Hamiltonian PDEs with strong potentials

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Heisenberg model

- 1-d ferromagnetic chain:

$s_n \in S^2$: spin vector

- Dynamical law:

$$\dot{s}_n = s_n \times (s_{n+1} + s_{n-1})$$

- Continuum approximation:

$$s_n = u(n\epsilon) \quad \Rightarrow \quad u_t = \epsilon^2 u \times u_{xx} + O(\epsilon^3)$$

where ϵ: lattice length. In the scale $\tau = \epsilon^2 t$,

$$u_\tau = u \times u_{xx} \qquad \text{(LL)}$$

• **Local well-posedness**:

• (LL): Visintin 85, Sulem, Sulem, Bardos, 86; Zhou, Guo, Tan, 91; *et. al.*

• Schrödinger maps: Ding, Wang 98, 01; Nahmod, Stefanov, Uhlenbeck 03, McGahagan, Shatah 03, Ko, Shatah 03; *et. al.*

* Approach 1: Vanishing viscousity.

$$u_t = u \times \Delta u + \epsilon(\Delta u + |\nabla u|^2 u)$$

* Approach 2: Vanishing 'wave'

$$\epsilon^2 u_{tt} + u \times u_t - \Delta u - |\nabla u|^2 u = 0$$

$$(\text{(LL)} \Leftrightarrow u \times u_t - \Delta u - |\nabla u|^2 u = 0)$$

Schrödinger map

- Given a Kaȟler manifold $\mathcal{M}$ with

$<\cdot,\cdot>$: metric structure

D: covariant differentiation

$\imath : T\mathcal{M} \to T\mathcal{M}$: complex structure.

- Hamiltonian

$$H(u) = \int \Sigma_j |u_{x_j}|^2 + G(u)dx, \qquad u \in H^1(R^n, \mathcal{M})$$

- Schrödinger map:

$$u_t = \imath \frac{\partial H}{\partial u} = \imath(-\Sigma_j D_{x_j} u_{x_j} + \nabla G(u))$$

- For (LL), $\mathcal{M} = S^2$, $\imath(u) = u\times$, and $G = 0$.

- Global existence: Kenig, Ponce, Vega, 93; Hayashi, Hirata 98; Chang, Shatah, Uhlenbeck, 00; *et. al.*.

- $x \in \mathbb{R}^1$: global existence;

- $x \in \mathbb{R}^2$: global existence radial or equivariant solutions with small energy;

- $x \in \mathbb{R}^2$: there exist self-similar and equivariant blow-up solutions in *weak* energy space (Shatah, Z);

- $x \in \mathbb{R}^2$: Blow-up with finite energy?

• **References:**

• (LL): Visintin 85, Sulem, Sulem, Bardos, 86, Zhou, Guo, Tan, 91; *et. al.*

• general Schrödinger maps:

* Local well-posedness: Ding, Wang 98, 01; Nahmod, Stefanov, Uhlenbeck 03, McGahagan, Shatah 03, Ko, Shatah 03; *et. al.*

* Global existence: Kenig, Ponce, Vega, 93; Hayashi, Hirata 98; Chang, Shatah, Uhlenbeck, 00; *et. al.*

- Anti-ferromagnetic chain:

$$s_{2n} + s_{2n+1} = O(\epsilon), \quad s_k \in S^2$$

- Continuum approximation:

$$s_{2n} = u(2n\epsilon), \qquad s_{2n+1} = v((2n+1)\epsilon)$$

$$\Rightarrow \begin{cases} u_t = u \times (-\epsilon^2 u_{xx} - \epsilon v_x + v) + O(\epsilon^3) \\ v_t = v \times (-\epsilon^2 v_{xx} + \epsilon u_x + u) + O(\epsilon^3) \\ u + v = O(\epsilon) \end{cases} \tag{AF}$$

- No motion in $O(1)$ time scale.

