Generalised Fractional-Black-Scholes Equation: pricing and hedging

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Outline

- Lévy processes
- Fractional calculus
- Fractional-Black-Scholes

Definition 1 Lévy process. *Let $X(t)$ be a random variable dependent on time t. Then the stochastic process*

$$X(t),\ \textit{for } 0 < t < \infty \textit{ and } X(0) = 0,$$

is a Lévy process iff it has independent and stationary increments.

Theorem 1 Lévy-Khintchine representation. *Let X(t) be a Lévy process. Then the natural logarithm of the characteristic function can be written as*

$$\ln \mathbb{E}[e^{i\theta X(t)}] = ait\theta - \frac{1}{2}\sigma^2 t\theta^2 + t\int \left(e^{i\theta x} - 1 - i\theta x \mathbf{I}_{|x|<1}\right) W(dx),$$

where $a \in \mathbb{R}$, $\sigma \geq 0$, $\mathbf{I}$ is the indicator function and the Lévy measure W must satisfy

$$\int_{\mathbb{R}/0} \min\{1, x^2\} W(dx) < \infty. \qquad (1)$$

Which Lévy process? Why?

- Brownian motion; Bachelier.
- α-Stable or Lévy Stable; Mandelbrot.
- Jump Diffusion; Merton.
- GIG and Generalised Hyperbolic Distribution; Barndorff-Nielsen.
- Variance Gamma; Madan et al.

- CGMY, Carr et al.
- KoBol, Tempered Stable; Koponen.
- FMLS; Carr and Wu.
- others.

When specifying a particular Lévy process we are basically asking how do we want to specify the 'behaviour' of the jumps, in other words how is the Lévy density $w(x)$ (ie $W(dx) = w(x)dx$) chosen. For example

- Size and sign of jumps
- Frequency of jumps
- Existence of moments
- **Simplicity**

The CGMY process

A simple answer is then to consider a Lévy density of the form

$$w_{CGMY}(x) = \begin{cases} C|x|^{-1-Y}e^{-G|x|} & \text{for } x < 0, \\ Cx^{-1-Y}e^{-Mx} & \text{for } x > 0, \end{cases}$$

and the log of the characteristic function is given by

$$\Psi_{CGMY}(\theta) = tC\Gamma(Y)\{(M - i\theta)^Y - M^Y + (G + i\theta)^Y - G^Y\}.$$

Here $C > 0$, $G \geq 0$, $M \geq 0$ and $Y < 2$.

The Damped-Lévy process

$$w_{DL}(x) = \begin{cases} Cq\,|x|^{-1-\alpha}\,e^{-\lambda|x|} & \text{for } x < 0, \\ Cpx^{-1-\alpha}e^{-\lambda x} & \text{for } x > 0, \end{cases}$$

and the natural logarithm of the characteristic equation is given by

$$\Psi_{DL}(\theta) = t\kappa^{\alpha}\left\{p(\lambda - i\theta)^{\alpha} + q(\lambda + i\theta)^{\alpha} - \lambda^{\alpha} - i\theta\alpha\lambda^{\alpha-1}(q-p)\right\},$$

for $1 < \alpha \leq 2$ and $p + q = 1$.

The Lévy-Stable process

Is a pure jump process with Lévy density

$$w_{LS}(x) \;=\; \begin{cases} Cq\,|x|^{-1-\alpha} & \text{for } x<0, \\ Cpx^{-1-\alpha} & \text{for } x>0, \end{cases}$$

Hence the log of the characteristic function is $\Psi(\theta) =$

$$\begin{cases} -\kappa^{\alpha}|\theta|^{\alpha}\{1 - i\beta\ \mathrm{sign}(\theta)\tan(\alpha\pi/2)\} & \text{for } \alpha \neq 1, \\ -\kappa|\theta|\left\{1 + \frac{2i\beta}{\pi}\ \mathrm{sign}(\theta)\ln|\theta|\right\} & \text{for } \alpha = 1, \end{cases}$$

here $C > 0$ is a scale constant, $p \geq 0$ and $q \geq 0$, with $p + q = 1$ and $\beta = p - q$ is the skewness parameter.

Fractional Integrals

For an n-fold integral there is the well known formula

$$\int_a^x \int_a^x \cdots \int_a^x f(x)dx = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)dt.$$

Note that since $(n-1)! = \Gamma(n)$ the expression above may have a meaning for non-integer values of n.

Definition 2 **The Riemann-Liouville Fractional Integral.** *The fractional integral of order* $\alpha > 0$ *of a function* $f(x)$ *is given by*

$$D_{a^+}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\xi)^{\alpha-1} f(\xi)d\xi,$$

and

$$D_{b^-}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\xi-x)^{\alpha-1} f(\xi)d\xi.$$

Definition 3 The Riemann-Liouville Fractional Derivative.

$$D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-\xi)^{n-\alpha-1}f(\xi)d\xi,$$

and

$$D^{\alpha}_{b-}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^b (\xi-x)^{n-\alpha-1}f(\xi)d\xi.$$

The Fourier Transform View

Note that if we let $a = -\infty$ and $b = \infty$ we have

$$\mathcal{F}\{D^{\alpha}_{+}f(x)\} = (-i\xi)^{\alpha}\widehat{f}(\xi)$$

and

$$\mathcal{F}\{D^{\alpha}_{-}f(x)\} = (i\xi)^{\alpha}\widehat{f}(\xi).$$

The Lévy-Stable Fractional-Black-Scholes. Under the physical measure the price process follows a geometric LS process

$$d(\ln S) = \mu dt + \sigma dL_{LS},$$

where $L \sim S_\alpha(dt^{1/\alpha}, \beta, 0)$ with $1 < \alpha < 2$, $-1 \leq \beta \leq 1$ and $\sigma > 0$.

