D}}^{2k-1}(X,\mathbf{R}(k))$$

$$\simeq \frac{H^{2k-2}(X,\mathbf{C})}{F^k H^{2k-2}(X,\mathbf{C}) + H^{2k-2}(X,\mathbf{R}(k))}$$

$$\underset{\simeq}{\xrightarrow{\pi_{k-1}}} H^{k-1,k-1}(X,\mathbf{R}(k-1))$$

$$\simeq H^{n-k+1,n-k+1}(X,\mathbf{R}(n-k+1))^\vee,$$

is given by

$$r_{k,1}(\xi) = \frac{1}{(2\pi\sqrt{-1})^{n-k+1}} \sum_j \int_{Z_j} w \log|f_j|$$

The formula for the real regulator can be derived from the diagram:

$$\xi \leftrightarrow f \otimes Z \quad : \quad \Sigma_Z = |\mathrm{div}_Z(f)| \bigcup Z_{\mathrm{sing}}$$

$$U_Z = Z \backslash \Sigma_Z \quad : \quad U_X = X \backslash \Sigma_Z$$

$$j_X : U_X \hookrightarrow X \quad : \quad j_Z : U_Z \hookrightarrow U_X$$

$$\begin{array}{ccc}
CH^k(X,1) & \to & H_{\mathcal{D}}^{2k-1}(X, \mathbf{R}(k)) \\
j_X^* \downarrow & & \downarrow j_X^* \\
CH^k(U_X,1) & \to & H_{\mathcal{D}}^{2k-1}(U_X, \mathbf{R}(k)) \\
j_{Z,*} \uparrow & & \uparrow j_{Z,*} \\
CH^1(U_Z,1) & \to & H_{\mathcal{D}}^{1}(U_Z, \mathbf{R}(1)) \\
\| \wr & & \left(\frac{df}{f}, \log|f|\right) \\
H^0(U_Z, \mathcal{O}_{U_Z}^{\times}) & \ni f^{\nearrow} &
\end{array}$$

(III) Case $m = 2$

We give an explicit construction, in special cases, of a regulator

$$\mathrm{cl}_{k,2} : CH^k(X, 2) \to H_{\mathcal{D}}^{2k-2}(X, \mathbf{A}(k)),$$

where

$$H_{\mathcal{D}}^{2k-2}(X, \mathbf{A}(k)) \underset{\text{torsion}}{\overset{\text{mod}}{\simeq}} \frac{H^{2k-3}(X, \mathbf{C})}{F^k H^{2k-3}(X, \mathbf{C}) + H^{2k-3}(X, \mathbf{A}(k))}$$

$$\simeq \frac{F^{n-k+1} H^{2n-2k+3}(X, \mathbf{C})^\vee}{H_{2n-2k+3}(X, \mathbf{A}(n-k))}.$$

modulo torsion considerations, by working with null homologous higher Chow cycles.

Recall that $CH^k(X, 2) = H_{\mathrm{Zar}}^{k-2}(X, \underline{\mathcal{K}}_{k,X}^M)$ can be computed from the Gersten-Milnor resolution. Let $Z \subset X$ be a subvariety of codimension $k-2$, and $f, g \in \mathbf{C}(Z)^\times$. By passing to a blow-up of Z, we can assume that $f, g : Z \to \mathbf{P}^1$ are morphisms. Put $\gamma := (f \times g)^{-1}([0, \infty]^2)$, where $f \times g : Z \to \mathbf{P}^1 \times \mathbf{P}^1$. We observe the following (vague form):

Let Z be smooth projective of dimension m and let $w \in E_{Z,d-\text{closed}}^{2\dim Z\ -2}$. Fix $f, g \in \mathbf{C}(Z)^\times$. Then

$$\int_Z \frac{df}{f} \wedge \frac{dg}{g} \wedge w = (*) \int_\gamma w$$

Now assume given a class $\{\xi\} \in CH^k(X,2)$:

$$\xi := \sum_\alpha \prod_{i_\alpha} \{f_{i_\alpha}, g_{i_\alpha}\} \otimes Z_\alpha \stackrel{T}{\mapsto} 0 = \sum_{\alpha, i_\alpha, D\subset X} \frac{f_{i_\alpha}^{\nu_D(g_{i_\alpha})}}{g_{i_\alpha}^{\nu_D(f_{i_\alpha})}} \otimes D.$$

According to Deligne cohomology product calculations, the composite map

$$CH^k(X,2) \to H_{\mathcal{D}}^{2k-2}(X, \mathbf{Z}(k)) \to F^k \bigcap H^{2k-2}(X, \mathbf{A}(k))$$

$$\to F^k \bigcap H^{2k-2}(X \backslash \{ \bigcup_{\alpha, i_\alpha} \{ |(f_{i_\alpha})_{Z_\alpha}| \cup |(g_{i_\alpha})_{Z_\alpha}| \},$$

corresponds to the current defined by

$$w \in E_{X,d-\text{closed}}^{2n-2k+2} \mapsto \sum_{\alpha, i_\alpha} \int_{Z_\alpha} \frac{df_{i_\alpha}}{f_{i_\alpha}} \wedge \frac{dg_{i_\alpha}}{g_{i_\alpha}} \wedge w.$$

For K_2, we have the following Abel-Jacobi map regulator prescription for null-homologous K_2 cycles:

Theorem. In the situation as above, let

$$\gamma := \sum_{\alpha, i_\alpha} \big(f_{i_\alpha} \times g_{i_\alpha}\big)^{-1}([0,\infty]^2),$$

be the torsion cycle associated to $\xi \in CH^k(X,2)$. Assume that it is zero, and choose ζ such that $\partial\zeta = \gamma$. Then

(i) Suppose $\mathbf{A} = \mathbf{Z}$, and $k \geq 3$. Then for any $\omega \in H^{n,n-2k+3}(X)$,

$$\mathrm{cl}_{k,2}(\xi)(\omega) = \frac{1}{(2\pi\sqrt{-1})^{n-k}} \int_\zeta \omega.$$

This determines a map:

$$CH^k(X,2) \to \frac{H^{0,2k-3}(X)}{H^{2k-3}(X,\mathbf{Z}(k))} \simeq \frac{H^{n,n-2k+3}(X)^\vee}{H_{2n-2k+3}(X,\mathbf{Z}(n-k))}.$$

(ii) Concerning the real regulator

$$r_{k,2} : CH^k(X,2) \to H^{2k-2}_{\mathcal{D}}(X, \mathbf{R}(k))$$

$$\simeq \left\{H^{k-1,k-2}(X) \oplus H^{k-2,k-1}(X)\right\} \bigcap H^{2k-3}(X, \mathbf{R}(k-1)),$$

$$r_{k,2}(\xi)(w) = \frac{1}{(2\pi\sqrt{-1})^{n-k+2}} \left(\sum_i \int_{Z_i} \left[\log|f_i| \pi_1 \left(\frac{dg_i}{g_i} \right) \right.\right.$$

$$\left.\left. - \log|g_i| \pi_1 \left(\frac{df_i}{f_i} \right) \right] \wedge w \right).$$

[Here $\pi_1 : \mathbf{C} \to \mathbf{R}(1)$ is the projection.]

