THE TRANSCENDENTAL PART OF THE REGULATOR MAP FOR K_1 ON A MIRROR FAMILY OF $K3$ SURFACES

PEDRO LUIS DEL ANGEL AND STEFAN MÜLLER-STACH

Abstract. We compute the transcendental part of the normal function corresponding to the Deligne class of a cycle in K_1 of a mirror family of quartic $K3$ surfaces. The resulting multi-valued function does not satisfy the hypergeometric differential equation of the periods and we conclude that the cycle is indecomposable for most points in the mirror family. The occuring inhomogenous Picard-Fuchs equations are related to Painlevé VI type differential equations.

1. The regulator map and Picard-Fuchs equations

In this paper we study the first non-classical higher K-group $K_1(X)$ for a smooth complex projective surface X. It was conjectured by H. Esnault around 1995 that certain elements in this group can be detected in the transcendental part of the Deligne cohomology group $H^3_{\mathcal{D}}(X,\mathbb{Z}(2))$ via the regulator (Chern class) map. The transcendental part of the regulator map is defined as an Abel-Jacobi type integral of holomorphic two-forms over non-closed real 2-dimensional chains in X associated to these elements. At that time is was only known that one could detect such classes in the complementary $(1,1)$-part of Deligne cohomology (see e.g. [16]). The goal of our paper is to show that Esnault's conjecture is true by looking at the differential equations which are satisfied by the normal functions arising from such classes in a family of surfaces. It turns out that the resulting equations for Abel-Jacobi type integrals with parameters are strongly connected to a generalization of Painlevé VI type differential equations.
The higher K-groups $K_1(X), K_2(X), \ldots$ of an algebraic variety X were defined around 1970 by D. Quillen [19]. Later Bloch [3] showed that on smooth quasi-projective varieties all their graded pieces with respect to the γ-filtration may be computed as

$$\mathrm{gr}^p_\gamma K_n(X)_\mathbb{Q} \cong CH^p(X,n)_\mathbb{Q}$$

where $CH^p(X,n)$ are Bloch's higher Chow groups [3]. This isomorphism gives an explicit presentation of higher K-groups modulo torsion via algebraic cycles.
Let us look more closely at the particular case of $K_1(X)$ for a smooth complex projective surface X. There it is known that $CH^1(X,1)=\mathbb{C}^\times$ and $CH^p(X,1)=0$ for $p\geq 4$. The remaining interesting parts of K_1 are therefore $CH^2(X,1)$ and $CH^3(X,1)$. The last group consists of zero cycles on $X\times\mathbb{P}^1$ in good position and therefore the map

$$\tau: CH^2(X)\otimes_Z \mathbb{C}^\times \to CH^3(X,1), \quad x\otimes a\mapsto (x,a)$$

is surjective. Therefore, the complexity of $CH^3(X,1)$ is governed by the complexity of $CH^2(X)$ which is fairly understood by Mumford's theorem resp. Bloch's conjecture. We say that

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$CH^3(X,1)$ is decomposable. For $CH^2(X,1)$ the situation is quite different and the complex geometry of X plays an essential role in the understanding of it. The natural map

$$\tau : CH^1(X) \otimes_Z \mathbb{C}^\times \to CH^2(X,1), \quad D \otimes a \mapsto D \times \{a\}$$

is neither surjective nor injective in general. In the literature there are several examples where the cokernel of τ is non-trivial modulo torsion and even infinite dimensional, see [4], [12], [16] and [24]. The kernel of τ is related but not equal to $\mathrm{Pic}^0(X) \otimes \mathbb{C}^\times$ even modulo torsion by [21, thm. 5.2]. Note that the cokernel of τ is a *birational invariant* (by localization) and hence vanishes for rational surfaces and, in fact, for all surfaces with geometric genus $p_g(X) = 0$ and Kodaira dimension ≤ 1. The Bloch conjecture would imply that it vanishes also for all surfaces of general type which satisfy $p_g(X) = 0$. One way to study $CH^2(X,1)$ is to look at the Chern class maps

$$c_{2,1} : CH^2(X,1) \to H^3_{\mathcal{D}}(X,\mathbb{Z}(2)) = \frac{H^2(X,\mathbb{C})}{H^2(X,\mathbb{Z}) + F^2H^2(X,\mathbb{C})}. \tag{1.1}$$

The decomposable cycles (the image of τ) are mapped to the subgroup

$$\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{C}^\times \subseteq \frac{H^2(X,\mathbb{C})}{H^2(X,\mathbb{Z}) + F^2H^2(X,\mathbb{C})}$$

generated by the *Néron-Severi group* $\mathrm{NS}(X) \subset H^2(X,\mathbb{Z})$ of all divisors in X. It is known [16] that the image of $c_{2,1}$ is at most countable modulo this subgroup, so that the image of $\mathrm{coker}(\tau)$ in Deligne cohomology modulo $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{C}^\times$ is at most countable. One conjectures that even $\mathrm{coker}(\tau)$ itself is countable.