- Motion in time scale $O(\frac{1}{\epsilon})$?

$$\begin{cases} u_t = u \times (-\epsilon u_{xx} - v_x + \frac{v}{\epsilon}) \\ v_t = v \times (-\epsilon v_{xx} + u_x + \frac{u}{\epsilon}) \\ u + v = O(\epsilon) \end{cases} \qquad \text{(AF)}$$

Theorem 1 *(Shatah, Z) Assume* $u + v \equiv 0$ *at* $t = 0$. *As* $\epsilon \to 0$, *(AF) converges to the* σ-*model (a wave map)*

$$w_{tt} - w_{xx} + (|w_t|^2 - |w_x|^2)w = 0, \qquad w \in S^2. \qquad (\sigma)$$

Hamiltonian Structure

• Kähler manifold: $\mathcal{M} = S^2 \times S^2$ with the symplectic map $\imath$

$$\imath(X,Y) = (u \times X, v \times Y), \qquad (X,Y) \in T_{(u,v)}\mathcal{M}.$$

• Hamiltonian: for $(u,v) : \mathbb{R} \to \mathcal{M}$,

$$H_\epsilon(u,v) = \frac{1}{2}\int \epsilon|u_x|^2 + \epsilon|v_x|^2 + 2u_x \cdot v + \frac{1}{\epsilon}|u+v|^2 dx$$

• **Remark.** $u + v \equiv 0$ at $t = 0 \Rightarrow \frac{1}{\epsilon}H_\epsilon = O(1)$.

• $\mathcal{L} = \{(u,-u) \mid u \in S^2\}$: a Lagrangian submanifold of $\mathcal{M}$, i.e.

$$\imath T_p\mathcal{L} \perp T_p\mathcal{L}, \qquad \forall p \in \mathcal{L}.$$

General setting

- $\mathcal{M}$: Kaȟler manifold

- Hamiltonian: for $u : R^d \to \mathcal{M}$,

$$H_\epsilon(u) = \int \epsilon \Sigma_j |u_{x_j}|^2 + \Sigma_j < W_j(u), u_{x_j} > + \frac{1}{\epsilon} G(u) dx$$

where W_j are vector fields.

- Schrödinger map:

$$u_t = \imath(-\epsilon \Sigma_k D_{x_k} u_{x_k} + \Sigma_k B_k u_{x_k} + \frac{1}{\epsilon} \nabla G(u)) \quad \text{(SM)}$$

where B_k are skew-symmetric operators

$$B_k(u) = DW_k(u)^* - DW_k(u)$$

Main result

$$u_t = \imath(-\epsilon\Sigma_k D_{x_k} u_{x_k} + \Sigma_k B_k u_{x_k} + \frac{1}{\epsilon}\nabla G(u)) \tag{SM}$$

Assume B_k and the potential G satisfy

(A1) $G \geq 0$, $\mathcal{L} = \{G = 0\}$ is a Lagrangian submanifold of $\mathcal{M}$;

(A2) D^2G commutes with $\imath B_k$ on $\mathcal{L}$;

(A3) At $p \in \mathcal{L}$, $D^2G|_{\imath T_p\mathcal{L}} \geq 1$ and dominates B_k.

Theorem 2 *(Shatah, Z) For $u(0,\cdot) : R^n \to \mathcal{L}$, as $\epsilon \to 0$, the limit of (SM) is a generalized wave map equation targeted on $\mathcal{L}$.*

* $u(0,\cdot) : R^n \to \mathcal{L} \Rightarrow \frac{1}{\epsilon}H_\epsilon = O(1)$.

A toy problem

- $\mathcal{M} = C^1 = \{u+iv \mid u, v \in \mathbb{R}\}, \qquad \mathcal{L} = \mathbb{R}^1 \subset C^1;$

- Hamiltonian: for $u + iv : \mathbb{R} \to \mathcal{M}$,

$$H_\epsilon(u, v) = \frac{1}{2} \int \epsilon u_x^2 + \epsilon v_x^2 + \frac{v^2}{\epsilon} dx.$$

- Schrödinger map:

$$\begin{cases} u_t = -\epsilon v_{xx} + \frac{v}{\epsilon} \\ v_t = \epsilon u_{xx} \end{cases} \quad \Rightarrow u_{tt} = -\epsilon^2 u_{xxxx} + u_{xx}$$

- Formal limit:

$$u_{tt} - u_{xx} = 0$$

Sketch of the proof

- Basic ideas:

1. Split the tangential and normal directions
2. Wave equation type energy estimate;
3. Elliptic estimate for the normal variable.