The formula for the real regulator can be derived from a similar diagram:

$$\xi \leftrightarrow \{f,g\}\otimes Z \quad : \quad \Sigma_Z = |\mathrm{div}_Z(f)|\bigcup|\mathrm{div}_Z(g)|\bigcup Z_{\mathrm{sing}}$$

$$U_Z = Z\backslash\Sigma_Z \quad : \quad U_X = X\backslash\Sigma_Z$$

$$j_X : U_X \hookrightarrow X \quad : \quad j_Z : U_Z \hookrightarrow U_X$$

$$\begin{array}{ccc}
CH^k(X,1) & \rightarrow & H^{2k-1}_{\mathcal{D}}(X,\mathbf{R}(k)) \\
j_X^* \downarrow & & \downarrow j_X^* \\
CH^k(U_X,1) & \rightarrow & H^{2k-1}_{\mathcal{D}}(U_X,\mathbf{R}(k)) \\
j_{Z,*} \uparrow & & \uparrow j_{Z,*} \\
CH^2(U_Z,2) & \rightarrow & H^2_{\mathcal{D}}(U_Z,\mathbf{R}(2)) \\
||\wr & & \left(\frac{df}{f}\wedge\frac{dg}{g},\log|f|\pi_1(\frac{dg}{g})-\log|g|\pi_1(\frac{df}{f})\right) \\
 & & || \\
H^0(U_Z,\mathcal{O}^\times_{U_Z})^{\otimes 2} & \rightarrow & \left(\frac{df}{f},\log|f|\right)\bullet\left(\frac{dg}{g},\log|g|\right) \\
\{f,g\} & \rightarrow & H^1_{\mathcal{D}}(U_Z,\mathbf{R}(1))\bigcup H^1_{\mathcal{D}}(U_Z,\mathbf{R}(1))
\end{array}$$

Some known results

(I) Abel-Jacobi for Threefolds

The formalism of the conjectured Bloch-Beilinson filtration on Chow groups, together with comparisons to other filtrations, leads to the following (Jannsen, Esnault, Nori):

Conjecture.

$$\ker\big(\Phi_2 : CH^2_{\mathrm{hom}}(X;\mathbf{Q}) \to J^2(X)_{\mathbf{Q}}\big) \subset CH^2_{\mathrm{alg}}(X),$$

where $CH^k_{\mathrm{alg}}(X) \subset CH^k(X)$ is the subgroup of cycles algebraically equivalent to zero.

On the other hand, S. Saito shows that if a Bloch-Beilinson exists, then

$$\ker\big(\Phi_2\big|_{CH^2_{\mathrm{alg}}(X)} : CH^2_{\mathrm{alg}}(X) \to J^2(X)\big)$$

is controlled by $H^{0,2}(X)$. In particular, $H^{0,2}(X) = 0 \Rightarrow \ker(\Phi_2) = 0$. This leads us to:

Conjecture. Let $X/_{\mathbf{C}}$ be a smooth threefold, and assume that $H^{0,2}(X) = 0$. Then

$$\Phi_2 : CH^2_{\mathrm{hom}}(X;\mathbf{Q}) \hookrightarrow J^2(X)_{\mathbf{Q}},$$

is injective. [In particular, if X is a rigid CY threefold, then $CH^2_{\mathrm{alg}}(X) = 0.$†]

† Similarly, if X is a sufficiently general Calabi-Yau threefold, then $CH^2_{\mathrm{alg}}(X) = 0$.

Theorem. [Griffiths, Clemens, Green, Müller-Stach, Voisin] Let $X \subset \mathbf{P}^4$ be a (smooth) threefold of degree d. If $d \leq 4$, then $\Phi_2 : CH^2_{\text{hom}}(X) \xrightarrow{\sim} J^2(X)$ is an isomorphism. Now assume that X is general. If $d \geq 6$ then $\text{Im}\big(\Phi_2\big)$ is torsion. If $d = 5$, then $\text{Im}\big(\Phi_2\big) \otimes \mathbf{Q}$ is countably infinite dimensional.

Recall the definition of the Griffiths group:

$$\text{Griff}^k(X) = \frac{CH^k_{\text{hom}}(X)}{CH^k_{\text{alg}}(X)}.$$

Theorem. [Voisin] If X is a general Calabi-Yau threefold, then $\text{Im}\big(\Phi_2\big)$ is a countably infinite dimensional, when tensored over $\mathbf{Q}$. In particular, since $\Phi_2(CH^k_{\text{alg}}(X)) = 0$, it follows that $\text{Griff}^2(X; \mathbf{Q})$ is [countably] infinite dimensional over $\mathbf{Q}$.

(II) The real regulator on $CH^k(X,1)$

Put

$$r_{k,m} = \mathrm{cl}_{k,m} : CH^k(X,m) \to H^{2k-m}_{\mathcal{D}}(X, \mathbf{R}(k)),$$

where

$$H^{2k-m}_{\mathcal{D}}(X,\mathbf{R}(k)) \simeq \frac{H^{2k-m-1}(X,\mathbf{R}(k-1))}{\pi_{k-1}\big(F^k H^{2k-m-1}(X,\mathbf{C})\big)},$$

($\pi_j : \mathbf{C} \to \mathbf{R}(j)$). For example, if $\xi = \sum_j f_j \otimes Z_j \in CH^k(X,1)$ (i.e. $m = 1$), then

$$H^{2k-1}_{\mathcal{D}}(X,\mathbf{R}(k)) \simeq H^{k-1,k-1}(X,\mathbf{R}(k-1))$$

$$\simeq \{H^{n-k+1,n-k+1}(X,\mathbf{R}(n-k+1))\}^{\vee}.$$

$$r_{k,1}(\xi)(\omega) = \frac{1}{(2\pi\sqrt{-1})^{n-k+1}} \sum_j \int_{Z_j} \omega \log|f_j|.$$

We introduce the following notation. The product

$$\mathcal{K}^M_{i,X} \otimes \mathcal{K}^M_{j,X} \to \mathcal{K}^M_{i+j,X},$$

induces a product:

$$\underbrace{H^0_{\mathrm{Zar}}(X, \mathcal{K}_{1,X})}_{=H^0(X,\mathcal{O}^\times_X)=\mathbf{C}^\times} \otimes H^{k-1}_{\mathrm{Zar}}(X, \mathcal{K}_{k-1,X}) \to H^{k-1}_{\mathrm{Zar}}(X, \mathcal{K}_{k,X}),$$

hence a product:

$$CH^1(X,1) \otimes CH^{k-1}(X) \to CH^k(X,1).$$

Definition. (i) The subgroup of decomposables is given by:

$$CH^k_{\mathrm{dec}}(X) := \mathrm{Im}\Big(CH^1(X,1) \otimes CH^{k-1}(X) \to CH^k(X,1)\Big).$$

(ii) the space of indecomposables is given by:

$$CH^k_{\mathrm{ind}}(X,1;\mathbf{Q}) = \left\{\frac{CH^k(X,1)}{CH^k_{\mathrm{dec}}(X,1)}\right\} \otimes \mathbf{Q}.$$

Example [Müller-Stach, Nori, Green, Coombes, Srinivas]: Let $X \subset \mathbf{P}^3$ be a smooth surface of degree d. If $d \leq 3$, then $r_{2,1} : CH^2(X,1) \to H^{1,1}(X, \mathbf{R}(1))$ is surjective; moreover $CH^2_{\text{ind}}(X,1;\mathbf{Q}) = 0$. Now assume that X is general. If $d \geq 5$, then $\text{Im}(r_{2,1})$ is "trivial".