The Chern class maps $c_{2,1}$ are defined as follows: let $Z = \sum a_j Z_j \in CH^2(X,1)$ be a cycle. Each Z_j is an integral curve and inherits a rational map $f_j : Z_j \to \mathbb{P}^1$ from the projection map $X \times \mathbb{P}^1 \to \mathbb{P}^1$. Let γ_0 be a path on $\mathbb{P}^1$ connecting 0 with ∞ along the real axis, then $\gamma := \cup \gamma_j := \cup f_j^{-1}(\gamma_0)$ is a closed homological 1-cycle, Poincaré dual to a cohomology class in $F^2H^3(X,\mathbb{Z})$ and therefore torsion, see [16]. If we assume that $\gamma = 0$ (for example if $b_1(X) = 0$) then we write $\gamma = \partial\Gamma$ for a real piecewise smooth 2-chain Γ. The defining property of $c_{2,1}(Z)$ is, as a current, i.e. a linear functional on differentiable complex valued 2-forms on X

$$c_{2,1}(\alpha) = \frac{1}{2\pi i} \sum_j \int_{Z_j - \gamma_j} \log(f_j)\alpha + \int_\Gamma \alpha. \tag{1.2}$$

Now let X be a projective K3 surface. Then $p_g(X) = 1$, $b_2(X) = 22$ and $b_1(X) = 0$. The intersection form on $H^2(X,\mathbb{Z})$ is known to be the unimodular form $2E_8 \oplus 3H$, where H is the 2-dimensional standard hyperbolic form.

The Néron-Severi lattice $NS(X) \subset H^2(X,\mathbb{Z})$ has an orthogonal complement $T(X) \subset H^2(X,\mathbb{Z})$. In particular there is a well-defined morphism

$$\mathrm{Coker}(\tau) \to \frac{T(X) \otimes \mathbb{C}^\times}{F^2}.$$

If we have an arbitrary smooth family $f : X \to B$ of complex algebraic surfaces over a quasiprojective complex variety B, and an algebraic family of cycles $Z_b \in CH^2(X_b,1)$ for all $b \in B$, then we may define the *normal function*

$$\nu(b) := c_{2,1}(Z_b) \in \frac{T(X_b) \otimes \mathbb{C}^\times}{F^2}.$$

One can easily show that ν is a holomorphic (however multivalued) section of the corresponding family of generalized tori $T(X_b) \otimes \mathbb{C}^\times / F^2$. Coming back to the case of K3-surfaces: there the canonical bundle ω_X is trivial, hence the group $H^{0,2}(X) = H^0(X, \Omega^2_X)^* = \mathbb{C}$ is 1-dimensional and generated by the dual of ω_X. In a smooth algebraic family X_b of K3-surfaces, the composition of the regulator with the projection onto

$$\frac{H^{0,2}(X_b)}{\mathrm{Im} H_2(X_b, \mathbb{Z})}$$

produces a multivalued holomorphic function on B, denoted by $\bar{\nu}(b)$, which has poles at all b where the family degenerates (proof see below). It is given by the formula

$$\bar{\nu}(b) = \int_{\Gamma_b} \omega_{X_b},$$

since the integral of ω_X over any effective divisor vanishes. If $\mathcal{D}_{\mathrm{PF}}$ denotes the Picard-Fuchs differential operator of the Gauß-Manin connection associated to the family X_b of K3-surfaces, then $\mathcal{D}_{\mathrm{PF}}$ annihilates all periods of the family. Therefore we obtain the following result:

Lemma 1.1. *Let $B \subset \bar{B}$ be smooth compactification of B. Then with the notation above, $\mathcal{D}_{\mathrm{PF}}(\bar{\nu})$ extends to a single-valued meromorphic function on $\bar{B}$ with poles only along degeneracies of X_b, and therefore satisfies a differential equation*

$$\mathcal{D}_{\mathrm{PF}}(\bar{\nu}(b)) = g(b), \tag{1.3}$$

where g is a rational function in $b \in \bar{B}$.

The proof is given in the appendix. Altogether we have obtained a map:

$$\{\text{Families of Cycles in } CH^2(X_b, 1)\} \longrightarrow \{\text{Differential Equations/Rational Functions}\} \tag{1.4}$$

For each such family of K3-surfaces it sends a family of cycles to the equation $\mathcal{D}_{\mathrm{PF}}\bar{\nu} = g$ resp. the rational function g, which is the same information on a given family. One should view the resulting solutions $\bar{\nu}(b)$ as *new transcendantal functions* arising from the family of K-theoretic cycles in $CH^2(X_b, 1)$. If g is a non-trivial function, then $\bar{\nu}$ and hence ν is a non-flat section of the family of Deligne cohomology groups of X_b. In [16] the relationship between the infinitesimal behaviour of such normal functions and the mixed Hodge structure of the total space X was already investigated.

This situation is very reminiscent of a method developed by Richard Fuchs [10] in the case of the Legendre family $y^2 = x(x-1)(x-t)$ of elliptic curves and investigated further in the work of Manin [14, page 134]. In particular there is a strong connection with differential equations of a generalized form of type Painlevé VI (loc. cit.).

There exists a *formula* to derive g: there is a so-called *inhomogenous Picard-Fuchs* equation

$$\mathcal{D}_{\mathrm{PF}}\omega_X = d_{rel}\beta \tag{1.5}$$

before integration over Γ, where β is a section of the vector bundle of (meromorphic) 1-forms in the fibers of the family $f: X \to B$. We say that Γ *does not depend on b* if it can be defined as real semi-algebraic subset via *flat coordinates*, i.e. coordinate functions which are horizontal with respect to the Gauß-Manin connection, and such that the defining inequalities of Γ are polynomials not depending on b. This shows on one hand that for closed Γ the periods satisfy the Picard-Fuchs equation, and on the other hand for non-closed Γ (not depending on b) with $\partial\Gamma = \gamma$ we get

$$g(b) = \mathcal{D}_{\mathrm{PF}} \int_\Gamma \omega_X = \int_\Gamma d_{rel}\beta = \int_\gamma \beta. \tag{1.6}$$