- Assume $\mathcal{M} = C^m$ and $x \in \mathbb{R}^1$.

$$u_t = i(-\epsilon u_{xx} + Bu_x + \frac{1}{\epsilon}\nabla G(u)) \qquad \text{(SM)}$$

- Tubular coordinate system near $\mathcal{L}$:

- $G(p, n) =< a(p)n, n > +O(n^3)$, where $a \geq 1$.

• Energy estimate:

$$(\partial_t - B\imath\partial_x)(\text{SM}) \Rightarrow$$

$$u_{tt} - (\imath B + \imath B)u_{tx} - B^2 u_{xx} = -\epsilon^2 u_{xxxx} - \imath D^2 G \imath u_{xx} + \partial_x(D^2 G(u_x)) + R_1(\frac{n}{\epsilon}, \epsilon\partial^2 u, \partial u, u)$$

where $u = p + n$ and R_1 is smooth.

• Elliptic estimates for $\frac{n}{\epsilon}$: apply normal projection operator P to (SM) $\Rightarrow$

$$-\epsilon^2 (P\partial_x)^2 n + an = R_2(\epsilon\partial u) + O(|n|^2)$$

Wave equation

$$\begin{cases} u_{tt} - \Delta u + \frac{1}{\epsilon^2}\nabla V(u) = 0, & u(t,x) \in R^m \\ u(0,x) = f(x), \quad u_t(0,x) = g(x) \end{cases} \tag{W}$$

- **Assumption**: $v \geq 0$ and

$$M = \{u \mid V(u) = 0\} \subset \mathbb{R}^m$$

is a smooth submanifold.

- **Question**: What are the singular limits of solutions with *bounded energy*, i.e.

$$u \rightharpoonup u^* = ? \quad \text{as} \quad \epsilon \to 0$$

* Remark: In the phase space $TR^m = R^{2m}$, $\{V = 0\}$ is an *anisotropic* submanifold.

$$
\begin{cases} u_{tt} - \Delta u + \frac{1}{\epsilon^2}\nabla V(u) = 0, & u(t,x) \in R^m \\ u(0,x) = f(x), & u_t(0,x) = g(x) \end{cases}
$$

(W)

- Energy (Hamiltonian):

$$
H = \int \frac{1}{2}|u_t|^2 + \frac{1}{2}|\nabla_x u|^2 + \frac{1}{\epsilon^2} V(u) dx
$$

* Bounded energy $\Leftrightarrow V(f) \equiv 0$.

- **Energy conservation**: As $\epsilon \to 0$

$H = O(1) \implies |V(u)| \to 0$, i.e. $u^* = \lim u \in M$

- Question: What governs the dynamics of u^* on M?

* Example: Pendulum

- Stiff spring:

- Rigid bar:

X

* Example: σ model (Shatah, 88)

$$u_{tt} - u_{xx} + \frac{1}{\epsilon^2}(|u|^2 - 1)u = 0, \quad u(t,x) \in \mathbb{R}^3$$

It implies

$$\partial_t(u_t \times u) - \partial_x(u_x \times u) = 0$$

$$\Rightarrow \partial_t(u_t^* \times u^*) - \partial_x(u_x^* \times u^*) = 0$$

$$\Rightarrow \begin{cases} (u_{tt}^* - u_{xx}^*) \times u^* = 0, & u^*(t,x) \in S^2 \\ u^*(0,\cdot) = f, & u_t^*(0,\cdot) = P(f)g \end{cases}$$

$$u_{tt}^* - u_{xx}^* + (|u_t^*|^2 - |u_x^*|^2)u^* = 0, \quad u^* \in S^2 \ (\sigma)$$

$P(u)$: the projection to the tangent space of S^2 at $u \in S^2$.