And under the risk-neutral measure (McCulloch) it follows

$$d(\ln S) = (r - \beta\sigma^\alpha \sec(\alpha\pi/2))dt + dL_{LS} + dL_{DL}$$

where dL_{LS} and dL_{DL} are independent.

$$\begin{aligned} rV \;=\; & \frac{\partial V(x,t)}{\partial t} + (r - \beta\sigma^\alpha \sec(\alpha\pi/2))\frac{\partial V(x,t)}{\partial x} - \kappa_2^\alpha \sec(\alpha\pi/2) D_+^\alpha V(x,t) \\ & + \kappa_1^\alpha \sec(\alpha\pi/2)\left(V(x,t) - e^x D_-^\alpha e^{-x} V(x,t)\right), \end{aligned}$$

where

$$\kappa_2^\alpha = \frac{1-\beta}{2}\sigma^\alpha \quad \text{and} \quad \kappa_1^\alpha = \frac{1+\beta}{2}\sigma^\alpha.$$

Two cases: classical Black-Scholes and the fractional FMLS

Case $\alpha = 2$, Black-Scholes

$$rV(x,t) = \frac{\partial V(x,t)}{\partial t} + (r - \sigma^2)\frac{\partial V(x,t)}{\partial x} + \sigma^2 \frac{\partial^2 V(x,t)}{\partial x^2}.$$

Case $\alpha > 1$ and $\beta = -1$, FMLS

$$\begin{aligned} rV(x,t) &= \frac{\partial V(x,t)}{\partial t} + (r + \sigma^\alpha \sec(\alpha\pi/2))\frac{\partial V(x,t)}{\partial x} \\ &\quad -\sigma^\alpha \sec(\alpha\pi/2) D_+^\alpha V(x,t). \end{aligned}$$

Proposition 1 CGMY Fractional-Black-Scholes equation.

Let the risk-neutral log-stock price dynamics follow a CGMY process

$$d(\ln S) = (r - w)dt + dL_{CGMY}. \quad (2)$$

The value of a European-style option with final payoff $\Pi(x, T)$ *satisfies the following fractional differential equation*

$$\begin{aligned} rV(x,t) &= \frac{\partial V(x,t)}{\partial t} + (r-w)\frac{\partial V(x,t)}{\partial x} \\ &\quad + \sigma(M^Y + G^Y)V(x,t) \\ &\quad + \sigma e^{Mx} D_-^Y \left(e^{-Mx} V(x,t) \right) \\ &\quad + \sigma e^{-Gx} D_+^Y \left(e^{Gx} V(x,t) \right), \end{aligned}$$

where $\sigma = C\Gamma(-Y)$.

Proof

1

$$V(x,t) = e^{-r(T-t)}\mathbb{E}_t[\Pi(x_T,T)].$$

2

$$V(x,t) = \frac{e^{-r(T-t)}}{2\pi}\mathbb{E}_t\left[\int_{-\infty+i\nu}^{\infty+i\nu} e^{-ix_T\xi}\widehat{\Pi}(\xi)d\xi\right].$$

3

$$\widehat{V}(\xi,t) \;\; = \;\; e^{-r(T-t)}e^{-i\xi\mu(T-t)}e^{(T-t)\Psi(-\xi)}\widehat{\Pi}(\xi).$$

4

$$\frac{\partial\widehat{V}(\xi,t)}{\partial t} = (r + i\xi\mu - \Psi(-\xi))\widehat{V}(\xi,t)$$

with boundary condition $\widehat{V}(\xi,T) = \widehat{\Pi}(\xi,T)$.

Dynamic Hedging: Delta hedging, Delta-Gamma hedging, Variance minimisation.