The last equality uses a version of Stokes theorem for currents since some of the differential forms involved will in general have integrable singularities. Hence Stokes theorem for currents (see [11, chap. 3]) also implies that β is integrable over γ. In general Γ depends also on b, and then there will be an additional contribution from the derivatives of the boundaries of the integral. In the case of the Legendre family $y^2 = x(x-1)(x-t)$ of elliptic curves, β is a meromorphic function (0-form)

$$\frac{y}{2(x-t)^2},$$

by [10, p. 310],[14, p. 76]. Manin has put these equations into a more formal context (so-called μ-equations) so that one can understand the sections and operators in a coordinate-free way in terms of certain locally free sheaves on B. This plays also a role in his work on the functional Mordell conjecture. Furthermore, after uniformizing the elliptic curves, the inhomogenous Picard-Fuchs equation is equivalent to a version involving the Weierstraß $\mathfrak{p}$-function [14, p. 137]:

$$\frac{d^2 z}{d\tau^2} = \frac{1}{(2\pi i)^2} \sum_{j=0}^{3} \alpha_j \mathfrak{p}_z(z + \frac{T_j}{2}, \tau),$$

where α_j are constants parametrizing the family of differential equations and $(T_0, \dots, T_3) = (0, 1, \tau, 1+\tau)$ are the vertices of the fundamental parallelogram. In this way the transcendental aspect of the solutions and also the connection to integrable systems becomes apparent, see [14, p. 139]. In the future we hope to investigate further the transcendental properties of our solutions (using again uniformization) and study the attached integrable systems.
The rest of this article is devoted to a particular solution of the inhomogenous Picard-Fuchs equation for a certain family of K3 surfaces introduced in section 2. In section 3 we deduce Esnault's conjecture from the non-vanishing of the $\mathcal{D}_{\mathrm{PF}}(\bar{\nu})$ in the special case $b = \sqrt{-1}$. In section 4 we study a certain Shioda-Inose model of X_b which has isomorphic transcendental cohomology. This leads to an explicit computation of β in this case.

2. An example: a mirror family of K3-surfaces

We will study the one-dimensional family of K3-surfaces given by the quartic equations

$$X_b := \{(x,y,z,w) \in \mathbb{P}^3 \mid f(x,y,z,w) = xyz(x+y+z+bw) + w^4 = 0\}. \tag{2.1}$$

with $b \in \mathbb{P}^1$. Note that this surface, for general b, is not smooth but has six singular points defining a rational singularity of type A_3. The six points are ([17, sect.4]):

$$\begin{aligned} P_1 &= (0,1,-1,0), & P_2 &= (1,-1,0,0) \\ P_3 &= (1,0,-1,0), & P_4 &= (1,0,0,0) \\ P_5 &= (0,1,0,0), & P_6 &= (0,0,1,0) \end{aligned}$$

The minimal resolution of the singularities defines a generically smooth family of K3-surfaces. In [17] the following theorem was shown:

Theorem 2.1. (Narumiya/Shiga) *The family X_b has the following properties:*

1. *It arises as a mirror family from the dual of the simplest polytope P of dimension three. The dual mirror family is the family of all quartic K3-surfaces.*
2. *The rank of* $\mathrm{Pic}(X)$ *is* ≥ 19 *for all* $b \in \mathbb{P}^1 \setminus \{0, \pm 4, \infty\}$ *and equal to* 19 *for very general b (see loc.cit. §4.).*
3. *$T(X_b)$ has signature* $(2,1)$ *for* $b \in \mathbb{P}^1 \setminus \{0, \pm 4, \infty\}$.
4. *The periods of X_b satisfy the Picard-Fuchs equation*
$$(1-u)\Theta^3 - \frac{3}{2}u\Theta^2 - \frac{11}{16}u\Theta - \frac{3}{32}u = 0$$
(where $\Theta = u\frac{d}{du}$) of the generalized Thomae hypergeometric function [22]
$$F_{3,2}(\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, 1; u)$$
and where we set $u := (\frac{4}{b})^4$.
5. *In other words, the Picard-Fuchs equation is given by*

$$(1-u)u^2\Phi''' + 3u(1-\frac{3}{2}u)\Phi'' + (1-\frac{51}{16}u)\Phi' - \frac{3}{32}\Phi = 0. \tag{2.2}$$

6. *The mirror map of the family X_b is given by the arithmetic Thompson series $T(q)$ of type* **2A** *in the classification of Conway and Norton* [5]*:*
$$T(q) = \frac{1}{q} + 8 + 4372q + 96256q^2 + 124002q^3 + 10698752q^4 + \dots$$

Proof. We refer to [17] for more details, but we sketch the proof of (4) and (5) since this is crucial. (1)-(3) follow from the construction. In particular six A_3-singularities give rise to 18 independent cohomology classes of type (1,1) so that the Picard number is ≥ 19. Since (5) is an easy corollary of (4), we prove (4). In [17] the periods are computed as power series in $1/b$ and the differential equation in (4) follows from [22]. As in [17] we consider the new affine equation
$$f(x,y,z) = xyz(x+y+z+1) + 1/b^4 = 0$$
obtained by substituting $w' := bw$ and setting $w' = 1$. The periods are integrals of the form
$$I(b) = \frac{1}{2\pi i}\int_{|x|=|y|=|z|=1/4} \frac{dxdydz}{xyz(x+y+z+1)+1/b^4}.$$
On the other hand one has a geometric series expansion at $b = \infty$:
$$\frac{1}{xyz(x+y+z+1)+1/b^4} = \sum_{n=0}^{\infty} \frac{(-1)^n b^{-4n}}{(xyz)^{n+1}(x+y+z+1)^{n+1}}.$$
Changing the order of summation and integration, we obtain
$$\begin{aligned} I(b) =& \frac{1}{2\pi i}\int_{|x|=|y|=|z|=1/4} \sum_{n=0}^{\infty} \frac{(-1)^n b^{-4n} dxdydz}{(xyz)^{n+1}(x+y+z+1)^{n+1}} \\ =& \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{|x|=|y|=|z|=1/4} \frac{(-1)^n b^{-4n} dxdydz}{(xyz)^{n+1}(x+y+z+1)^{n+1}} \end{aligned}$$

