# THE GUTZWILLER TRACE FORMULA
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ABSTRACT. We'll sketch below a proof of the Gutzwiller trace formula based on the "symplectic category" ideas of [We] and [Gu-St],. We'll review these ideas in §1 and in §2 give a brief account, based on these ideas, of the theory of oscillatory functions. In §3 we'll discuss a key ingredient in the proof of the Gutzwiller formula, the lemma of stationary phase, and in §4 another key ingredient: a formula for the phase function of a Hamiltonian flow. Finally in §5 we'll show how to prove the Gutzwiller theorem using these results.

## 1. The category *Symp*

The objects in this category are symplectic manifolds, pairs $(M, \omega)$ where $M$ is an even-dimensional manifold and $\omega \in \Omega^2(M)$ a symplectic form. In these notes we will usually write "$M$" for "$(M, \omega)$" dropping the $\omega$, however, we will denote by $M^-$ the pair $(M, -\omega)$.

For most of the categories that one encounters in category theory "morphisms" are synonymous with "maps". For the applications of symplectic geometry to semi-classical analysis one needs a much larger class of morphisms. Given symplectic manifolds, $M_1$ and $M_2$, one has to allow the morphisms from $M_1$ to $M_2$ to be Lagrangian submanifolds

$$\Gamma \subseteq M_1^- \times M_2$$

a.k.a., *canonical relations.* This makes composition of morphisms a bit of a problem. Given a pair of canonical relations, $\Gamma_i \subseteq M_i \times M_{i+1}$, $i = 1, 2$, their relation theoretic composition is defined by

$$(p_1, p_3) \in \Gamma_2 \circ \Gamma_1 \Leftrightarrow (p_i, p_{i+1}) \in \Gamma_i\,, \quad i = 1, 2, \text{ for some } p_2 \in \Gamma_2$$

just as for compositions of mappings. However for $\Gamma_2 \circ \Gamma_1$ to be a submanifold of $M_1^- \times M_3$ one has to impose transversality (or cleanness) assumptions on $\Gamma_1$ and $\Gamma_2$, and hence compositions aren't always well-defined. In other words the symplectic category is not really a category at all but just a "category". (For ways of removing the stigma of these quotation marks see [Ca-Dh-We] or [Wehr-Wo].) We will use double arrow notation

$$\Gamma : M_1 \Rightarrow M_2$$

for these morphisms to distinguish them from maps. (Occasionally, however, a morphism will be a map, i.e., a symplectomorphism.)

Some features of the category, *Symp*:

(1) This is a pointed category, the unique point object in this category, "*pt*", being the (unique up to isomorphism) connected zero-dimensional symplectic manifold consisting of a single point.
(2) The morphisms, $\Lambda : pt \Rightarrow M$ are just the Lagrangian submanifolds of $M$.
(3) To every morphism, $\Gamma : M_1 \Rightarrow M_2$, corresponds a transpose morphism,

$$\Gamma^t : M_2 \Rightarrow, \text{ where } (p,q) \in \Gamma \Leftrightarrow (q,p) \in \Gamma^t .$$

From now on we'll confine ourselves for the most part to symplectic manifolds of the form, $M = T^*X$, i.e., cotangent bundles. This will create some notational problems since

$$T^*(X_1 \times X_2) = T^*X_1 \times T^*X_2 \neq (T^*X_1)^- \times T^*X_2 .$$

On the other hand, $T^*X_1 \cong (T^*X_1)^-$ via the symplectomorphism, $(x,\xi) \to (x,-\xi)$, and we'll implicitly, whenever required, make the identification.

We will also, in our cotangent bundle version of *Symp*, restrict ourselves to Lagrangian manifolds and canonical relations which are *exact*. Recall that the symplectic form on $T^*X$ is the exterior derivative of a one-form, $-\alpha_X$, where $\alpha_X = \sum \xi_i \, dx_i$.