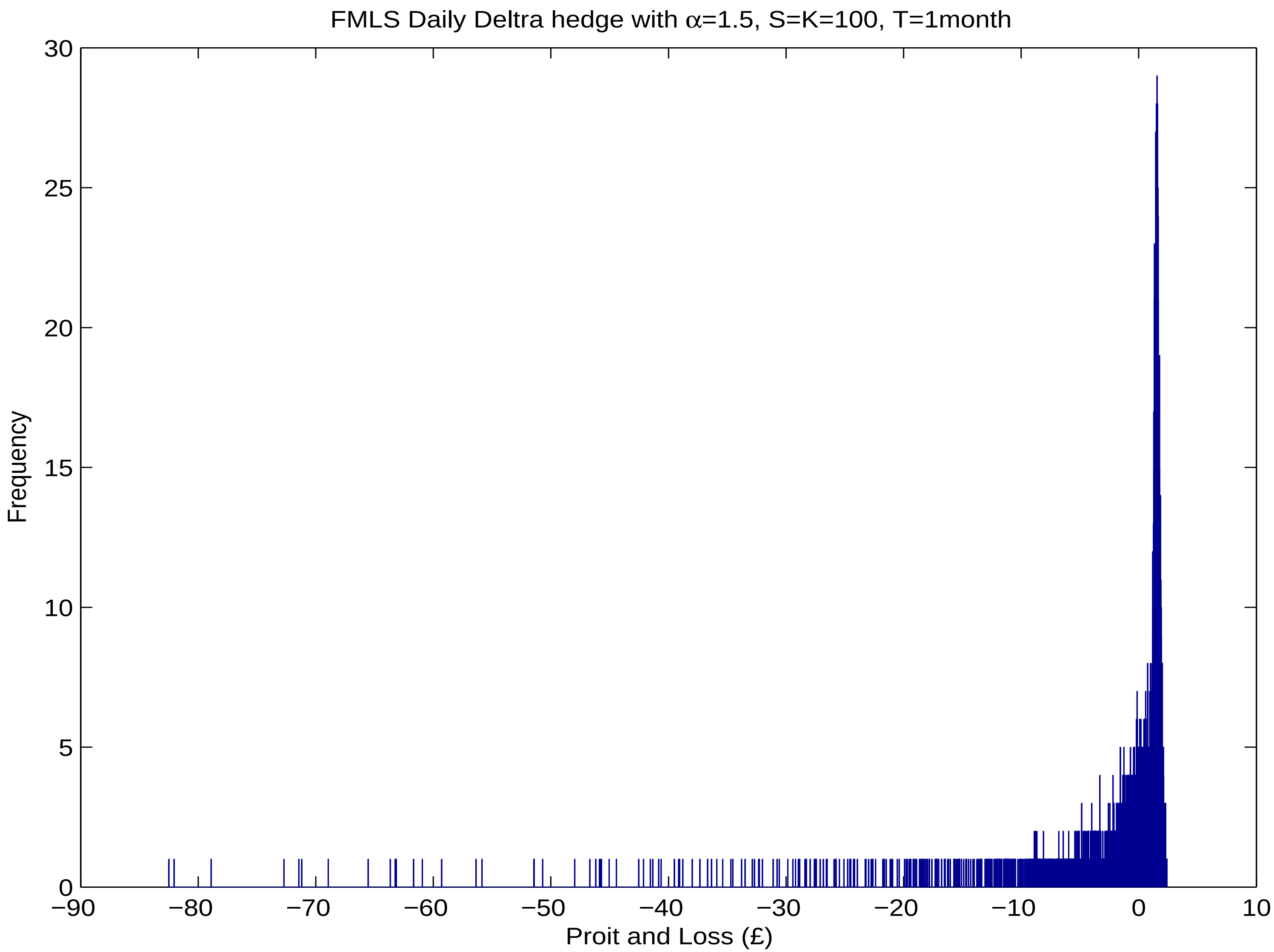
The Taylor Expansion View

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 + \ldots.$$

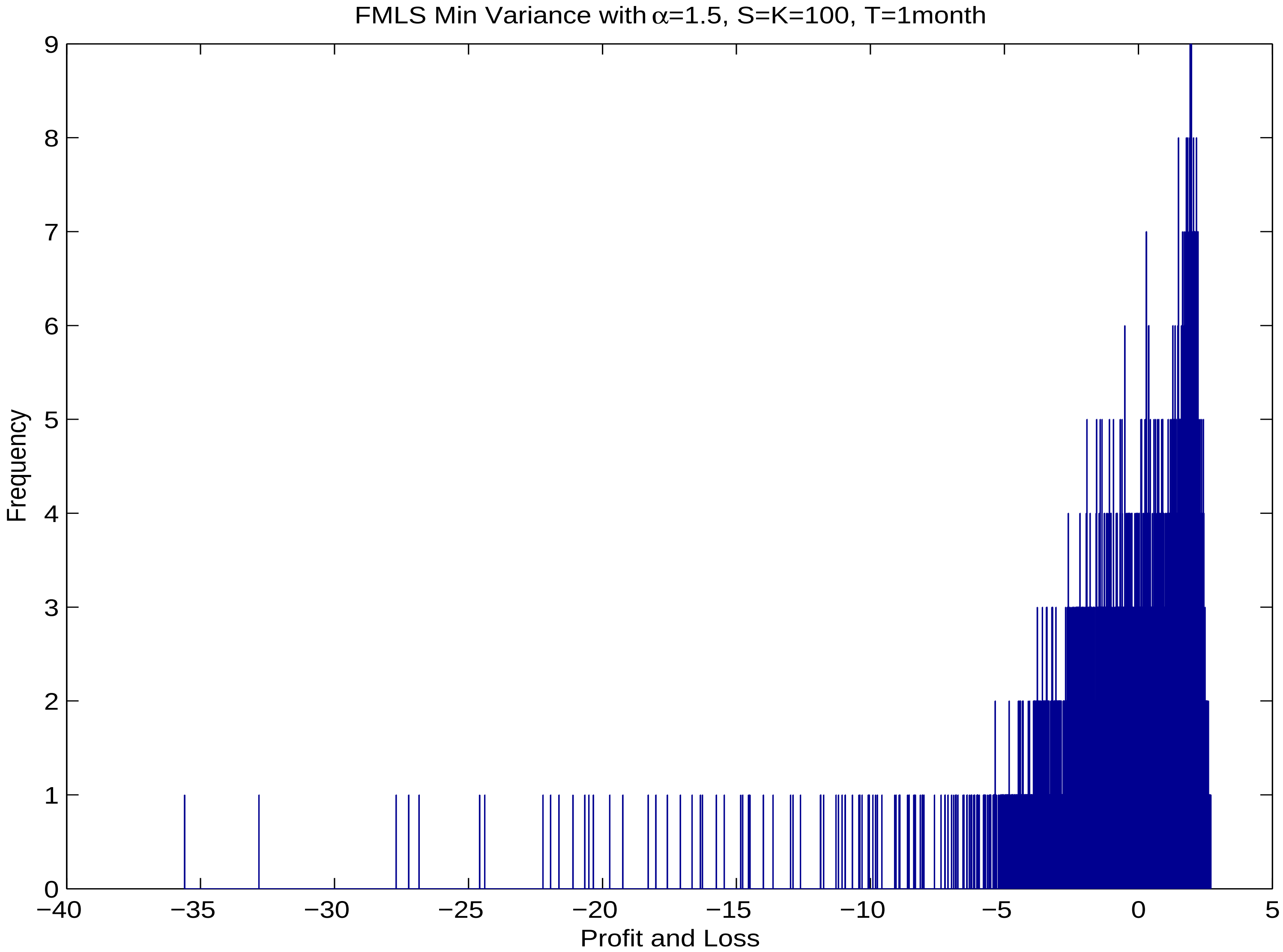
Portfolio $P(S,t) = V_1(S,t) - \Delta S - bV_2(S,t)$