Now one can apply 3 times the residue theorem and gets

$$I(b) = (2\pi i)^2 \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} b^{-4n}.$$

Observing the identity involving Pochhammer symbols

$$\frac{(4n)!}{(n!)^4} = \frac{(\frac{1}{4})_n (\frac{2}{4})_n (\frac{3}{4})_n}{(1)_n (1)_n (1)_n} (4^4)^n,$$

we have shown that

$$I(b) = (2\pi i)^2 F_{3,2}(\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, 1; (\frac{4}{b})^4)$$

Substituting $u := (\frac{4}{b})^4$, one gets a multiple of the functions $F_{3,2}(\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, 1; u)$ which satisfy a differential equation of order three precisely of the type described in (4) resp. (5) by [22]. □

Corollary 2.2. *In b-coordinates, the Picard-Fuchs equation can be written as*

$$((\frac{b}{4})^4 - 1)(\frac{b}{4})^3 \Phi''' + \frac{3}{4}(\frac{b}{4})^2 (1 + (\frac{b}{4})^4)\Phi'' + \frac{1}{16}\frac{b}{4}((\frac{b}{4})^4 - 6)\Phi' + \frac{3}{32}\Phi = 0 \tag{2.3}$$

Proof. Use chain rule. □

To make the following computations easier, we follow [17] and perform the following birational coordinate change (written in affine coordinates):

$$X = xy, \quad Y = i(\frac{bxy}{2} + \frac{1 + xyz}{z}), \quad Z = yz.$$

Then X, Y, Z are affine coordinates and define the family of surfaces S_b in Weierstraßform:

$$S_b : \quad Y^2 = X(X^2 + X(Z + \frac{1}{Z} - \frac{b^2}{4}) + 1)$$

as an elliptic fibration over $\mathbb{P}^1$ in the Z-coordinate. The inverse transformation is given by

$$x = -2\frac{iX(1 + ZX)}{(-2Y + ibX)Z}, \quad y = \frac{1}{2}\frac{iZ(-2Y + ibX)}{1 + ZX}, \quad z = -2\frac{i(1 + ZX)}{-2Y + ibX},$$

in affine coordinates. The following is taken from [17] (with a slight correction):

Lemma 2.3. *The surfaces S_b are ramified coverings of $\mathbb{P}^1 \times \mathbb{P}^1$ (in X, Z coordinates). In X, Y, Z coordinates, the canonical holomorphic two form on S_b is given up to an non-zero constant by*

$$\omega = \frac{dXdZ}{YZ} = \frac{dXdY}{X^2(Z - \frac{1}{Z})}, \tag{2.4}$$

where X, Z are flat coordinates and

$$Y = Y(b) = \sqrt{P(X, Z)} = \sqrt{X(X^2 + X(Z + \frac{1}{Z} - \frac{b^2}{4}) + 1)}.$$

Proof. In the x, y, z coordinate system the holomorphic two form is given up to a constant by $\frac{dxdy}{f_z}$ in affine (x, y, z)-coordinates with $w = 1$. Then, using the coordinate transformations above, one computes that

$$\omega = \sqrt{-1} \cdot \frac{dxdy}{f_z} = \frac{dXdY}{X^2(Z - \frac{1}{Z})} = \frac{dXdZ}{YZ},$$

since $2YdY = X^2(1 - \frac{1}{Z^2})dZ + \frac{\partial P}{\partial X}dX$. □

Now, if we apply the Picard-Fuchs-Operator $\mathcal{D}_{\mathrm{PF}}$ from Cor.2.2 to $\frac{dXdZ}{YZ}$, we get an expression of the form

$$\mathcal{D}_{\mathrm{PF}} \frac{dXdZ}{YZ} = K(X, Z) \cdot \frac{dXdZ}{Y^7 Z}$$

where $K(X, Z)$ is a polynomial function in X, Z.

3. The normal function and the Picard-Fuchs equation

On any elliptic surface, the easiest way to find cycles in K_1 is to use fibers. However sometimes configurations coming from Néron fibers (degenerate into a union of $\mathbb{P}^1$'s) do have trivial class in K_1 as was already observed by Beilinson in [1]. But one can use one smooth fiber together with a bunch of sections (rational) and rational curves in degenerate fibers. In our example let us take the following cycles: denote by S_b the surface defined by the equation

$$S_b : \quad ZY^2 = X(X^2Z + (Z^2 + 1 - Z\frac{b^2}{4})X + Z) \tag{3.1}$$

Let C_b be the smooth elliptic fibre over $Z = 1$ of this surface. Its defining equation is hence

$$C_b : \quad Y^2 = X(X^2 + (2 - \frac{b^2}{4})X + 1)$$

The quadratic term $X^2 + (2 - \frac{b^2}{4})X + 1$ in the right hand side has two negative real roots if b is purely imaginary, for example $b = \sqrt{-1}$. The points $X = 0$ and $X = \infty$ are rationally equivalent on C_b after taking a multiple of two, since they are ramification points. The real line from 0 to ∞ does not hit the other ramification points by this observation. The surface X_b in this birational model has 0 and $X = \infty$ as sections. The fiber over $Z = 0$ on S_b decomposes into three rational curves with multiplicity counted. Hence one can construct a cycle in $CH^2(S_b, 1)$ for general b by using C_b, the two sections and the degenerate fibers and appropriate rational functions on all curves. In X, Z coordinates the region Γ is given by the real square $0 \le Z \le 1, 0 \le X \le \infty$. For $b = \sqrt{-1}$ we make the following observation:

Lemma 3.1. *For $b = \sqrt{-1}$ all coefficients occuring in $\mathcal{D}_{\mathrm{PF}} \frac{dXdZ}{YZ} = K(X, Z) \cdot \frac{dXdZ}{Y^7 Z}$ are positive integers, i.e. all coefficients of $K(X, Z)$ and all coefficients of $Y^2 = X(X^2 + X(Z + \frac{1}{Z} - \frac{b^2}{4}) + 1)$.*

Proof. Here the rules of differentiating are $\frac{\partial X}{\partial b} = \frac{\partial Z}{\partial b} = 0$ and $\frac{\partial Y}{\partial b} = \frac{b}{4} \frac{1}{Y^3}$. This implies that odd derivatives of $1/Y$ get multiplied by even powers of b. Now if we look at the coefficients of equation (2.3), we see that the coefficients at Ψ''' and Ψ' become positive, since $(\frac{b}{4})^4 - 1$ and $(\frac{b}{4})^4 - 6$ are negative rational numbers and get multiplied with $(\frac{b}{4})^6$, resp. $(\frac{b}{4})^2$ which are both also negative rational numbers. The coefficients at Ψ and Ψ'' involve already 4-th powers of

b and hence positive. Consequently all coefficients occuring are positive. Using any computer algebra program this can be verified and indeed one has:

$$\begin{aligned}\mathcal{D}_{\mathrm{PF}}\frac{dX\,dZ}{YZ} =& \frac{dX\,dZ}{8192}(349951\,X^3\,Z^3 + 85952\,X\,Z^3 + 85952\,X^5\,Z^3 + 171904\,X^4\,Z^4 + 171904\,X^4\,Z^2 \\ & + 85952\,X^3\,Z^5 + 171904\,X^2\,Z^4 + 294912\,X^2\,Z + 294912\,X\,Z^2 + 98304\,Z^3 \\ & + 98304\,X^3 + 294912\,X^4\,Z + 909352\,X^2\,Z^3 + 909352\,X^3\,Z^2 + 98304\,X^6\,Z^3 \\ & + 294912\,X^5\,Z^4 + 294912\,X^5\,Z^2 + 909352\,X^4\,Z^3 + 294912\,X^4\,Z^5 + 909352\,X^3\,Z^4 \\ & + 294912\,X\,Z^4 + 98304\,X^3\,Z^6 + 294912\,X^2\,Z^5 + 85952\,X^3\,Z + 171904\,X^2\,Z^2)/ \\ & (4\,X^2\,Z + 4\,X\,Z^2 + 4\,X + X\,Z + 4\,Z)^{7/2}\,\sqrt{XZ}\end{aligned}$$

This completes the proof. ∎

In particular, if we integrate over the positive region Γ, we get a positive and non-zero integral. Since the boundary of Γ is defined as the rectangle $0 \le Z \le 1, 0 \le X \le \infty$ and X, Z are flat with respect to the connection, we say that Γ does not depend on b (see introduction) and this suffices to show that the normal function is non-trivial. So we have proved the Esnault's conjecture (see [16]):

Corollary 3.2. *The projected normal function $\bar{\nu}(b)$ does not satisfy the Picard-Fuchs equation*

$$((\frac{b}{4})^4 - 1)(\frac{b}{4})^3\Phi''' + \frac{3}{4}(\frac{b}{4})^2(1 + (\frac{b}{4})^4)\Phi'' + \frac{1}{16}\frac{b}{4}((\frac{b}{4})^4 - 6)\Phi' + \frac{3}{32}\Phi = 0 \tag{3.2}$$

In particular, it is not a rational multiple of a period for all but a countable number of values b. For those b, the corresponding cycle Z_b has no integer multiple which is decomposable modulo $Pic(X_b) \otimes \mathbb{C}^*$.

The main open problem remains to find 1-form β such that $d\beta = \mathcal{D}_{\mathrm{PF}}\frac{dXdZ}{YZ}$. We will compute such a β for the Kummer type model of these K3 surfaces in the next section.

4. The solution of $\mathcal{D}\omega = d\beta$

In [17] you can find the description of a 2:1 map $\pi : S_b \to S'_b$ onto a Kummer surface S'_b which has a birational model with the equation

$$u^2 = s(s-1)(s - (\frac{\nu+1}{\nu-1})^2)t(t-1)(t-\nu^2), \tag{4.1}$$

where ν and b are related via the algebraic equation

$$b^2 = -4 \cdot \frac{(\nu^2+1)^2}{\nu(\nu^2-1)}.$$

We prefer to use this equation for a computation of the solution of $\mathcal{D}\omega = d\beta$, since it is slightly easier but we do not loose essential information. In this description we see that the transcendental part of $H^2(S_b)$ also denoted by $T(S_b)$ has a Inose-Shioda structure (in the sense of [15]) and is therefore related to a variation of a family of elliptic curves. In fact there are two isogenous elliptic curves $E_1(\nu)$ and $E_2(\nu)$ with equations