Given a canonical relation,

$$\Gamma \subseteq (T^*X_1)^- \times T^*X_3$$

we'll say that it's exact if

$$\iota_\Gamma^*(-(pr_1)^*\alpha_{X_1} + (pr_2)^*\alpha_{X_2}) = d\psi \tag{1.1}$$

for some $\psi \in \mathcal{C}^\infty(\Gamma)$, and in what follows we'll restrict ourselves to canonical relations with this property. Moreover, the function, $\psi$, in (1.1) will play an important role in the applications below, and to emphasize this fact we'll henceforth write canonical relations as pairs, $(\Gamma, \psi)$.

**Examples**

(1) Given $\varphi \in \mathcal{C}^\infty(X)$ let

$$\Lambda_\varphi = \{(x,\xi) \in T^*X \, , \quad \xi = d\varphi_x\} \, .$$

Then

$$\iota_\Lambda^* = d\psi$$

where $\psi(x,\xi) = \varphi(x)$

(2) Let $\pi : Z \to X$ be a fibration and let $\Gamma$ be the subset of $(T^*Z)^- \times T^*X$ defined by

$$(z, \eta, x, \xi) \in \Gamma \Leftrightarrow \pi(z) = x \text{ and } \eta = (d\pi_z)^*\xi \,.$$

Via the identification $(T^*Z)^- \times T^*X \cong T^*Z \times T^*X$ this is just the conormal bundle of the graph $\pi$ in $Z \times X$ and hence is a morphism

$$\Gamma : T^*Z \Rightarrow T^*X \,. \tag{1.2}$$

Now let $\varphi$ be in $\mathcal{C}^\infty(Z)$ and let

$$\Lambda_\varphi : pt \Rightarrow T^*Z \tag{1.3}$$

be the Lagrangian manifold in example 1. Then if (1.3) and (1.2) are composable one gets a canonical relation

$$\Gamma \circ \Lambda_\varphi : pt \Rightarrow T^*X$$

i.e., a Lagrangian submanifold, $\Lambda = \Gamma \circ \Lambda_\varphi$, of $T^*X$. It's easy to check (exercise) that

$$\iota_\Lambda^* \alpha_X = d\psi \tag{1.4}$$

where $\psi(x, \xi) = \varphi(z)$ if $(z, \eta, x, \xi) \in \Gamma$.

## 2. Oscillatory functions

Let $\Lambda_\varphi \subseteq T^*X$ be the Lagrangian manifold in example 1. In quantum mechanics one attaches to $\Lambda$ a "de Broglie function"

$$a e^{\frac{i\varphi}{\hbar}} \tag{2.1}$$

with amplitude $a \in \mathcal{C}^\infty(X)$ and phase $\varphi$. Thus, as $h \to 0$, the phase part of (2.1) becomes more and more oscillatory and hence, from the macro-perspective, more and more fuzzy and ill-defined.

In semi-classical analysis one replaces these functions by a slightly larger class of functions: functions of the form

$$a(x, h) e^{\frac{i\varphi}{\hbar}} \tag{2.2}$$

for which the amplitude also depends on $h$. However, one requires $a(x, h)$ to have an asymptotic expansion in powers of $\hbar$:

$$a(x, h) \sim \sum_{i=-k_0}^{\infty} a_i(x) h^i \tag{2.3}$$

for some $k_0 \in \mathbb{N}$.

One can also associate a class of oscillatory functions to Lagrangian manifolds of the type described in example 2 by requiring them to be "superpositions" of functions of type (2.2). More explicitly, suppose that each fiber, $\pi^{-1}(x)$, of the fibration, $\pi : Z \to X$ is equipped with a volume form, $\mu_x$. Then if $a(z,h)e^{\frac{i\varphi(z)}{h}}$ is an oscillatory function on $Z$ of the type above and $a(z,h)$ is compactly supported in fiber directions one can define an oscillatory function, $\pi_* a e^{\frac{i\varphi}{h}}$, on $X$ by defining it pointwise by

$$(\pi_* a e^{\frac{i\varphi}{h}})(x) = \int_{\pi^{-1}(x)} e^{\frac{i\varphi}{h}} a\mu_x \,. \tag{2.4}$$

One can prove:

**Theorem 2.1.** *The space of functions (2.4) is intrinsically defined, depending only on the pair,* $(\Lambda, \psi)$ *where* $\iota_\Lambda^* \alpha_X = d\psi$.