Conjecture. [Hodge-$\mathcal{D}$ for $K3$ surfaces] If $d = 4$, then

$$CH^2(X,1) \otimes \mathbf{R} \to H^{1,1}(X, \mathbf{R}(1)),$$

is onto, and this is more generally true for $K3$ surfaces with general moduli. [Similarly, for a general $K3$ surface X, $CH^2_{\text{ind}}(X,1;\mathbf{Q})$ is expected to be (countably) infinite dimensional over $\mathbf{Q}$.]

For the following, we use the description, that modulo torsion†

$$H^3_{\mathcal{D}}(X, \mathbf{Z}(2)) \simeq \frac{H^2(X, \mathbf{C})}{H^{2,0}(X) + H^2(X, \mathbf{Z}(2))},$$

and hence, if we quotient out by the (Neron-Severi) Hodge group $Hg^1(X) = H^{1,1}(X, \mathbf{Z})$, we have (again, modulo torsion):

$$\frac{H^3_{\mathcal{D}}(X, \mathbf{Z}(2))}{Hg^1(X) \otimes \mathbf{C}^\times} \simeq \frac{H^2(X, \mathbf{C})}{H^{2,0}(X) + Hg^1(X) \otimes \mathbf{C} + H^2(X, \mathbf{Q}(2))}.$$

---

† Recall:

$$H^3_{\mathcal{D}}(X, \mathbf{Z}(2)) \Big/ \left\{ \frac{H^2(X, \mathbf{C})}{H^{2,0}(X) + H^2(X, \mathbf{Z}(2))} \right\} \simeq H^3_{\text{tor}}(X, \mathbf{Z}(2)).$$

For $K3$ surfaces X, $H^3_{\mathrm{tor}}(X,\mathbf{Z}) = 0$, hence $CH^2_{\mathrm{hom}}(X,1) = CH^2(X,1)$; moreover there is an induced map

$$\underline{c}_{2,1} : CH^2_{\mathrm{ind}}(X,1) \to H^3_{\mathcal{D}}(X,\mathbf{Z}(2))/Hg^1(X)\otimes\mathbf{C}^\times.$$

Some evidence...

(i) **Theorem**. (i) (Lewis-Gordon) Let X be a general special Kummer surface. Then the Hodge-$\mathcal{D}$-conjecture holds for X.

(ii) (Müller-Stach) Let $X \subset \mathbf{P}^3$ be a general quartic surface containing a line. Then

$$\underline{c}_{2,1} : CH^2_{\mathrm{ind}}(X,1) \to H^3_{\mathcal{D}}(X,\mathbf{Z}(2))/Hg^1(X)\otimes\mathbf{C}^\times,$$

is non-trivial; moreover $CH^2_{\mathrm{ind}}(X,1;\mathbf{Q})$ is infinite dimensional.

(iii) (Voisin/Collino) Let $X \subset \mathbf{P}^3$ be a general quartic surface. Then $\underline{c}_{2,1} : CH^2_{\mathrm{ind}}(X,1) \to H^3_{\mathcal{D}}(X,\mathbf{Z}(2))/Hg^1(X)\otimes\mathbf{C}^\times$ is non-trivial.

Remark. One of the things that makes $K3$ surfaces different from surfaces of higher order, is the presence of rational curves. With this in mind, it is easy to construct a higher Chow cycle that varies with parameters, as follows: Let $X \subset \mathbf{P}^N$ be a general $K3$ surface. Recall that $\forall d \geq 2$, the linear system $\Gamma(\mathcal{O}_X(d))$ contains an irreducible rational nodal curve. Since the linear system is very ample, it follows that distinct irreducible rational curves $Q_1,\ Q_2 \subset X$ can be found such that $Q_1 \cap Q_2 \neq \emptyset$. If $d_j = \deg Q_j$, then we can construct $f \in \mathbf{C}(X)^\times$ such that $\mathrm{div}(f) = d_2 Q_1 - d_1 Q_2$. Let $D \subset X$ be a general curve (so that D varies with parameters and that $D \cap Q_{j,\mathrm{sing}} = \emptyset$). Then $\exists g_j \in \mathbf{C}(Q_j)^\times$ such that $f \otimes D + g_1 \otimes Q_1 + g_2 \otimes Q_2$ defines a higher Chow cycle; moreover this cycle can be made to deform with parameters. [More specifically, $\mathrm{div}_D(f)$ can be written in the form $\xi_1' + \xi_2'$, where $|\xi_1'| \subset Q_1$, $|\xi_2'| \subset Q_2$. Write $k_1 = \deg \xi_1'$, $k_2 = \deg \xi_2'$. Note that $k_1 + k_2 = 0$. Let $p \in Q_1 \cap Q_2$. Put $\xi_1 = \xi_1' - k_1\{p\}$, $\xi_2 = \xi_2' - k_2\{p\}$. Then $\mathrm{div}_D(f) = \xi_1 + \xi_2$, where $\deg \xi_1 = \deg \xi_2 = 0$.]

