The Néron model of an abelian variety

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1 Fibered products

Definition 1 *Let S be a scheme and X, Y S-schemes. The* **fibered product** *of X and Y over S is an S-scheme, denoted*

$$X \times_S Y,$$

together with projection S-morphisms

$$p_1 : X \times_S Y \longrightarrow X, \ \ p_2 : X \times_S Y \longrightarrow Y,$$

such that for any other S-scheme Z and any S-morphisms $f : Z \longrightarrow X$, $g : Z \longrightarrow Y$, there exists a unique S-morphism $h : Z \longrightarrow X \times_S Y$ such that

$$p_2 \circ g = p_1 \circ f = h.$$

Intuitively, $X \times_S Y$ might be thought of as the set of ordered pairs $(x, y) \in X \times Y$ with the same image in S.

The fiber product exists and is unique up to isomorphism. Note that if X, Y, S are varieties, then $X \times_S Y$ need not be a variety. Thus the category of (S-)schemes resolves this difficulty.

Examples of fibered products

1. **Product of affine schemes:** Let A, B, R be rings such that $X = \operatorname{Spec} A, Y = \operatorname{Spec} B$ are $S = \operatorname{Spec} R$-schemes. Then
$$X \times_S Y = \operatorname{Spec}(A \otimes_R B).$$

2. **Extension of scalars:** Let R be a ring and let X be a scheme over $\operatorname{Spec} R$. We say that X is a "scheme over R". Let $f : R \longrightarrow R'$ be a ring homomorphism. Then f induces a morphism
$$f^* : \operatorname{Spec} R' \longrightarrow \operatorname{Spec} R.$$
We **extend scalars on** X by forming the $\operatorname{Spec} R'$-scheme
$$X \times_{\operatorname{Spec} R} \operatorname{Spec} R' =: X \times_R R'.$$

2 Fibers of a morphism

Definition 2 *Let $f : X \longrightarrow Y$ be a morphism of schemes and let $y \in Y$ be a point (not necessarily closed). Then y corresponds to a morphism*

$$\operatorname{Spec} k(y) \longrightarrow Y,$$

where $k(y)$ is the residue field of y. The **fiber of f over y** *is by definition the scheme*

$$X_y := X \times_Y \operatorname{Spec} k(y).$$

Note that if y is a geometric point and if X, Y are varieties, then X_y need not be irreducible or reduced; thus schemes allow us to discuss "multiple fibers".

Definition 3 *A* **family of schemes** *is the set of fibers of a morphism of schemes*

$$f : X \longrightarrow Y.$$

Let us assume that Y is an irreducible variety.

1. *Let $\eta \in Y$ be the generic point of Y. Then*

$$X_\eta := X \times_Y \operatorname{Spec} k(Y)$$

   is called the **generic fiber** *of the family, where $k(\eta) = k(Y)$ is the function field of Y.*

2. *Let $y \in Y$ be a closed point. Then*

$$X_y := X \times_Y \operatorname{Spec} k(y)$$

   is called the **special fiber at** *y of the family.*

Note that:

- if Y is an algebraic curve and X/K is a variety, then the special fibers are defined over K, while the generic fiber is defined over the function field $K(Y)$;
- if $Y = \operatorname{Spec} \mathbb{Z}$, then the special fibers are defined over a field of different characteristic $p \neq 0$, while the generic fiber is defined over $\mathbb{Q}$.

3 Models and good reduction

For the rest of this lecture, K will denote a *number field* or the *function field* $k(C)$ of a smooth, affine curve (these situations correspond to the *arithmetic*, respectively *geometric* flavors of the theory).

Let X/K be a smooth projective variety.