In other words it doesn't depend on the choice of $(Z, \pi)$ and only depends on the choice of $\varphi$ to the extent that $\varphi$ and $\psi$ are related by (2.3). For a proof of this result see [Gu-St], Chapter ?. We also show in this chapter that this result allows one to attach a class of oscillatory functions to any Lagrangian pair, $(\Lambda, \psi)$. We'll call a function of the form (2.4) an oscillatory function with *micro-support* on $\Lambda$ and *phase* $\psi$.

## 3. The lemma of stationary phase

Let $(\Lambda, \psi)$ be an exact Lagrangian submanifold of $T^*X$, $Y$ a submanifold of $X$ and $\mu$ a volume form on $Y$. Given an oscillatory function, $f(h)$, on $X$ with micro-support on $\Lambda$ and phase, $\psi$, the integral

$$\int_Y \iota_Y^* f\mu = I(h) \tag{3.1}$$

is an oscillatory "constant" and in this section we'll describe how to compute its phase. Let $\Lambda_0$ be the conormal bundle of $Y$ in $T^*X$. Then by a basic property of conormal bundles $\iota_{\Lambda_0}^* \alpha_X = 0$. Suppose now that $\Lambda$ and $\Lambda_0$ intersect cleanly and that their intersection, $W = \Lambda \cap \Lambda_0$ is connected. Then

$$\iota_W^* \alpha = \iota_W^* \iota_{\Lambda_0}^* \alpha = 0 = \iota_W^* \, d\psi \,,$$

so $\psi$ is constant on $W$, and one has:

**Theorem 3.1. (lemma of stationary phase)**

The oscillatory constant (3.1) is an expression of the form

$$I(h) = c(h) e^{\frac{i\psi(p_0)}{h}} \tag{3.2}$$

where $p_0$ is any point on $W$ and $c(h)$ has an asymptotic expansion

$$\sum_{i=k}^{-\infty} c_i h^i \,. \tag{3.3}$$

For a proof of this result and for some details about how one computes the terms in the asymptotic series (3.3) we refer again to [Gu-St].

## 4. Canonical relations generated by Hamiltonian flows

Let $(M, \omega)$ be a symplectic manifold and $v = v_H$, $H \in \mathcal{C}^\infty(M)$, a Hamiltonian vector field, We'll prove

**Theorem 4.1.** *The set*

$$\Lambda = \{(p, (\exp tv)(p), t, \tau), , \quad p \in M, t \in \mathbb{R}, \tau = H(p)\} \tag{4.1}$$

*is a Lagrangian submanifold of* $M^- \times M \times (T^*\mathbb{R})^-$.

*Proof:*

For fixed $t \in \mathbb{R}$, $\Lambda_t$ = graph $\exp tv_H$ is a Lagrangian submanifold of $M^- \times M$ and hence an isotropic submanifold of $M^- \times M \times (T^*\mathbb{R})^-$. Now note that the tangent space to $\Lambda$ at $\lambda = (p, q, t, \tau = H(\iota))$, $q = (\exp tv)(p)$, is spanned by $T_{p,q}\Lambda_t$ and $v_H(q) + \frac{\partial}{\partial t} =: \mathcal{W}(q, t)$ and that

$$\iota(\mathcal{W})(\omega_M - dt \wedge d\tau)_{q,t} = dH_q - (d\tau)_{q,t}$$

and hence is zero when we set $\tau = H$.