(III) Regulators examples on $CH^k(X,2)$

Example: X = elliptic curve. Consider the real regulator

$$r : CH^2(X,2) \to H^1(X,\mathbf{R}(1)),$$

given by

$$\omega \in H^1(X,\mathbf{R}(1)) \simeq H^1(X,\mathbf{R})^\vee$$

$$\mapsto \int_X \big[\log|g| d\log|f| - \log|f| d\log|g|\big] \wedge \omega.$$

Alternatively, up to a real isomorphism, this is the same as

$$\frac{1}{2\pi\sqrt{-1}} \int_X \left[\log|g|\pi_1\left(\frac{df}{f}\right) - \log|f|\pi_1\left(\frac{dg}{g}\right)\right] \wedge \omega.$$

Incidently, if we assume for the moment that X is any compact Riemann surface, this latter formula is the same as the following homological version. Fix $p \in X$, and consider any loop γ in X based at p. Then:

$$\gamma \mapsto \frac{1}{2\pi}\mathrm{Im}\bigg(\int_\gamma \log g\frac{df}{f} - \log|f(p)|\int_\gamma \frac{dg}{g}\bigg).$$

Here are some details....

$$\pi_1\left(\frac{df}{f}\right) = \pi_1(d\log f) = \sqrt{-1}d\arg(f); \mathrm{Re}\left(\frac{df}{f}\right) = d\log|f|$$

$$\begin{aligned}
r(\{f,g\}) &= \frac{1}{2\pi}\mathrm{Im}\left[\int_\gamma (\log g)\frac{df}{f} - \log|f(p)|\int_\gamma \frac{dg}{g}\right] \\
&= \frac{1}{2\pi\sqrt{-1}}\pi_1\left[\int_\gamma (\log g)\frac{df}{f} - \log|f(p)|\int_\gamma \frac{dg}{g}\right] \\
&= -\frac{1}{2\pi}\log|f(p)|\arg_\gamma(g) + \frac{1}{2\pi}\int_\gamma \arg(g)d\log|f| \\
&\quad + \frac{1}{2\pi\sqrt{-1}}\int_\gamma \log|g|\pi_1\left(\frac{df}{f}\right),
\end{aligned}$$

Next,

$$d\big(\arg(g)\log|f|\big) = \frac{\log|f|}{\sqrt{-1}}\pi_1\left(\frac{dg}{g}\right) + \arg(g)d\log|f|.$$

$$\text{By Stokes' Thm}: \int_\gamma d\big(\arg(g)\log|f|\big) = \log|f(p)|\arg_\gamma(g).$$

Therefore:

$$r(\{f,g\}) = \frac{1}{2\pi}\int_\gamma \big[\log|g|d\arg(f) - \log|f|d\arg(g)\big]$$

$$= \frac{1}{2\pi\sqrt{-1}}\int_X \left[\log|g|\pi_1\left(\frac{df}{f}\right) - \log|f|\pi_1\left(\frac{dg}{g}\right)\right]\wedge\omega,$$

where $\omega := [\gamma]$ is the Poincaré dual of γ.

Here is Bloch's construction on $K_2(X)$: Assume given $f,\ g \in \mathbf{C}(X)^\times$ such that $\Sigma := |\mathrm{div}(f)| \cup |\mathrm{div}(g)|$ are points of order N in $Pic(X)$. Then

$$T\big(\{f,g\}^N\big) \in \coprod \mathbf{C}^\times \quad \text{and } \mapsto 0 \in Pic(X) \otimes \mathbf{C}^\times.$$

Thus there exists $\{h_i\} \in \mathbf{C}(X)^\times$ and $\{c_i\} \in \mathbf{C}^\times$ such that $\{f,g\}^N \prod\{h_i, c_i\} \in CH^2(X,2)$.

[Note that if $h \in \mathbf{C}(X)^\times$ and $c \in \mathbf{C}^\times$, then

$$T\{h,c\}) = \mathrm{div}(h) \otimes c]$$

Clearly this construction takes advantage of the existence of a dense subset of torsion points on X.

Bloch shows that the real regulator is non-trivial for general elliptic curves, and indeed A. Collino shows that the regulator image of $CH^2(X,2)$ for a general elliptic curve X is infinite dimensional (over $\mathbf{Q}$). For curves X of genus $g > 1$, the problem of constructing classes in $CH^2(X,2)$ seems to be related to the fact that under the Abel-Jacobi mapping $\Phi : X \to J^1(X), \quad p \mapsto \{p - p_0\}$, the inverse image of the torsion subgroup, $\Phi^{-1}(J^1(X)_{\mathrm{tor}})$, is finite. This is due to Raynaud's *mixed characteristic* proof of the Mumford-Manin conjecture. From a different perspective, and using infinitesimal methods, A. Collino proves that for a general curve X of genus $g > 1$, the image of the regulator map $CH^2(X,2) \to H^2_{\mathcal{D}}(X, \mathbf{Z}(2))$ is torsion, hence the real regulator image is zero.

On the flip side, one can arrive at an *analytic* proof of the following (weaker) version of the Mumford-Manin conjecture:

Let $X \subset \mathbf{P}^2$ be a general curve of degree $d \geq 3$, and consider the Abel-Jacobi map $X \hookrightarrow J^1(X)$. Put $\Gamma = X \cap J^1_{\mathrm{tor}}(X)$. Then Γ is dense in $X \Leftrightarrow d = 3$.

The basic idea of proof is this. Let $[z_0, z_1, z_2]$ be homogeneous cordinates for $\mathbf{P}^2$, and let $(x, y) = (z_1/z_0, z_2/z_0)$ be corresponding affine cordinates. Via a degeneration argument, one can show that:

$$\int_X \Big[\log|y| d\log|x| - \log|x| d\log|y|\Big] \wedge \omega \neq 0,$$

for a suitable choice of real 1-form ω on X, and for general such $X \subset \mathbf{P}^2$. It is notationally more convenient to write this as:

$$\int_X \Big[\log|g| d\log|f| - \log|f| d\log|g|\Big] \wedge \omega \neq 0,$$

for some $f,\ g \in \mathbf{C}(X)^\times$. Now assume that Γ is dense in X, and choose 0-cycles $\xi_f,\ \xi_g$ "close to" $\mathrm{div}(f)$ and $\mathrm{div}(g)$ respectively, such that $\xi_f,\ \xi_g$ are supported on Γ. Thus for some integer $N > 0$, we can write $N\xi_f = (\tilde{f})$, $N\xi_g = (\tilde{g})$ for some $\tilde{f},\ \tilde{g} \in \mathbf{C}(X)^\times$, and for which

$$\int_X \Big[\log|\tilde{g}| d\log|\tilde{f}| - \log|\tilde{f}| d\log|\tilde{g}|\Big] \wedge \omega \neq 0.$$

This leads us to the setting where we may assume that the divisor sets of f and g are torsion points, of order N say. Thus we can extend $\{f, g\}$ to a class $\underline{\xi} \in CH^2(X, 2)$, for which $r(\underline{\xi}) \neq 0$. However, as we mentioned above, from the work of A. Collino, the real regulator image is zero for a general curve $X \subset \mathbf{P}^2$ of degree ≥ 4, [and more generally for a general curve of genus $g > 1$]. Thus in our case, we have established that Γ dense in $X \Rightarrow d = 3$. The converse statement is obvious.

Rigid CY threefolds and Modularity

Suppose X is a rigid CY threefold. Then

$$J^2(X) \simeq \frac{H^{3,0}(X)^\vee}{H_3(X, \mathbf{Z})}$$

is an elliptic curve.

Conjecture. Suppose further that X is defined over $\mathbf{Q}$. Then $J^2(X)$ is defined over $\mathbf{Q}$ as well. Hence the modularity of $J^2(X)$ (à la Wiles) implies the modularity of X.