If K is a number field, let R be its ring of integers and

$$S := \operatorname{Spec} R.$$

If $K = k(C)$, let R be the coordinate ring of C. Let C^{sch} be the scheme associated to C and

$$S := C^{sch} = \operatorname{Spec} R.$$

Fix an embedding

$$i : X \longrightarrow \mathbb{P}^n_K.$$

One can show that $\mathbb{P}^n_K$ is the generic fiber of the scheme

$$\mathbb{P}^n_S \longrightarrow S.$$

In fact, $R[x_0, \ldots, x_n] \otimes_R K \simeq K[x_0, \ldots, x_n]$. We then take $\mathcal{X}$ to be the Zariski closure of $i(X)$ inside $\mathbb{P}^n_S$. Thus $\mathcal{X}$ is an S-scheme, which is projective (i.e. all the fibers of $\mathcal{X} \longrightarrow S$ are projective varieties) and whose generic fiber

$$\mathcal{X}_\eta := \mathcal{X} \times_S \operatorname{Spec} K$$

is isomorphic to X. This is a particular case of what we call *a model for X over S*. Note that $\mathcal{X}$ may have many "bad" fibers, i.e. fibers which are reducible and/or non-reduced.

Definition 4 *Let K, S be as above and let X/K be a variety. A* **model for X over S** *is a scheme*

$$\mathcal{X} \longrightarrow S$$

whose generic fiber is isomorphic to X.

The concept of a model generalizes two well-known constructions in geometry/arithmetic:

- if S is a curve, then $\mathcal{X}$ is a deformation parametrized by S;
- if $S = \operatorname{Spec} \mathbb{Z}$, then $\mathcal{X}$ is a family of schemes (namely its special fibers) $\mathcal{X}_p = \mathcal{X} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec}(\mathbb{Z}/p\mathbb{Z})$. The scheme $\mathcal{X}_p$ is the "reduction" of X modulo p.

Example

Let $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ be an affine variety defined over $\mathbb{Q}$ by equations $f_1, \ldots, f_t$. We can assume without loss of generality that $f_1, \ldots, f_t \in \mathbb{Z}[x_1, \ldots, x_n]$. Then $\mathcal{X} = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_t)$ is a model for X over $\operatorname{Spec} \mathbb{Z}$. The special fiber at a prime p is defined by $f_1, \ldots, f_t \in \mathbb{Z}/p\mathbb{Z}[x_1, \ldots, x_n]$, and the generic fiber is X.

We have:

Lemma 5 *Let K, S be as above. Let X/K be a variety and let*

$$\mathcal{X} \longrightarrow S$$

be a projective model for X (i.e. all its fibers are projective schemes). Then every K-rational point

$$\operatorname{Spec} K \longrightarrow X$$

of X extends to an S-morphism

$$S \longrightarrow \mathcal{X}.$$

In other words, there is a natural 1:1 correspondence

$$X(K) \leftrightarrow \mathcal{X}(S).$$

Note that there are many different models for a given X. We would like to consider the one with "optimal" features.

Definition 6 *A smooth projective variety X/K has* **good reduction at** $x \in X$ *if there exists a projective model of X over $\mathcal{O}_x$ whose special fiber is smooth. If such a model does not exist, we say that X has* **bad reduction at** x.

Proposition 7 *Let K, S be as above. Let X/K be a smooth projective variety. Then:*

1. *X has good reduction at all but finitely many points;*

2. *let $T \subset S$ be the finite set of points where X has bad reduction. Let S_T be $\operatorname{Spec}(R_K)_T$ if K is a number field and $C \backslash T$ if $K = k(C)$. Then there exists a projective model of X over S_T all of whose fibers are smooth.*

For a subvariety X of $\mathbb{P}^n_K$ defined by equations with K-coefficients, the intuitive approach to construct a model would be to "clear denominators" in order to obtain equations which are defined over R. These equations define a subvariety $\mathcal{X}$ of $\mathbb{P}^n_R$ (this corresponds to the construction described on page 4), which is a model for X. In the case of elliptic curves, we have the following result. We state the next result in the case $K = \mathbb{Q}$, but it holds in general for a discrete valuation ring R with field of fractions K.

Proposition 8 *Let E be an elliptic curve over $\mathbb{Q}$, defined by the equation*

$$f(x,y) = y^2 - x^3 - ax - b \in \mathbb{Z}[x,y].$$

1. *$f(x,y)$ defines a scheme $W \subseteq \mathbb{P}^2_{\mathbb{Z}}$. Let W^0 be the largest subscheme of W which is smooth over $\operatorname{Spec} \mathbb{Z}$.*

2. $W(\mathbb{Z}) = E(\mathbb{Q})$. *If* W *is smooth, then* $W(\mathbb{Z}) = W^0(\mathbb{Z}) = E(\mathbb{Q})$.