Q.E.D.

Suppose now that $\omega$ is exact, i.e., $\omega = -d\alpha$ for $\alpha \in \Omega^1(M)$. Then the symplectic form on $M^- \times M \times (T^*\mathbb{R})^-$ is exact and is equal to $-d\tilde{\alpha}$ where

$$\tilde{\alpha} = -(pr_1)^*\alpha + (pr_2)^*\alpha - \tau\, dt\,. \tag{4.2}$$

**Theorem 4.2.** *Let* $\iota_\Lambda : M \times \mathbb{R} \to M^- \times M \times (T^*\mathbb{R})^-$ *be the map*

$$\iota_\Lambda(p, t) = (p, (\exp tv)(p), t, \tau) \tag{4.3}$$

*where* $\tau = H(p) = H((\exp tv)(p))$. *Then*

$$\iota_\Lambda^* \tilde{\alpha} = d\psi \tag{4.4}$$

*where*

$$\psi = \int_0^1 (\exp sv)^*(v)\alpha - tH\,. \tag{4.5}$$

*Proof:*

By (4.2) and (4.3)

$$\begin{aligned} \iota_\Lambda^* \alpha &= -\alpha + (\exp tv)^*\alpha + (\exp tv)^*\iota(v)\alpha\, dt - H\, dt \\ &= \int_0^1 (\exp sv)^*\alpha\, ds + (exptv)^*\iota(v)\alpha\, dt - H\, dt\,. \end{aligned} \tag{4.6}$$

We can rewrite the first term on the right as

$$\begin{aligned}
&\int_0^t (\exp sv)^* L_v\alpha\, ds \\
= \;& \int_0^t (\exp sv)^*\, d_M\iota(v)\alpha\, ds + \int_0^t (\exp sv)^*\iota(v)\, d_M\alpha \\
= \;& (d_{M\times\mathbb{R}})\int_0^t (\exp sv)^*\iota(v)\alpha\, ds - \frac{d}{dt}\left(\int_0^t (\exp sv)^*\iota(v)\alpha\, ds\right) dt \\
& - \left(\int_0^t ds\right) dH \\
= \;& (d_{M\times\mathbb{R}})\int_0^t (\exp sv)^*\iota(v)\alpha\, ds - ((\exp tv)^*\iota(v)\alpha)\, dt - t\, dH \\
= \;& d\psi - (\exp tv)^*\iota(v)\alpha\, dt + H\, dt
\end{aligned}$$

so the last two terms cancel the last two terms in (4.6) leaving us with $\iota_\Lambda^*\alpha = d\psi$.
Q.E.D.

We conclude with a few remarks about periodic trajectories of $v_H$. Suppose $\gamma(t) = (\exp tv)(p)$, $-\infty < t < \infty$, is a periodic trajectory of period $T : \gamma(0) = \gamma(T)$. Then the map, $\exp Tv : M \to M$, has a fixed point at $p$ and its derivative

$$d(\exp Tv)_p : T_pM \to T_pM \tag{4.7}$$

maps the subspace, $dH_p = 0$, of $T_pM$ and the vector, $v(p)$, onto itself. Since the subspace of $T_pM$ spanned by $v(p)$ is contained in the subspace, $dH_p = 0$, one gets from (4.7) a linear map, $P_\gamma$, of the quotient space onto itself.

**Definition**

The trajectory, $\gamma$, is non degenerate if $\det(I - P_\gamma) \neq 0$.

## 5. The Gutzwiller trace formula

Let $X$ be an $n$-dimensional Riemannian manifold and let

$$S_h = -h^2\Delta_X + V \tag{5.1}$$

be the Schrödinger operator on $X$ with potential $V \in \mathcal{C}^\infty(X)$. As a self-adjoint operator on $L^2(X)$, $S_h$ generates a one-parameter group of unitary transformations

$$\exp\frac{it}{h}S_n\,, \quad -\infty < t < \infty\,. \tag{5.2}$$

This can be viewed as the *quantization* of the one-parameter group of canonical transformations

$$\exp tv_H : T^*X \to T^*X\,, \quad -\infty < t < \infty \tag{5.3}$$

where

$$H(x,\xi) = |\xi|^2 + V(x)\,, \tag{5.4}$$

the connection between (5.2) and (5.3) being given by the following result.