$$E_1(\nu) :\ u_1^2 = s(s-1)(s-\nu^2), \quad E_2(\nu) :\ u_2^2 = t(t-1)(t - (\frac{\nu+1}{\nu-1})^2) \tag{4.2}$$

together with a Nikulin involution (see [23]) on the abelian surface $A = E_1 \times E_2$ such that the associated Kummer surface is S'_b and one has an isomorphism $T(S_b) \cong T(S'_b)$ under π_*. This explains in addition why the periods of S_b are squares of other hypergeometric functions related to the family $E_1(\nu)$ resp. $E_2(\nu)$. More details about the birational map can be found in [17]. Further instances where Inose-Shioda structures and modular forms come up can be found in [8]. Let us now compute the Picard-Fuchs equation of the family $E_1(\nu)$. If we let $\lambda = \nu^2$, we have $\frac{\partial\nu}{\partial\lambda} = \frac{1}{2\nu} = \frac{1}{2\sqrt{\lambda}}$ and therefore for any function Φ we have the transformation rules

$$\frac{\partial\Phi}{\partial\lambda} = \frac{\partial\Phi}{\partial\nu} \cdot \frac{\partial\nu}{\partial\lambda} = \frac{\partial\Phi}{\partial\nu}\frac{1}{2\sqrt{\lambda}} = \frac{\partial\Phi}{\partial\nu}\frac{1}{2\nu}$$

and for the second derivative

$$\frac{\partial^2\Phi}{\partial\lambda^2} = \frac{1}{4\nu^2}\frac{\partial^2\Phi}{\partial\nu^2} - \frac{1}{4\nu^3}\frac{\partial\Phi}{\partial\nu}.$$

Plugging this into the standard hypergeometric Picard-Fuchs equation

$$\lambda(1-\lambda)\Phi''(\lambda) + (1-2\lambda)\Phi'(\lambda) - \frac{1}{4}\Phi(\lambda) = 0,$$

we get the new equation

$$(1-\nu^2)\Phi''(\nu) + \frac{1-3\nu^2}{\nu}\Phi'(\nu) - \Phi(\nu) = 0$$

and the inhomogenous variant (equality of 1-forms)

$$(1-\nu^2)\frac{\partial^2}{\partial\nu^2}\omega(s) + \frac{1-3\nu^2}{\nu}\frac{\partial}{\partial\nu}\omega(s) - \omega(s) = 2d_{\mathrm{rel}}\frac{\sqrt{s(s-1)(s-\nu^2)}}{(s-\nu^2)^2} = 2d_{\mathrm{rel}}\frac{s^2(s-1)^2}{\sqrt{s(s-1)(s-\nu^2)}^3},$$

where

$$\omega(s) = \frac{ds}{\sqrt{s(s-1)(s-\nu^2)}}.$$

The rational normalized version of this equation is

$$\frac{\partial^2}{\partial\nu^2}\omega(s) + \frac{1-3\nu^2}{\nu(1-\nu^2)}\frac{\partial}{\partial\nu}\omega(s) - \frac{1}{1-\nu^2}\omega(s) = \frac{2}{1-\nu^2}d_{\mathrm{rel}}\frac{\sqrt{s(s-1)(s-\nu^2)}}{(s-\nu^2)^2}, \tag{4.3}$$

In a similar way we use the substitution $\lambda = (\frac{\nu+1}{\nu-1})^2$ and get the formula $\frac{\partial\nu}{\partial\lambda} = -\frac{(\nu-1)^3}{4(\nu+1)}$ and hence

$$\frac{\partial\Phi}{\partial\lambda} = \frac{\partial\Phi}{\partial\nu}\cdot\frac{\partial\nu}{\partial\lambda} = -\frac{(\nu-1)^3}{4(\nu+1)}\frac{\partial\Phi}{\partial\nu}, \quad \frac{\partial^2\Phi}{\partial\lambda^2} = \frac{(\nu-1)^6}{16(\nu+1)^2}\frac{\partial^2\Phi}{\partial\nu^2} + \frac{(\nu-1)^5(\nu+2)}{8(\nu+1)^3}\frac{\partial\Phi}{\partial\nu}$$

Combining all this, we get the equation

$$\frac{\partial^2}{\partial\nu^2}\omega(t) + \frac{\nu^2-2\nu-1}{\nu(\nu^2-1)}\frac{\partial}{\partial\nu}\omega(t) + \frac{1}{\nu(\nu-1)^2}\omega(t) = -\frac{2}{\nu(\nu-1)^2}d_{\mathrm{rel}}\frac{\sqrt{t(t-1)(t-(\frac{\nu+1}{\nu-1})^2}}{(t-(\frac{\nu+1}{\nu-1})^2)^2} \tag{4.4}$$

for

$$\omega(t) = \frac{dt}{\sqrt{t(t-1)(t-(\frac{\nu+1}{\nu-1})^2)}}.$$

We have to compute a sort of convolution product of these two equations in the following sense: set

$$\omega = \omega(s) \wedge \omega(t) = \frac{dsdt}{\sqrt{s(s-1)(s-\nu^2)t(t-1)(t-(\frac{\nu+1}{\nu-1})^2)}},$$

and notice that we have the product formula:

$$\frac{\partial^3}{\partial \nu^3}\omega = \frac{\partial^3}{\partial \nu^3}\omega(s) \wedge \omega(t) + 3\frac{\partial^2}{\partial \nu^2}\omega(s) \wedge \frac{\partial}{\partial \nu}\omega(t) + 3\frac{\partial}{\partial \nu}\omega(s) \wedge \frac{\partial^2}{\partial \nu^2}\omega(t) + \omega(s) \wedge \frac{\partial^3}{\partial \nu^3}\omega(t),$$