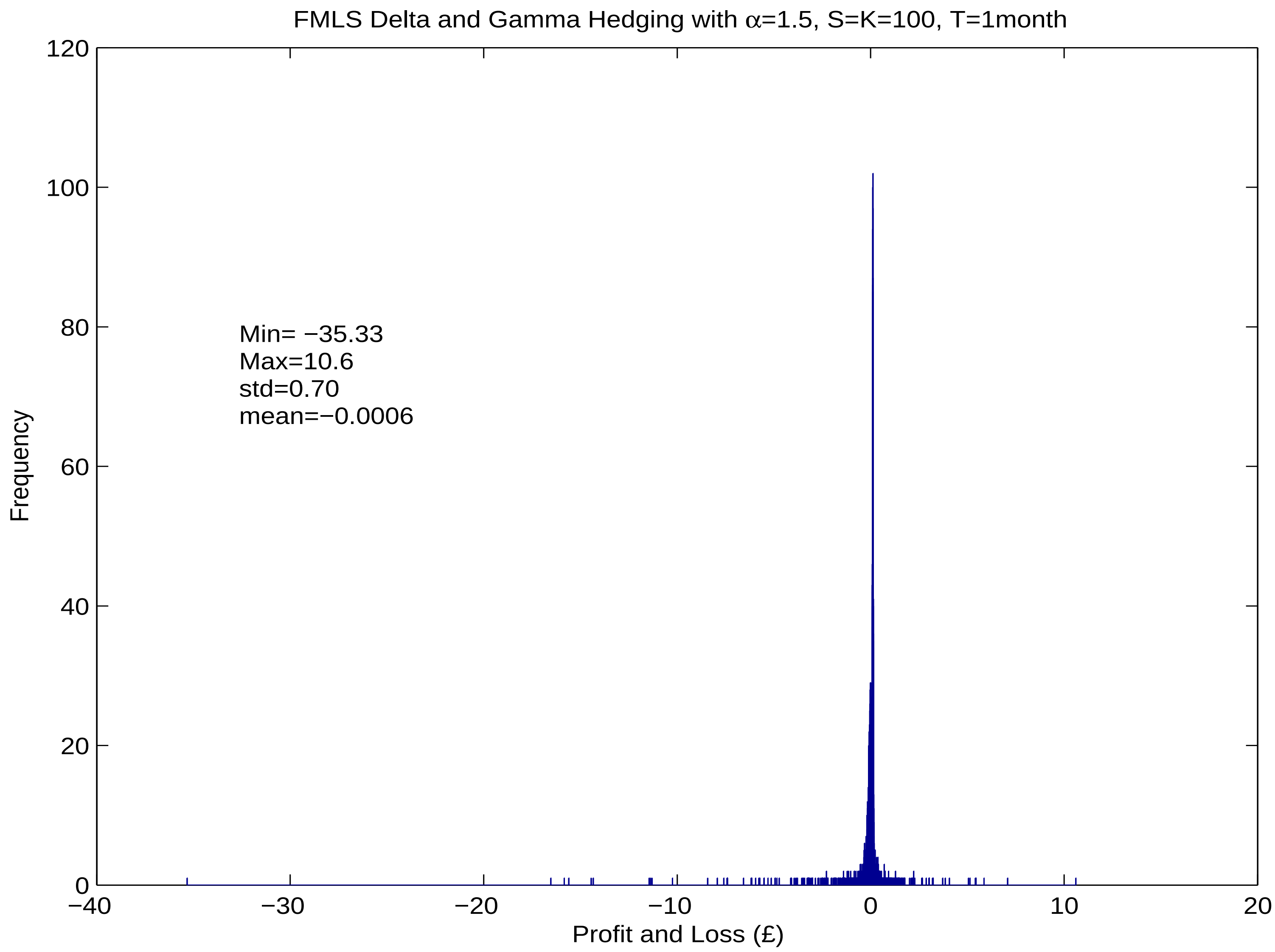
$$\Delta = \frac{\partial V_1(S,t)}{\partial S} - \frac{\partial V_2(S,t)}{\partial S}b,$$