One way to understand $J^2(X)$ is via the Abel-Jacobi map $\Phi_2 : CH^2_{\mathrm{hom}}(X) \to J^2(X)$. Note that $\Phi_2(CH^2_{\mathrm{alg}}(X)) = 0$ (and conjecturally $CH^2_{\mathrm{alg}}(X) = 0$ as well). Thus we are really looking at $\Phi_2 : \mathrm{Griff}^2(X) \to J^2(X)$. The difficulty here is finding a cycle.

For a rigid Calabi-Yau threefold, there is a s.e.s.:

$$0 \to J^2(X) \to H^4_{\mathcal{D}}(X, \mathbf{Z}(3)) \to H^4_{\mathrm{tor}}(X, \mathbf{Z}(3)) \to 0.$$

Let

$$CH^3_{\mathrm{hom}}(X, 2) = \ker : CH^3(X, 2) \to H^4_{\mathrm{tor}}(X, \mathbf{Z}(3)).$$

Then another approach is to consider the induced map

$$c_{3,2} : CH^3_{\mathrm{hom}}(X, 2) \to J^2(X).$$

A twisted variant of the higher Chow groups

[Joint work with Brent Gordon]

We indicate that using a twisted version of the Gersten-Milnor complex leads to surjectivity results for the image of the corresponding twisted regulator, for Abelian varieties of dimension at most four. First we introduce the notion of a flat holomorphic line bundle, i.e., a pair $(L, ||\ ||_L)$, where L is a line bundle on X and $||\ ||_L$ is a flat metric on L, i.e. the corresponding curvature form is zero (hence L has zero Chern class). Then we define the group

$$\underline{z}^k(X,1) := \left\{ \sum_i (\sigma_i, ||\ ||_{L_i}) \otimes Z_i \ \middle|\ \begin{matrix} \operatorname{codim}_X Z_i = k-1, \\ L_i/Z_i \text{ flat}, \\ \sigma_i \in \operatorname{Rat}^*(L_i), \\ \sum_i \operatorname{div}(\sigma_i) = 0 \end{matrix} \right\},$$

where $\operatorname{div}(\sigma)$ is the divisor associated to the meromorphic section $\sigma \in \operatorname{Rat}^*(L)$, and $\operatorname{Rat}^*(L)$ denotes the non-zero rational sections of L over E, and where we view holomorphic line bundles in terms of their algebraic counterparts in the Zariski topology.

Let $E_X^{p,q}$ be the (global) complex-valued C^∞-forms of Hodge type (p,q) on X.

Proposition. *Notation as above. Let* $\xi \in \underline{z}^k(X,1)$, *and* $\omega \in E^{n-k+1,n-k+1}_{X,d-\text{closed}}$. *The current defined by*

$$\omega \mapsto \frac{1}{(2\pi\sqrt{-1})^{n-k+1}} \sum_i \int_{Z_i} \omega \log ||\sigma_i||_{L_i},$$

determines a well-defined map [twisted regulator]

$$\underline{r} : \underline{z}^k(X,1) \longrightarrow H^{2k-1}_{\mathcal{D}}(X, \mathbf{R}(k)).$$

Restatement: *Recall that* $H^{2k-1}_{\mathcal{D}}(X, \mathbf{R}(k)) \simeq$

$$H^{k-1,k-1}(X, \mathbf{R}(k-1)) \simeq H^{n-k+1,n-k+1}(X, \mathbf{R}(n-k+1))^\vee,$$

and that

$$H^{p,q}(X) \simeq \frac{E^{p,q}_{X,d-\text{closed}}}{\partial\overline{\partial} E_X^{p-1,q-1}}.$$

Then the current defined by

$$\omega \mapsto \underline{r}(\xi)(\omega) := \frac{1}{(2\pi\sqrt{-1})^{n-k+1}} \sum_i \int_{Z_i} \omega \log ||\sigma_i||_{L_i},$$

is $\partial\overline{\partial}$*-closed.*

Proof. Let $\omega \in \partial\overline{\partial}E_X^{n-k,n-k}$. Then we can write $\omega = d\overline{\partial}\eta$ for some $\eta \in E_X^{n-k,n-k}$. Also let

$$\xi = \sum_i (\sigma_i, || \ ||_{L_i}) \otimes Z_i \in \underline{z}^k(X,1)$$

be given as above, and consider the corresponding integral

$$\underline{r}(\xi)(\omega) = \frac{1}{(2\pi\sqrt{-1})^{n-k+1}} \sum_i \int_{Z_i} \omega \log ||\sigma_i||_{L_i}.$$

By Stokes' theorem and a standard calculation (below):

$$\int_{Z_i} (d\overline{\partial}\eta) \log ||\sigma_i||_{L_i} = \int_{Z_i} \overline{\partial}\eta \wedge d \log ||\sigma_i||_{L_i}$$

$$= \int_{Z_i} \overline{\partial}\eta \wedge \partial \log ||\sigma_i||_{L_i} = \int_{Z_i} d\eta \wedge \partial \log ||\sigma_i||_{L_i},$$

where the latter two equalities follow by Hodge type, and the former uses

$$d\big((\overline{\partial}\eta) \log ||\sigma||_L\big) = (d\overline{\partial}\eta) \log ||\sigma||_L - \overline{\partial}\eta \wedge d \log ||\sigma||_L.$$

More specifically, by taking ϵ-tubes about the components $D \subset \operatorname{div}(\sigma)$, and using that $\dim D = n-k$ and $\overline{\partial}\eta \in E_X^{n-k,n-k+1}$, and that $\log ||\sigma||$ is locally L^1, we find

$$\lim_{\epsilon \to 0} \int_{\mathrm{Tube}_\epsilon((\sigma))} (\overline{\partial}\eta) \log ||\sigma||_L = 0.$$

Note that $\overline{\partial}\eta \wedge \overline{\partial} \log ||\sigma||_L \in E^{n-k,n-k+2}_{X,L^1}$ (locally L^1 forms) and that $\dim Z_i = n-k+1$. Thus we are left with an integral of the form $\int_Z d\eta \wedge \partial \log ||\sigma||_L$ as indicated above.