**Theorem 5.1.** *Suppose that for some $\epsilon > 0$ $H^{-1}([-\epsilon,\epsilon])$ is compact. Then for $\rho \in \mathcal{C}_0^\infty(-\epsilon,\epsilon)$ the Schwartz kernel, $e_\rho(x,y,t,h)$, of the operator, $\exp\frac{itS_n}{h}\rho(S_n)$ is an oscillatory function with micro-support on the Lagrangian manifold (4.1) and the phase function (4.5).*

A proof of this can be found in [Di-Sj] and (hopefully) in the final version of [Gu-St].

Now suppose that there are only a finite number of periodic trajectories of the vector field, $v_H$, lying on the energy surface $H = 0$ and having period $0 < a < T < b$. In addition suppose that these trajectories, which we'll denote by $\gamma_i$, $i = 1,\dots,N$, are all non-degenerate. Then one has:

**Theorem 5.2. (The Gutzwiller trace formula)**

*For $f \in \mathcal{C}_0^\infty(a,b)$ the trace of the operator*

$$\int f(t)\exp\frac{itS_h}{h}\rho(S_h)\,dt \tag{5.5}$$

*has an asymptotic expansion*

$$\sum_{i=1}^{N} c_i(h)e^{\frac{iS\gamma i}{h}} \tag{5.6}$$

*where the $c_i$'s are asymptotic series in $h$ and*

$$S_{\gamma_i} = \int_{\gamma_i}\alpha\,. \tag{5.7}$$

*Proof:*

By Theorem 5.1 the trace of (5.5) is the integral of $f(t)e_\rho(x,y,t)$ over the submanifold

$$Y = \Delta_X \times \mathbb{R}$$

of $X \times X \times \mathbb{R}$. The conormal bundle of $Y$ in $(T^*X)^- \times T^*X \times (T^*\mathbb{R})^-$ is the set of points $(x,\xi,y,\eta,t,\tau)$ with

$$x = y\,,\quad \xi = \eta\,,\quad \tau = 0\,, \tag{5.8}$$

and the intersection of this set with the set (4.1) is the set of points in $(T^*X)^- \times T^*X \times (T^*\mathbb{R})^-$ satisfying

$$(\exp tv)(x,\xi) = (y,\eta) \tag{5.9}$$

and

$$\tau = H(x, \xi) = 0 \,. \tag{5.10}$$

Hence if we apply the lemma of stationary phase to the integral

$$\int_Y f(t) e_\rho(x, x, t) \, dx \, dt \,, \tag{5.11}$$

$dx$ being the Riemannian volume form on $X$, we get by (4.4)–(4.5) an asymptotic expansion of the form (5.6)–(5.7).

## References

[Ca-Dh-We] A. Cattaneo, B. Dherin and A. Weinstein, "The cotangent microbundle category I", arXiv 0712-1385.

[Di-Sj] M. Dimassi and J. Sjostrand, *Spectral Asymptotics in the Semi-classical Limit*, London Math. Society Lecture Note Series, 1999.

[Gu-St] V. Guillemin and S. Sternberg, *Semi-classical Analysis*, a manuscript in progress.[1]

[We] A. Weinstein, "Symplectic geometry", Bull. AMS, vol. 5, pp.1–14 (1981).

[Wehr-Wo] K. Wehrheim and C.T. Woodward, "Functionality for Lagrangian correspondences in Floer theory", arXiv 0708.2851 (2007).

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[1]These notes can be downloaded from Shlomo's website at the Harvard math department.