$$b = \frac{\partial^2 V_1(S,t)/\partial S^2}{\partial^2 V_2(S,t)/\partial S^2}.$$



FMLS Daily Deltra hedge with α=1.5, S=K=100, T=1month
Frequency
30
25
20
15
10
5
0
−90
−80
−70
−60
−50
−40
−30
−20
−10
0
10
Proit and Loss (£)


FMLS Min Variance with α=1.5, S=K=100, T=1month
Frequency
9
8
7
6
5
4
3
2
1
0
−40
−35
−30
−25
−20
−15
−10
−5
0
5
Profit and Loss


FMLS Delta and Gamma Hedging with α=1.5, S=K=100, T=1month
Min= −35.33
Max=10.6
std=0.70
mean=−0.0006
Frequency
Profit and Loss (£)
120
100
80
60
40
20
0
−40
−30
−20
−10
0
10
20

FMLS Black-Scholes: the Taylor expansion view

$$dV(x,t) = \frac{\partial V(x,t)}{\partial t}dt + \frac{\partial V(x,t)}{\partial x}dx + \frac{1}{\Gamma(2-\alpha)}D_+^\alpha V(x,t)(dx)^\alpha + \cdots .$$

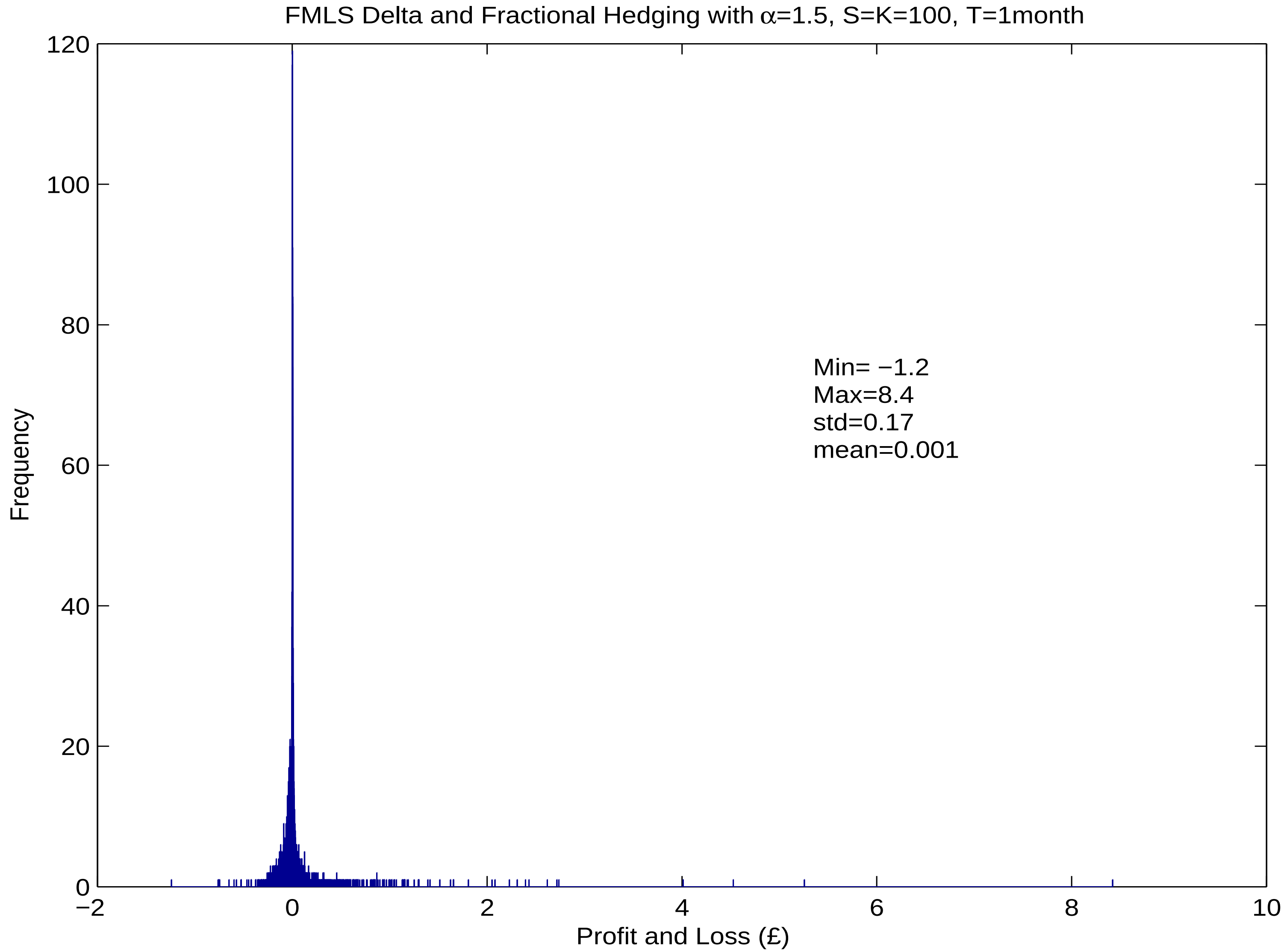
(Samko et al 1993).