Next,

$$d(\eta \wedge \partial \log ||\sigma||_L) = d\eta \wedge \partial \log ||\sigma||_L,$$

since $\overline{\partial}\partial \log ||\sigma||_L = 0$. Thus

$$\int_{Z_i} (d\overline{\partial}\eta) \log ||\sigma_i||_{L_i} = \int_{Z_i} d(\eta \wedge \partial \log ||\sigma_i||_L)$$

$$= \lim_{\epsilon \to 0} \int_{\mathrm{Tube}_\epsilon((\sigma_i))} \eta \wedge \partial \log ||\sigma_i||_L.$$

If we put $Z = Z_i$, for a given i, with z a local coordinate on Z_{reg}, then using the dictionary $|\sigma_\alpha| \leftrightarrow |z|$ we have the residue integral:

$$\lim_{\epsilon \to 0} \int_{|z|=\epsilon} \eta \wedge \partial \log |z|^2 = \lim_{\epsilon \to 0} \int_{|z|=\epsilon} \eta \wedge \frac{dz}{z}$$

$$= 2\pi\sqrt{-1} \int_{\{z=0\} \cap Z} \eta_{|\{z=0\} \cap Z},$$

where

$$\eta_{|\{z=0\} \cap Z} = \mathrm{Residue}_{\{z=0\} \cap Z}\big(\eta \wedge \frac{dz}{z}\big)$$

(i.e., taking "tubes" is dual to taking "residues"). Then by computing the residue, and linearity, we find that

$$\underline{r}(\xi)(\omega) = \tfrac{2\pi\sqrt{-1}}{2(2\pi\sqrt{-1})^{n-k+1}} \sum_D \Big[\Big(\sum_i \nu_D(\sigma_i)\Big) \int_D \eta\Big].$$

(We note that here remains the possibility that $Z = Z_i$ is singular along D. To remedy this, one may pass to a normalization of Z with the same calculations above.) Since $\sum_i \operatorname{div}(\sigma_i) = 0$, we have $\underline{r}(\xi)(\omega) = 0$ and the proposition is proved.

Generalizing the Gersten-Milnor Complex

One can show that there is a corresponding "twisted" Gersten-Milnor complex, with a corresponding group

$$\underline{CH}^k(X,1)$$

defined in terms of the cohomology of this complex. Likewise, in defining these cycle groups, we could also restrict ourselves to all line bundles in question, being restrictions of flat line bundles over X. This leads to cycle groups $\underline{\underline{z}}^k(X,1)$ and $\underline{\underline{CH}}^k(X,1)$. In the case where the irregularity $q(X)=0$, it can be shown that $\underline{\underline{CH}}^k(X,1) = CH^k(X,1)$, and that there is a twisted Gersten resolution for a twisted version $\underline{\underline{\mathcal{K}}}^M_{2,X}$ of $\mathcal{K}^M_{2,X}$, such that $\underline{\underline{CH}}^2(X,1) \simeq H^1_{\mathrm{Zar}}(X, \underline{\underline{\mathcal{K}}}^M_{2,X})$. We mention in passing:

Proposition *There are induced maps*

$$\underline{r}_{k,1} : \underline{CH}^k(X,1) \to \frac{H^{k-1,k-1}(X,\mathbf{R}(k-1))}{H^{2k-2}_{\mathrm{alg}}(X)\otimes\mathbf{R}}$$

$$\underline{\underline{r}}_{k,1} : \underline{\underline{CH}}^k(X,1) \to \frac{H^{k-1,k-1}(X,\mathbf{R}(k-1))}{H^{2k-2}_{\mathrm{alg}}(X)\otimes\mathbf{R}}.$$

We want to state some results in this direction. First some terminology. We say that the corresponding regulator $\underline{\underline{r}} : \underline{\underline{z}}^k(X,1) \to H^{k-1,k-1}(X,\mathbf{R}(k-1))$ is *surjective up to isogeny,* if there is a finite cover $\lambda : Y \to X$, such that the regulator $\underline{\underline{r}} : \underline{\underline{z}}^k(Y,1)_{\mathbf{R}} \longrightarrow H^{k-1,k-1}(X,\mathbf{R}(k-1))$ defined by $\xi \in \underline{\underline{z}}^k(Y,1)_{\mathbf{R}} \mapsto \underline{r}(\xi)(\lambda^* w)$, is surjective. In the theorem below, the word "general" means general with respect to Zariski open sets in the real analytic topology, i.e., where X is viewed as being defined over $\mathbf{R}$ via $X \to \mathrm{Spec}(\mathbf{C}) \to \mathrm{Spec}(\mathbf{R})$.

Theorem. *Assume that X is a general Abelian variety of dimension $n \leq 4$. Then*

(i) $\underline{r}_{k,1} : \underline{z}^k(X,1)_{\mathbf{R}} \longrightarrow H^{k-1,k-1}(X,\mathbf{R}(k-1))$ *is surjective for* $(k,n) = (2,2),\ (3,3)$.

(ii) $\underline{\underline{r}}_{k,1} : \underline{\underline{z}}^2(X,1)_{\mathbf{R}} \longrightarrow H^{k-1,k-1}(X,\mathbf{R}(k-1))$ *is surjective up to isogeny for* $(k,n) = (2,2),\ (3,3),\ (3,4)$.

There are two main ingredients in the proof of the above theorem. The first involves a specialization argument to the case where X is a product of elliptic curves. One makes use of the following:

Theorem. [Gordon-Lewis] *Assume that X is a general product of $n \leq 4$ elliptic curves, and $CH^k(X,1)$ is Bloch's higher Chow group as above. Then $r_{k,1} : CH^k(X,1)_{\mathbf{R}} \longrightarrow H^{k-1,k-1}(X,\mathbf{R}(k-1))$ is surjective for* $(k,n) = (2,2),\ (3,3)$, *and* $(3,4)$.

The other main ingredient uses the fact that Abelian varieties are dominated by products of curves. This is used in constructing classes in $\underline{z}^k(X,1)$ and $\underline{\underline{z}}^k(X,1)$.

Corollary I. *If X is a general Kummer surface, then the twisted regulator:*

$$\underline{r}_{2,1} : \underline{z}^2(X,1)_{\mathbf{R}} \to H^{1,1}(X,\mathbf{R}(1)),$$

is onto.

In the case of a general $K3$ surface X, we use a deformation argument from a general Kummer surface to deduce:

Corollary II. *For a general $K3$ surface X, the twisted regulator*

$$\underline{r}_{2,1} : \underline{z}^2(X,1)_{\mathbf{R}} \to H^{1,1}(X,\mathbf{R}(1)),$$

is likewise onto.

In the case of a $K3$ surface X, if we could also show that $\underline{\underline{r}}_{2,1} : \underline{\underline{z}}^2(X,1) \to H^{1,1}(X,\mathbf{R}(1))$ is onto, then using $q(X) = 0$, this would give us an affirmative answer to the Hodge-$\mathcal{D}$-conjecture for X.

Proof (outline) of Corollary I

A variant of the Hodge-$\mathcal{D}$-conjecture

for general Kummer surfaces

The basic idea is to see that both a general Abelian surface X, and X_2, the product of 2 general elliptic curves, live in some real analytic family $\mathcal{A} \to \mathcal{T}$ of Abelian surfaces. Then the strategy will be to perturb and reconstruct the "main terms" in the higher Chow cycles on X_2, along with enough algebraic cycles, so that they can be deformed, as twisted chains, continuously along (a path in) $\mathcal{T}$ in such a way as to keep their twisted regulator values linearly independent in $H^{1,1}(X, \mathbf{R}(1))$. After that, we attempt to complete these chains to elements of $\underline{z}^2(X, 1)$ whose regulator values remain either undisturbed or perturbed only by a sufficiently small amount. We then apply this information to the corresponding Kummer surface $X/\langle\sigma\rangle$, where σ is the corresponding involution on X.