Similar formulas hold for lower derivatives. Note that s, t are flat coordinates so that differentiating a differential form with respect to the coefficients is a well-defined procedure. Such formulas can be used to compute $\frac{\partial^3}{\partial \nu^3}\omega$ and obtaining a Picard-Fuchs differential operator $\mathcal{D}$ for ω together with a solution β of $\mathcal{D}\omega = d_{\rm rel}\beta$:

Lemma 4.1. *One has the following inhomogenous Picard-Fuchs equation involving 2-forms:*

$$\frac{\partial^3\omega}{\partial \nu^3} + 3\frac{2\nu+1}{\nu(\nu+1)}\frac{\partial^2\omega}{\partial \nu^2} + \frac{7\nu^4-6\nu^3-4\nu^2+6\nu+1}{(\nu-1)^2(\nu+1)^2\nu^2}\frac{\partial\omega}{\partial \nu} + \frac{\nu^4-2\nu^3-2\nu-1}{(\nu-1)^3(\nu+1)^2\nu^2}\omega = d_{\rm rel}\beta, \tag{4.5}$$

where

$$\beta = -2\frac{s(s-1)(2\nu^4+3\nu^3-\nu^2-3\nu s+s\nu^2-2s)}{\nu(s-\nu^2)^2(\nu^2-1)^2\sqrt{s(s-1)(s-\nu^2)}}\omega(t) + \frac{6}{1-v^2}\frac{\sqrt{s(s-1)(s-\nu^2)}}{(s-\nu^2)^2}\omega'(t) \tag{4.6}$$

$$2\frac{t(t-1)(2t\nu^4-7t\nu^3+7t\nu^2-t\nu-t-2\nu^4-7\nu^3-7\nu^2-\nu+1)}{\nu^2(\nu-1)^4(t-(\frac{\nu+1}{\nu-1})^2)^2(\nu^2-1)\sqrt{t(t-1)(t-(\frac{\nu+1}{\nu-1})^2)}}\omega(s)+6\frac{\sqrt{t(t-1)((t-(\frac{\nu+1}{\nu-1})^2)}}{\nu(\nu-1)^2(t-(\frac{\nu+1}{\nu-1})^2)^2}\omega'(s).$$

Proof. We denote the derivative of a function (or a form in flat coordinates) f of ν by f'. Let us carry out the computation in a more general setting: assume we have two Picard-Fuchs equations:

$$\omega''(s) - A_s\omega'(s) - B_s\omega(s) = d\beta_s,$$
$$\omega''(t) - A_t\omega'(t) - B_t\omega(t) = d\beta_t,$$

with functions β_s, β_t and A_s, B_s, A_t, B_t depending on ν. Now first note that

$$\omega'''(s) = A_s\omega''(s) + (A_s' + B_s)\omega'(s) + B_s'\omega(s) + \frac{d}{d\nu}d\beta_s$$

and furthermore

$$\frac{d}{d\nu}d\beta_s = d\beta_s',$$

by symmetry of mixed derivatives. A similar relation holds for t. By using the product formulas

$$\omega''' = \omega'''(s) \wedge \omega(t) + 3\omega''(s) \wedge \omega'(t) + 3\omega'(s) \wedge \omega''(t) + \omega(s) \wedge \omega'''(t),$$
$$\omega'' = \omega''(s) \wedge \omega(t) + 2\omega'(s) \wedge \omega'(t) + \omega(s) \wedge \omega''(t),$$

we obtain that

$$\omega''' - \frac{3}{2}(A_s + A_t)\omega'' = [A_s' + B_s + 3B_t - \frac{1}{2}A_s^2 - \frac{3}{2}A_sA_t]\omega'(s) \wedge \omega(t)$$

$$+[A_t' + B_t + 3B_s - \frac{1}{2}A_t^2 - \frac{3}{2}A_sA_t]\omega(s) \wedge \omega'(t) + [A_sB_s + A_tB_t - \frac{3}{2}(B_s + B_t)(A_s + A_t) + B_s' + B_t']\omega$$

$$+d_{\rm rel}[(\beta_s' - (\frac{1}{2}A_s + \frac{3}{2}A_t)\beta_s)\omega(t) + 3\beta_s\omega'(t) - (\beta_t' - (\frac{1}{2}A_t + \frac{3}{2}A_s)\beta_t)\omega(s) - 3\beta_t\omega'(s)]$$

does not involve anymore terms of the form $\omega'(s) \wedge \omega'(t)$. Now let

$$A := \frac{3}{2}(A_s + A_t),\ B := -\frac{1}{2}A_s^2 - \frac{3}{2}A_sA_t + A_s' + B_s + 3B_t,$$

$$C := -\frac{1}{2}A_sB_s - \frac{3}{2}(A_sB_t + A_tB_s) - \frac{1}{2}A_tB_t + B_s' + B_t'.$$

Then, assuming that we have the equality

$$-\frac{1}{2}A_s^2 - \frac{3}{2}A_sA_t + A_s' + B_s + 3B_t = -\frac{1}{2}A_t^2 - \frac{3}{2}A_sA_t + A_t' + B_t + 3B_s,$$

(this condition is equivalent to the fact that the elliptic curves $E_1(\nu)$ and $E_2(\nu)$ are isogenous), then we have the following inhomogenous Picard-Fuchs equation:

$$\omega''' - A\omega'' - B\omega' - C\omega = d_{\rm rel}\beta,$$

where β is the 1-form:

$$\beta := (\beta_s' - (\frac{1}{2}A_s + \frac{3}{2}A_t)\beta_s))\omega(t) + 3\beta_s\omega'(t) + (\beta_t' - (\frac{1}{2}A_t + \frac{3}{2}A_s)\beta_t))\omega(s) + 3\beta_t\omega'(s).$$