Therefore it seems natural, in the FMLS case, to delta and fractional-gamma hedge the portfolio $P(x,t) = V_1(x,t) - \Delta e^x - bV_2(x,t)$, hence

$$a = \frac{\partial V_1(x,t)}{\partial x}\frac{1}{e^x} - \frac{\partial V_2(x,t)}{\partial x}\frac{1}{e^x}b$$

and

$$b = \frac{e^x D_+^\alpha V_1(x,t) - \partial V_1(x,t)/\partial x D_+^\alpha e^x}{e^x D_+^\alpha V_2(x,t) - \partial V_2(x,t)/\partial x D_+^\alpha e^x}.$$


FMLS Delta and Fractional Hedging with α=1.5, S=K=100, T=1month
Frequency
Profit and Loss (£)
Min= −1.2
Max=8.4
std=0.17
mean=0.001
0
20
40
60
80
100
120
−2
0
2
4
6
8
10

In General might want to do...

$$rV(x,t) = \frac{\partial V(x,t)}{\partial t} + \mu \frac{\partial V(x,t)}{\partial x} + \mathcal{G}V(x,t), \tag{3}$$

where $\mathcal{G}$ is an operator containing the fractional derivatives.

$$P(x,t) = V_1(s,t;T_1) - ae^x - bV_2(s,t;T_2) \tag{4}$$

Therefore we require

$$a = \frac{\partial V_1(x,t)}{\partial x}\frac{1}{e^x} - \frac{\partial V_2(x,t)}{\partial x}\frac{1}{e^x}b$$