3. *The addition and inversion maps on* E *extend to*

$$W^0 \times_{\mathbb{Z}} W^0 \longrightarrow W^0 \quad \text{and} \quad W^0 \longrightarrow W^0$$

so W^0 *is a group scheme over* $\operatorname{Spec}\mathbb{Z}$. *The addition map further extends to*

$$W^0 \times_{\mathbb{Z}} W \longrightarrow W.$$

Remark 9

- *If* $E/\mathbb{Q}$ *has a good reduction at all* p, *then* W *is smooth over* $\operatorname{Spec}\mathbb{Z}$. *Then* $W = W^0$ *is a group scheme over* $\mathbb{Z}$ *and it is the Néron model of* $E/\mathbb{Q}$ *(to be discussed soon).*
- *If* $E/\mathbb{Q}$ *has a bad reduction at* p *(i.e. if* $\Delta \equiv 0 (\operatorname{mod} p)$*), then* E_p *has exactly one singular point (a node or a cusp). So the special fiber* $E_p = W \times_{\mathbb{Z}} (p) = W \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Z}/p\mathbb{Z})$ *has a singular point* γ_p *which we discard and* $W^0 = W \setminus \{\gamma_p\}$.
- *Summarizing,* $W \subseteq \mathbb{P}^2_{\mathbb{Z}}$ *is a closed subscheme with* $W(\mathbb{Z}) = E(\mathbb{Q})$. *In general* W *is not smooth. If* $W(\mathbb{Z})$ *contains a singular point, then in general* $W^0(\mathbb{Z}) \neq E(\mathbb{Q})$. *So* W^0 *is not large enough to be a Néron model for* E, *but* W *is too large, since the group law does not extend.*

4 The minimal model of a curve

Let K, S be as before and let X/K be a variety. We want to find the *best possible* model

$$\mathcal{X} \longrightarrow S$$

which should reflect some interesting arithmetic features of X.

Definition 10 *We keep the above setting.*

1. *A projective model* $\mathcal{X} \longrightarrow S$ *of* X/K *is called a* **relatively minimal model** *if:*
   (a) *it is regular;*
   (b) *every birational morphism* $\mathcal{X} \longrightarrow \mathcal{X}'$ *from* $\mathcal{X}$ *to another regular model* $\mathcal{X}'$ *is an isomorphism.*
2. *A projective model* $\mathcal{X} \longrightarrow S$ *of* X/K *is called* **minimal** *if for any other regular model* $\mathcal{X}'$, *there exists a birational morphism* $\mathcal{X}' \longrightarrow \mathcal{X}$.

Theorem 11 *(Abhyankar, Shafarevich)*
Let X/K be a curve of genus $g \geq 1$. Then there exists a unique (up to isomorphism) projective minimal model $\mathcal{X} \longrightarrow S$ of X.

Definition 12 *Let X/K be a variety and let $\mathcal{X} \longrightarrow S$ be a minimal model of X. Let $\mathfrak{p} \in S$. We say that X has* **semistable reduction at** $\mathfrak{p}$ *if the special fiber $\mathcal{X}_{\mathfrak{p}}$ is reduced and has only ordinary double points as singularities.*

Theorem 13 *Let K be as above. Let X/K be a smooth projective curve. Then there exists a finite extension L/Ksuch that X has semistable reduction at all places of L.*

5 The Néron model of an abelian variety

Let K, S be as before.

Definition 14 *Let X/K be a variety. A scheme $\mathcal{X} \longrightarrow S$ is a* **Néron model of** X/K *if*

1. *it is smooth over S;*
2. *satisfies the following universal property: for any smooth scheme $\mathcal{Y} \longrightarrow S$ with generic fiber Y/K and any morphism $f : Y/K \longrightarrow X/K$, there exists a unique morphism of S-schemes $\bar{f} : \mathcal{Y} \longrightarrow \mathcal{X}$ which extends f.*

Clearly, if a Néron model exists, then it is unique up to isomorphism.