Put $\tilde{z}^k(X,1) = \Gamma\big(\bigoplus_{\operatorname{codim}_X Z = k-1} \mathbf{C}(Z)^\times\big)$, and let $\xi = \{\sum_i f_i \otimes Z_i\} \in \tilde{z}^k(X,1)$. One has a regulator current $\tilde{r}_{k,1}(\xi)$ acting on $E^{n-k+1,n-k+1}_{X,d-\text{closed}}(\mathbf{R}(n-k+1))$, where $\dim X = n$, given by

$$\tilde{r}_{\mathcal{D}}(\xi)(w) \;=\; \frac{1}{(2\pi\sqrt{-1})^{n-k+1}} \sum_i \int_{Z_i} w \log|f_i| \;;$$

however this current is not $\partial\overline{\partial}$-closed, unless $\xi \in z^k(X,1)$, i.e. $\sum_i (f_i) = 0$. Thus we have

$$\tilde{r}_{k,1} : \tilde{z}^k(X,1) \to \{E^{n-k+1,n-k+1}_{X,d-\text{closed}}(\mathbf{R}(n-k+1))\}^\vee.$$

We can impose on $\tilde{r}_{\mathcal{D}}$ to be a cohomological map by working with harmonic forms $\mathcal{H}^\bullet_X$, which is metric independent, albeit not functorial with respect to holomorphic pullbacks and pushforwards f^*, f_*. The inclusion $\mathcal{H}^{\bullet,\bullet}_X \subset E^{\bullet,\bullet}_{X,d-\text{closed}}$ gives us the composite

$$\begin{array}{ccccc}
\tilde{r}_{k,1} & : & \tilde{z}^k(X,1) & \to & \{E^{n-k+1,n-k+1}_{X,d-\text{closed}}(\mathbf{R}(n-k+1))\}^\vee \\
 & & & & \downarrow \\
 & & & & \{\mathcal{H}^{n-k+1,n-k+1}_X\}^\vee \\
 & & & & \downarrow\wr \\
 & & & & \mathcal{H}^{k-1,k-1}_X \\
 & & & & \downarrow\wr \\
 & & & & H^{k-1,k-1}(X,\mathbf{R}(k-1)).
\end{array}$$

The next step is to construct cycles on X_2 that can be made to vary continuously with parameters in a space of Abelian surfaces. In particular, there are four classes (two of them are transcendental, the other two are algebraic) of the form $g_i \otimes C_i,\ i = 1, 2, 3, 4$, where $C_i \subset X_2$ is irreducible, and $g_i \in \mathbf{C}(X_2)^\times$, such that g_i defines a function $g_i \in \mathbf{C}(C_i)^\times$, for all i, and where $\tilde{r}_{\mathcal{D}}(g_i \otimes C_i)$, $i = 1, 2, 3, 4$, defines a basis of $\mathcal{H}^{1,1}_{X_2}$. If X_2 is a member of a family of Abelian surfaces $\mathcal{A}$, then there is no guarantee that the classes $\{g_i \otimes C_i \mid i = 1, \ldots, 4\}$ deform in $\mathcal{A}$, but we can show that there are classes "close enough" that do deform. Since X_2 is a member of a family of Abelian surfaces $\mathcal{A}$ in some $\mathbf{P}^N$, which we assume is smooth over some base variety, we can consider a hyperplane section $H := \mathbf{P}^{N-1} \cap X_2 \subset X_2$ that now clearly deforms in $\mathcal{A}$. Set $g \otimes C = g_i \otimes C_i$ for any given $i = 1, 2, 3, 4$. Note that for $m >> 0$, $mH - C$ is very ample. Write $C' \sim_{\text{rat}} mH - C$, where $C' > 0$ can be assumed irreducible. Thus $C + C' \sim_{\text{rat}} mH$, hence $C + C'$, mH belong to the same linear system $|mH|$, which we can assume is complete, by choosing m large enough. Now let us choose $k \in \mathbf{C}(X_2)^\times$ so that $k \equiv 0$ on C', and $k \not\equiv 0$ on C. For $\epsilon > 0$, set

$$h_\epsilon = 1 \ + \ (g-1)\frac{k}{k+\epsilon} \overset{\epsilon \to 0}{\mapsto} g.$$

Note that $h_\epsilon \otimes C + h_\epsilon \otimes C' = h_\epsilon \otimes C + 1 \otimes C'$, and that this class will deform in the linear system $|mH|$ as well as in $\mathcal{A}$. Thus to be more precise, we can assume that for some $K \in |m\mathbf{P}^{N-1}|$, $K \cap \mathcal{A}$ is smooth (use Bertini), $K \cap X_2 = C$, and that the deformation of $g \otimes C$ will now involve a class of the sort $h \otimes \{K \cap \mathcal{A}\}$, for a given rational function h.

It follows easily from this construction that we may assume that the classes $g_i \otimes C_i, \ i = 1,2,3,4$ deform continuously (algebraically) in $\mathcal{A}$, over the base variety of $\mathcal{A}$.

Next, we construct a family of Abelian surfaces over a *real* base. X_2 is represented by the period matrix

$$\begin{bmatrix} 1 & 0 & \tau_1 & 0 \\ 0 & 1 & 0 & \tau_2 \end{bmatrix},$$

where $\mathrm{Im}(\tau_j) > 0$. More generally, consider

$$\begin{bmatrix} 1 & 0 & a_{11} + \sqrt{-1}b_{11} & a_{12} + \sqrt{-1}b_{12} \\ 0 & 1 & a_{21} + \sqrt{-1}b_{21} & a_{22} + \sqrt{-1}b_{22} \end{bmatrix},$$

where $a_{12} = a_{21}, \ b_{12} = b_{21}$, and $\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} > 0$. Then for $t \in [-1,1]$, set

$$Z_t = \begin{bmatrix} a_{11} + \sqrt{-1}b_{11} & t(a_{12} + \sqrt{-1}b_{12}) \\ t(a_{21} + \sqrt{-1}b_{21}) & a_{22} + \sqrt{-1}b_{22} \end{bmatrix}.$$

For general Z_1, Z_0 defines a general product of two elliptic curves X_2. Since the forms we where dealing with, for the Abelian varieties above, are harmonic to begin with, we deduce that $\tilde{r}_{\mathcal{D}}$ is surjective for a general (in the [real analytic] Zariski topology sense) Abelian surfaces.

The proof of the surjectivity of $\underline{r}_{2,1}$ for general Abelian surfaces (and hence for general Kummer surfaces) will be established in four steps below.

Step I: By a deformation argument, one argues that the surjectivity is true if we replace $\underline{z}^2(X,1)$ by the corresponding $\tilde{z}^2(X,1)$. The point here is that the currents arising from $\tilde{z}^2(X,1)$ are no longer closed, and hence one must necessarily replace $H^{1,1}(X)$ by the corresponding harmonic space $\mathcal{H}_X^{1,1}$. This is a consequence of the above discussion.