X

- Guess: for equation (W), $\lim u = u^* \in M$ is a *wave map*, i.e.

$$\begin{cases} D_t u_t^* - \Sigma_i D_{x_i} u_{x_i}^* = 0, & u^* \in M \\ u^*(0,\cdot) = f, & u_t^*(0,\cdot) = P(f)g \end{cases}$$

D: covariant differentiation.

- **ODE results:**

1. true if $g \in T_f M$ or $V'' \equiv$ const on M;

2. *not* necessarily true otherwise! u^* has an extra potential, depending on f and g.

References: Rubin and Ungar 1957, Takens 1980, Borneman and Schütte 1997, Borneman, 1998, Froese and Herbst 2000, ...

Decomposition

$$u_{tt} - u_{xx} + \frac{1}{\epsilon^2}\nabla\left\{a(u)(|u|^2 - 1)^2\right\} = 0 \quad u \in \mathbb{R}^3$$

$a(u)$: smooth and $a(u) \geq 1$.

- $u = p + dp, \quad n = p = \frac{u}{|u|} \in S^2,\ d = |u| - 1 \in \mathbb{R}$

- Tangential equation:

$$p_{tt} - p_{xx} + (|p_t|^2 - |p_x|^2)p = -2d_t p_t + 2d_x p_x - \frac{d^2}{\epsilon^2}\nabla a(p) + O(|d|)$$

- Normal equation:

$$d_{tt} - d_{xx} + \frac{2}{\epsilon^2}a(p)d = -|p_t|^2 + |p_x|^2 + O(|d|)$$

Main result

Theorem 3 *(Shatah-Z, 2003) Assume M is a hypersurface. Given smooth initial data f and g so that $V(f) \equiv 0$, locally, the solution of*

$$\begin{cases} u_{tt} - \Delta u + \frac{1}{\epsilon^2}\nabla V(u) = 0, \quad u(t,x) \in R^m \\ u(0,x) = f(x), \quad u_t(0,x) = g(x) \end{cases} \tag{W}$$

converges to u^* where (u^*, θ, E) solves

Lagrangian : $\frac{1}{2}(-|u_t|^2 + |u_x|^2 + Ea - E\theta_t^2 + E\theta_x^2)$

$$\begin{cases} D_t u_t^* - \Sigma_i D_{x_i} u_{x_i}^* + \frac{E}{2}\nabla a(u^*) = 0; \\ \theta_t E_t - \nabla_x \theta \cdot \nabla_x E + (\theta_{tt} - \Delta\theta)E = 0; \\ \theta_t^2 - |\nabla_x \theta|^2 = a(u^*). \end{cases} \tag{KR}$$

Here D is the covariant derivative on M and

$$a = D^2 G, \quad u^*(t,x) \in M, \quad \theta(t,x),\ E(t,x) \in \mathbb{R}.$$

•References: Ebin 1977, 1982, Shatah 1988, Keller and Rubinstein 1991, Freier 1996, Shatah and Zeng 02 ...

Periodic orbits for ODEs

- ODE version of (W):

$$u_{tt} + \nabla w(u) + \frac{1}{\epsilon^2}\nabla V(u) = 0, \quad u \in R^n \quad \text{(O)}$$

- Limit equation

$$D_t u_t^* + P(u^*)\nabla w(u^*) = 0, \quad u^* \in M. \quad \text{(GO)}$$

Theorem 4 *If (GO) has a nondegenerate T-periodic orbit, then (O) also has a T-periodic orbit for small nonresonant ϵ.*

- M does *not* have to be a hypersurface.
- If $w \equiv 0$, a periodic orbit is a closed geodesic.
- The nonresonance condition on ϵ depends on the orientation of this periodic orbit.
- Normally elliptic singular perturbation.