In our case $A_s = -\frac{1-3\nu^2}{\nu(1-\nu^2)}$, $B_s = \frac{1}{1-\nu^2}$, $A_t = -\frac{\nu^2-2\nu-1}{\nu(\nu^2-1)}$ and $B_t = -\frac{1}{\nu(\nu-1)^2}$. Therefore we get

$$A = -3\frac{2\nu+1}{\nu(\nu+1)}, \quad B = -\frac{7\nu^4 - 6\nu^3 - 4\nu^2 + 6\nu + 1}{(\nu-1)^2(\nu+1)^2\nu^2}, \quad C = -\frac{\nu^4 - 2\nu^3 - 2\nu - 1}{(\nu-1)^3(\nu+1)^2\nu^2}$$

and for β the expression above. This finishes the proof. □

5. Appendix

In this section we give the proof of the following lemma from the introduction:

Lemma 1.1: *In the situation of the introduction, $\mathcal{D}_{\rm PF}(\bar\nu)$ is a single-valued meromorphic function on B with poles only along degeneracies of X_b, and therefore satisfies a differential equation*

$$\mathcal{D}_{\rm PF}(\bar\nu) = g,$$

where g is a rational function in $b \in B$.

Proof. Assume that we have a family $f : \bar X \to \bar B$ of projective surfaces over a compact Riemann surface $\bar B$. Let $\Sigma \subseteq \bar B$ be the finite subset over which there are singular fibers. Let $h : X \to B$ be the smooth part of f. We may assume that the family is semi-stable (semi-stable reduction) and that there is a cycle $Z \in CH^2(X, 1)$ such that the restriction of Z to all fibers induces the family of cycles in $CH^2(X_b, 1)$ (both reductions require perhaps a finite cover of B which does not however change the assertion). The cycle Z has a class $c_{3,2}(Z) \in H^3_{\mathcal{D}}(X, \mathbb{Z}(2))$ in Deligne cohomology. By semi-stability $\Delta := f^{-1}\Sigma$ is a divisor with strict normal crossings and its Deligne cohomology can be computed via the logarithmic de Rham complex. Let $\mathcal{V}^2$ be the sheaf of transcendental cohomology classes in $R^2h_*\mathbb{C}$, a local system of rank $22 - \rho(X_b)$ for b general. The Deligne class vanishes in $F^3 \cap H^3(X_b, \mathbb{Z})$, since $b_3(X_b) = 0$ in our case, and therefore induces a holomorphic normal function $\nu \in H^0(B, \mathcal{V}^2 \otimes \mathcal{O}_B/F^2)$ over B. However since the family is semi-stable, there is a canonical extension of ν to a holomorphic section of the sheaf $R^2 f_* \Omega^*_{\bar{X}/\bar{B}}(log\Delta)$. This can be seen as follows: let

$$\mathbb{Z}_{\mathcal{D},X}(2) = Cone(Rj_*\mathbb{Z}(2) \to \Omega^*_{\bar{X}}(log\Delta)/F^2)[-1]$$

be the Beilinson-Deligne complex [9] of X using the inclusion $X \overset{j}{\hookrightarrow} \bar{X}$ and

$$\mathbb{Z}_{\mathcal{D},h}(2) = Cone(Rj_*\mathbb{Z}(2) \to \Omega^*_{\bar{X}/\bar{B}}(log\Delta)/F^2)[-1]$$

the relative Beilinson-Deligne complex of h. There is a natural surjection of complexes $\mathbb{Z}_{\mathcal{D},X}(2) \to \mathbb{Z}_{\mathcal{D},h}(2)$ which induces a morphism $H^3_{\mathcal{D}}(X, \mathbb{Z}(2)) \to H^0(\bar{B}, R^3 f_* \mathbb{Z}_{\mathcal{D},f}(2))$. Since all fibers of h satisfy $b_3(X_b) = 0$, we conclude that the image of this element in $H^0(\bar{B}, R^3 f_* Rj_* \mathbb{Z}(2))$ vanishes. Therefore the image of $c_{3,2}(Z)$ in $H^0(\bar{B}, R^3 f_* \mathbb{Z}_{\mathcal{D},f}(2))$ is coming (at least locally because of monodromy) from a class in $H^0(\bar{B}, R^2 f_* \Omega^*_{\bar{X}/\bar{B}}(log\Delta)/F^2)$ and is thus an extension of $\nu \in H^0(B, R^2 f_* \Omega^*_{X/B}/F^2)$ to $\bar{B}$. By construction it is meromorphic along Σ but still multi-valued with indeterminacies in the local system of integral cohomology classes. Now we apply the Picard-Fuchs operator. This makes $g(b)$ a single-valued complex function on B. $\mathcal{D}_{\mathrm{PF}}$ has meromorphic (rational) coefficients in b, since they are the coefficients of the characteristic polynomial of the Gauß-Manin connection, which has regular singular points along Σ by Deligne [6]. Therefore the resulting function $g(b)$ is holomorphic outside Σ, but can have higher order poles along Σ. By Chow's theorem any meromorphic function on $\bar{B}$ is rational. $\square$

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Pedro Luis del Angel, Cimat, Guanajuato, Mexico
E-mail address: luis@cimat.mx

Stefan Müller-Stach, Fachbereich Mathematik, 45117 Essen, Germany
E-mail address: mueller-stach@uni-essen.de