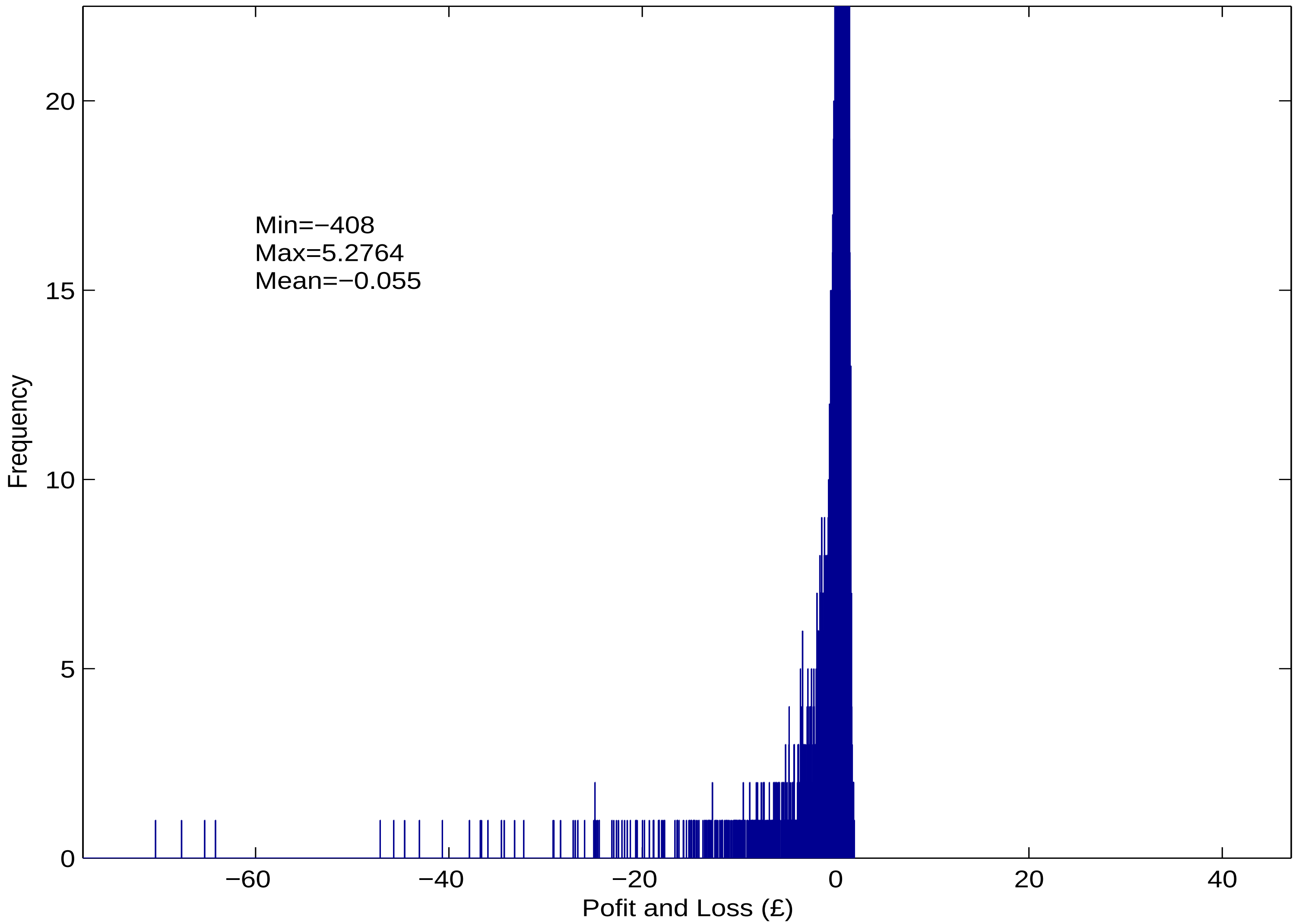
and

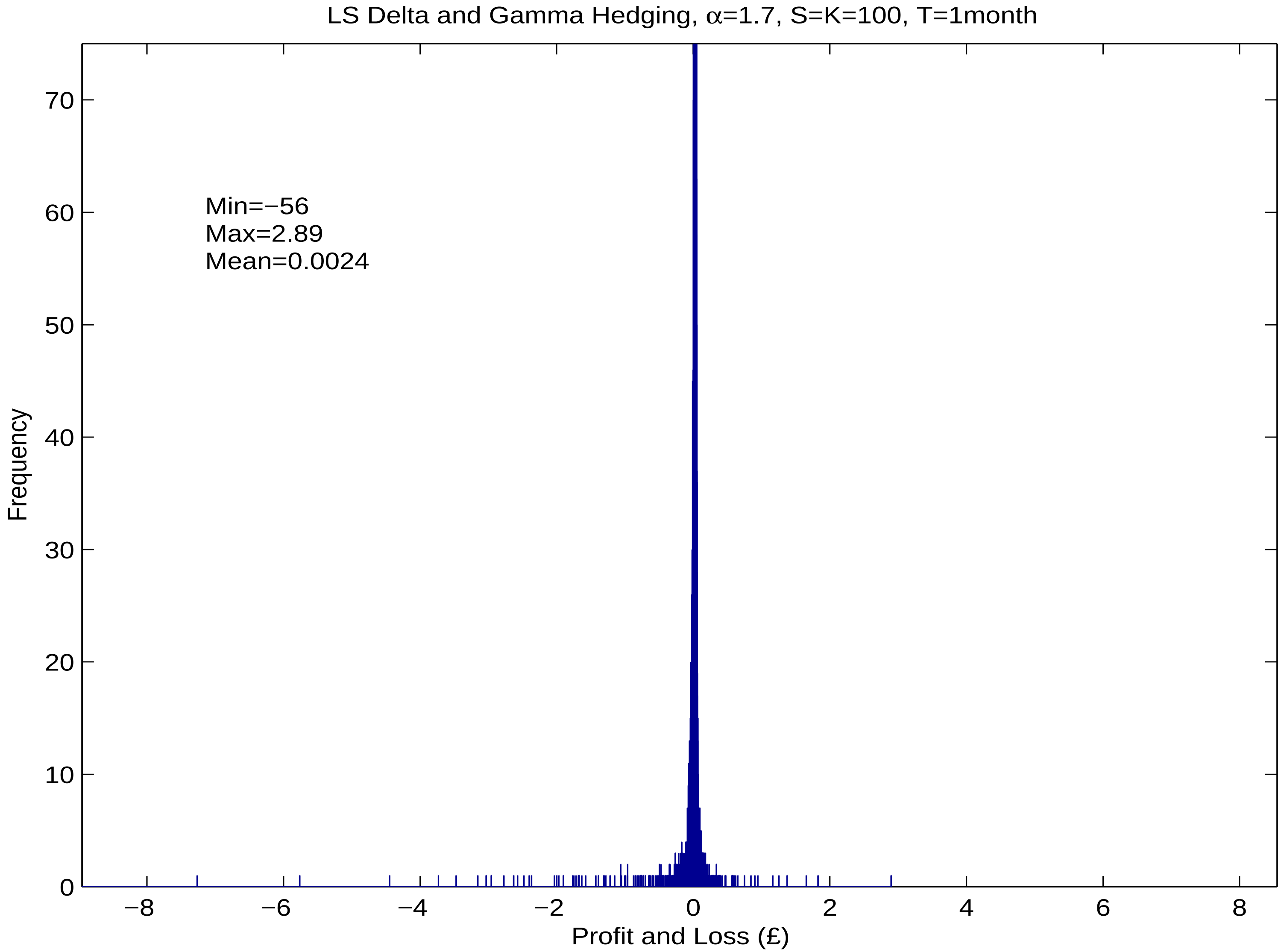
$$b = \frac{e^x \mathcal{G} V_1(x,t) - \partial V_1(x,t)/\partial x \mathcal{G} e^x}{e^x \mathcal{G} V_2(x,t) - \partial V_2(x,t)/\partial x \mathcal{G} e^x}$$