Step II: The basic idea in this step is to extend classes in $\tilde{z}^2(X,1)$ to those in $\underline{z}^2(X,1)$, for certain product varieties X, involving only "degenerate" classes. This will be carried out in the Lemma below:

Lemma. *Let $X = M \times N$ be a product of two curves, and let $\xi \in \tilde{z}^2(X,1)$ be given. Then ξ can be completed to an element $\underline{\xi} \in \underline{z}^2(X,1)$, involving only "degenerate" factors. More specifically, involving flat line bundle terms of the form $\mathcal{O}_M(p_1 - p_2)$ on $M \times q \simeq M$ and $\mathcal{O}_N(q_1 - q_2)$ on $p \times N \simeq N$.*

Proof. The basic idea is this. First, for $\xi \in \tilde{z}^2(X,1)$, we write $\sim$ to mean equivalent modulo cycles arising from degenerate factors. Now let $\xi \in \tilde{z}^2(X,1)$. Then $\operatorname{div} \xi = \sum_j \{(p_j, q_j) - (t_j, s_j)\} \in z^2_{\mathrm{rat}}(X)$. Then via the pullback of $\mathcal{O}_M(t_j - p_j)$ on $M \times q_j \simeq M$, $(p_j, q_j) - (t_j, s_j) \sim (t_j, q_j) - (t_j, s_j)$, and via the pullback of $\mathcal{O}_N(s_j - q_j)$ on $t_j \times N \simeq N$, $(t_j, q_j) - (t_j, s_j) \sim 0$.

Remark. The reader can easily check that the extensions of $\xi \in \tilde{z}^2(X,1)$ to $\underline{\xi} \in \underline{z}^2(X,1)$ given in the lemma above, also determine classes $\underline{\underline{\xi}} \in \underline{\underline{z}}^2(X,1)$.

Step III: In the previous step, we considered extending cycles on a products of two curves. If X is an Abelian surface, and $C_1, C_2 \subset X$ are general curves, then the Pontryagin map $P : C_1 \times C_2 \to X$ is surjective. If $\xi \in \tilde{z}^2(X,1)$ is given, then the pullback, $P^*\xi \in \tilde{z}^2(C_1 \times C_2, 1)$ is defined, and can be extended to a class $\underline{P^*\xi} \in \underline{z}^2(C_1 \times C_2, 1)$. We then push forward our construction to $\underline{z}^2(X,1)$.

We now explain the deformation argument for $\underline{z}^2(X,1)$ for an Abelian surface X. We first want to show that certain divisors on X_2 that are 'degenerate' with respect to the Hodge structure $H^{1,1}(X_2, \mathbf{Q}(1))$, deform in the family $\mathcal{A}$.

Key Lemma. *Assume given $X_2 := E_1 \times E_2$, a product of two general elliptic curves. Let Q be a divisor on X_2 with the property that* $\deg\big((Q \cdot \{p \times E_2\})_{X_2}\big) > 0$ *and* $\deg\big((Q \cdot \{E_1 \times q\})_{X_2}\big) > 0$*, for $(p,q) \in X_2$. Then for a sufficiently large integer $m >> 0$:*

$$mQ \sim_{\mathrm{rat}} \Gamma_1 + \Gamma_2,$$

where

$$\Gamma_1 = E_1 \times q_1 + \cdots + E_1 \times q_s$$

$$\Gamma_2 = p_1 \times E_2 + \cdots + p_r \times E_2,$$

and where $\{q_1, \ldots, q_s\}$ (resp, $\{p_1, \ldots, p_r\}$) are distinct in E_2 (resp. E_1).

Now recall the setting $X_2 \subset \mathcal{A} \subset \mathbf{P}^N$, where $\mathcal{A}$ is a family of Abelian surfaces over some base variety $\mathcal{T}$. Then recall for $m >> 0$, the linear system of hypersurface sections associated to $\mathcal{O}_{X_2}(m) := \mathcal{O}_{X_2} \otimes \mathcal{O}_{\mathbf{P}^N}(m)$ is complete. Thus any divisor associated to $\mathcal{O}_{X_2}(m)$ deforms in $\mathcal{A}$ over $\mathcal{T}$. Now given any $\xi \in \tilde{z}^2(X_2, 1)$, we can complete ξ to $\underline{\xi} \in \underline{z}^2(X_2, 1)$ involving (flat) line bundles over divisors of the form $\Gamma_1 + \Gamma_2$ as in the above Lemma, which can be assumed to deform in $\mathcal{A}$ over $\mathcal{T}$. Thus the M and N in the Lemma can be made to deform in $\mathcal{A}$ over $\mathcal{T}$, and thus it is easy to see that classes in $\underline{z}^2(X_2, 1)$ can be made to deform in $\mathcal{A}$ over $\mathcal{T}$ as well. [Note: Here we are using the Pontryagin sum map $M \times N \to X$.]

The procedure in the previous paragraph works well in preserving the rank of a 2×2 matrix of regulator values of transcendental classes in $H^{1,1}(X_2)$, under deformations in in $\mathcal{A}$ over general points in $\mathcal{T}$. This is because Γ_1, Γ_2 are 'degenerate' divisors with respect to the Hodge structure $H^{1,1}(X_2, \mathbf{Q}(1))$. However, we also have to take into consideration the deformation of the algebraic classes in X_2, which requires the preservation of the nonsingularity of a corresponding 4×4 matrix over general points in $\mathcal{T}$, via a deformation in $\mathcal{A}$ over $\mathcal{T}$ of the classes in X_2.

Consider a possibly reducible curve $\Gamma \subset X_2$ that deforms in $\mathcal{A}$ over $\mathcal{T}$, and a given h_ϵ as above on Γ, where $g = \lambda \in \mathbf{C}^\times \backslash \{S^1\}$, and where h_ϵ is employed to extend a given algebraic cycle supported on Γ. Note that we can write $\operatorname{div}(h_\epsilon) = (p_1 - q_1) + \cdots + (p_r - q_r)$, where $(p_j - q_j) \mapsto 0$ as $\epsilon \to 0$. On the same curve where h_ϵ is defined, construct a rational section σ_ϵ of the flat line bundle associated to the divisor $\operatorname{div}(\sigma_\epsilon) = (q_1 - p_1) + \cdots + (q_r - p_r)$, such that $\sigma_\epsilon \mapsto 1$ as $\epsilon \mapsto 0$. Then $\operatorname{div}(h_\epsilon) + \operatorname{div}(\sigma_\epsilon) = 0$, and hence for small $\epsilon > 0$, one can easily argue that a corresponding 4×4 matrix from X_2 deforms in a nonsingular way over general points in $\mathcal{T}$.

The details of the variant of the Hodge-$\mathcal{D}$-conjecture for general $K3$ surfaces are similar.