Theorem 15 *(Néron)*
Let K, S be as before. Let A/K be an abelian variety. Then there exists a Néron model $\mathcal{A} \longrightarrow S$ of A/K. Furthermore, $\mathcal{A}$ is a group scheme over S.

Conversely, an abelian scheme (i.e. a scheme such that every fiber is an abelian variety) $\mathcal{A} \longrightarrow S$ is the Néron model of its generic fiber.

Note that every point $P \in A(K)$ (viewed as a morphism $\operatorname{Spec} K \longrightarrow A$) extends to a morphism $S \longrightarrow \mathcal{A}$, however this may not be true for points in $A(L)$ for finite extensions L/K.

Let us also remark that for every $\mathfrak{p} \in S$, the special fiber $\mathcal{A}_{\mathfrak{p}}$ is an abelian variety iff A has good reduction at $\mathfrak{p}$.

The connected component of $\mathcal{A}_{\mathfrak{p}}$ is usually denoted by $\mathcal{A}_{\mathfrak{p}}^0$. This is an extension of an abelian variety over $k(\mathfrak{p})$ by a commutative affine group.

Definition 16 *An abelian variety A/K has* **semistable reduction at** $\mathfrak{p}$ *if $\mathcal{A}^0_{\mathfrak{p}}$ is an extension of an abelian variety by a torus T. It has* **split semistable reduction** *if the torus is isomorphic to $\mathbb{G}_m^s$ over the residue field $k(\mathfrak{p})$.*

Theorem 17 *Let K be as before. Let A/K be an abelian variety. Then there exists a finite extension L/K such that S has split semistable reduction at all places of L.*

<u>The minimal model and the Néron model of an elliptic curve</u>

Let $E/\mathbb{Q}$ be an elliptic curve defined by the minimal Weierstrass equation

$$f(x, y, z) = -y^2z - a_1xyz - a_3yz^2 + x^3 + a_2x^2z + a_4xz^2 + a_6z^3 = 0,$$

of discriminant Δ. Let $\mathcal{E}$ be the projective scheme associated to the graded ring $\mathbb{Z}[x, y, z]/(f)$.

As proven in [Si1], we have that the special fiber $\tilde{E}_p := \mathcal{E} \times \mathbb{F}_p$ is smooth iff $\Delta \not\equiv 0 (\mathrm{mod}\, p)$. If $\Delta \equiv 0 (\mathrm{mod}\, p)$, then $\tilde{E}_p$ has exactly one singular point.

In general, $\mathcal{E}$ will not be regular, hence it will not be a minimal model for $E/\mathbb{Q}$. However, if $\mathrm{ord}_p(\Delta) = 0$ or 1 for every prime p, then $\mathcal{E}$ is regular, hence a minimal model for $E/\mathbb{Q}$. In this case, the Néron model of $E/\mathbb{Q}$ is $\mathcal{E}$ with the one singular point removed from each of its bad special fibers.

N.B. In general, the scheme obtained by removing the singular points from the bad fibers is only the *connected component* of the Néron model.

For curves of higher genus, the relation between the minimal model of the curve and the Néron model of its Jacobian is much more complicated:

let X/K be a smooth projective curve of genus $g \geq 1$, J_X/K its Jacobian, $\mathcal{X} \longrightarrow S$ its minimal model, and $\mathcal{J} \longrightarrow S$ the Néron model of J_X. We have:

- the curve X/K has semistable reduction iff its Jacobian J_X/K has semistable reduction; if X/K has good reduction, then J_X/K also has good reduction, but the converse is not true in general;
- the connected component of $\mathcal{J}$ is isomorphic to $\mathrm{Pic}^0(\mathcal{X})$;
- the group of components of $\mathcal{J}$ can be easily computed from the intersection matrix of the components of the fiber of $\mathcal{X}$.

In general, we construct the Néron model of an abelian variety by following five main steps:

1. work locally, i.e. over DVR's R;
2. pass to the strict henselization R' of R;
3. resolve the singularities;
4. extend the birational group law obtained to an actual group law to obtain the Néron model over R';
5. descend to R (uses a method due to Grothendieck).

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