so the portfolio is both Delta and Fractional-Gamma-neutral, ie

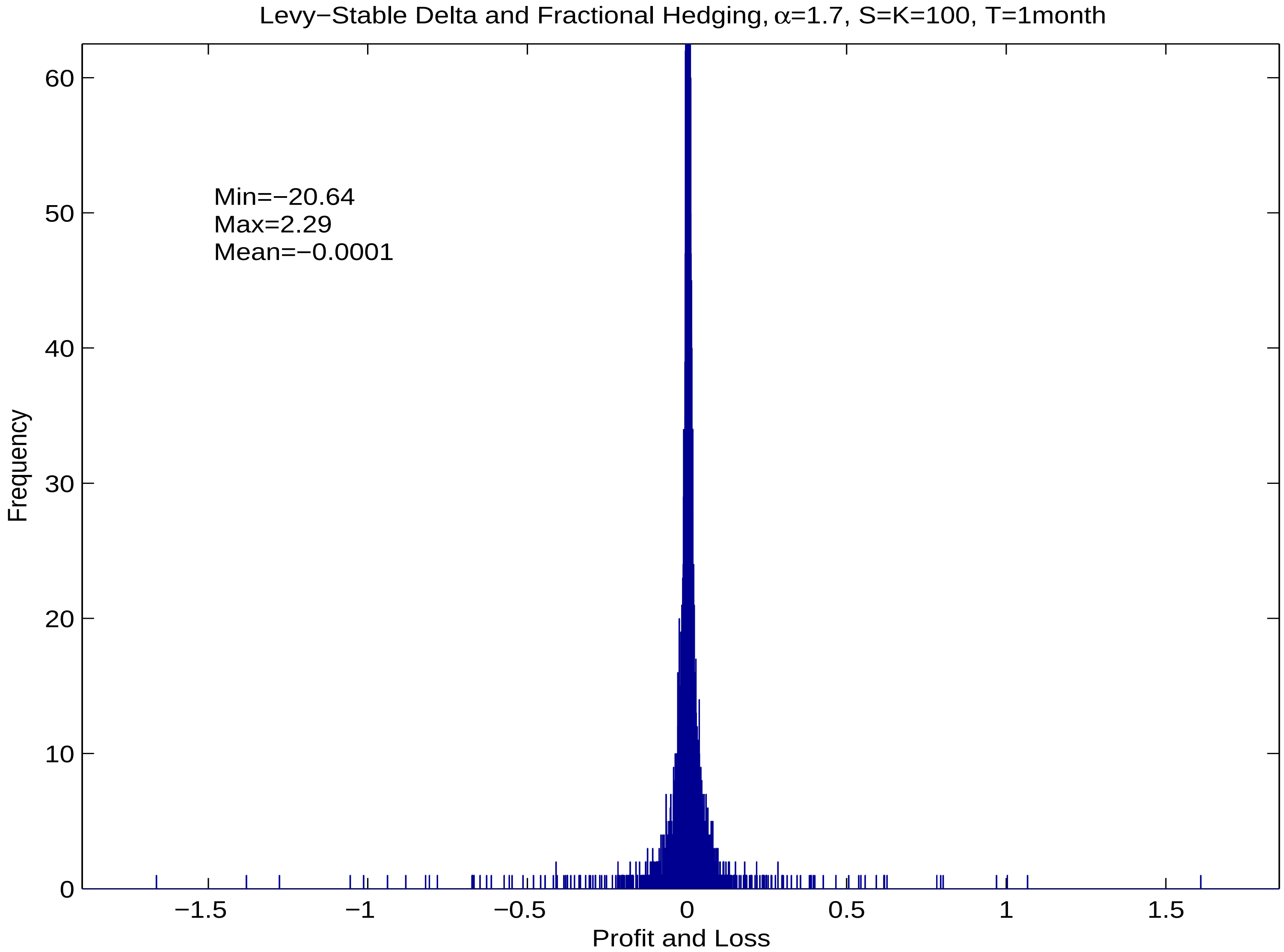
$$\frac{\partial P(x,t)}{\partial x} = 0 \quad \text{and} \quad \mathcal{G}P(x,t) = 0.$$

LS Delta Hedging, α=1.7, S=K=100, T=1month





LS Delta and Gamma Hedging, α=1.7, S=K=100, T=1month
Min=−56
Max=2.89
Mean=0.0024
Frequency
Profit and Loss (£)
0
10
20
30
40
50
60
70
−8
−6
−4
−2
0
2
4
6
8


Levy−Stable Delta and Fractional Hedging, α=1.7, S=K=100, T=1month
Min=−20.64
Max=2.29
Mean=−0.0001
Frequency
Profit and Loss
0
10
20
30
40
50
60
−1.5
−1
−0.5
0
0.5
1
1.5

CONCLUSIONS and FURTHER WORK

For Lévy processes with Lévy densities that have a polynomial singularity at the origin and exponential decay at the tails we can recast the pricing equation in terms of Fractional derivatives.

The non-local property of the fractional operators can be useful when dynamically hedging options.

Using well established numerical schemes for Fractional operators it might be possible to price American options. Moreover, for these processes we can derive Fractional Fokker-Planck equations that may also be used in the pricing of American options.