FIELDS-MITACS Conference

on the Mathematics of Medical Imaging

Photoacoustic and Thermoacoustic Tomography with a variable sound speed

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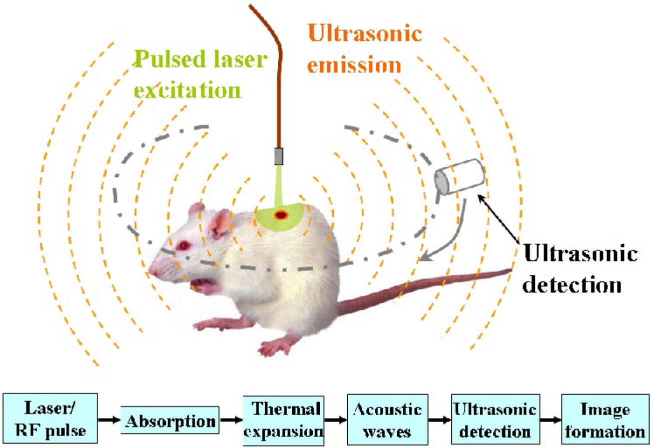

Pulsed laser excitation
Ultrasonic emission
Ultrasonic detection
Laser/ RF pulse
Absorption
Thermal expansion
Acoustic waves
Ultrasonic detection
Image formation

Wikipedia

First Step in PAT and TAT is to reconstruct $H(x)$ from $u(x,t)|_{\partial\Omega\times(0,T)}$, where u solves

$$\begin{gathered}(\partial_t^2 - c^2(x)\Delta)u = 0 \quad \text{on} \quad \mathbb{R}^n \times \mathbb{R}^+ \\ u|_{t=0} = \beta H(x) \\ \frac{\partial u}{\partial t}|_{t=0} = 0\end{gathered}$$

Second Step in PAT and TAT is to reconstruct the optical or electrical properties from $H(x)$ (internal measurements).

FIRST STEP: IP for Wave Equation

$c(x) > 0$: acoustic speed

$$\begin{cases} (\partial_t^2 - c^2\Delta)u &= 0 \quad \text{in } (0,T)\times \mathbf{R}^n, \\ u|_{t=0} &= f, \\ \partial_t u|_{t=0} &= 0. \end{cases}$$

f: supported in $\bar{\Omega}$. Measurements:

$$\Lambda f := u|_{[0,T]\times\partial\Omega}.$$

The problem is to reconstruct the unknown f from Λf.

Prior results

Constant Speed

Kruger; Agranovsky, Ambartsoumian, Finch, Georgieva-Hristova, Jin, Haltmeier, Kuchment, Nguyen, Patch, Quinto, Rakesh, Wang, Xu ...

Variable Speed (Numerical Results)

Burgholzer,Georgieva-Hristova, Grun, Haltmeir, Hofer, Kuchment, Nguyen, Paltauff, Wang, Xu... (Time reversal)

Partial Data

Problem is uniqueness, stability and reconstruction with measurements on a part of the boundary. There were no results so far for the variable coefficient case, and there is a uniqueness result in the constant coefficients one by Finch, Patch and Rakesh (2004).

Ω=ball, constant speed

$c = 1$, Ω: unit ball, $n = 3$. Explicit Reconstruction Formulas (Finch, Haltmeier, Kunyansky, Nguyen, Patch, Rakesh, Xu, Wang).

$g(x,t) = \Lambda f$, $x \in S^{n-1}$. In 3D,

$$f(x) = -\frac{1}{8\pi^2}\Delta_x \int_{|y|=1} \frac{g(y,|x-y|)}{|x-y|} \mathrm{d}S_y.$$

$$f(x) = -\frac{1}{8\pi^2} \int_{|y|=1} \left(\frac{1}{t}\frac{d^2}{dt^2} g(y,t)\right)\Bigg|_{t=|y-x|} \mathrm{d}S_y.$$

$$f(x) = \frac{1}{8\pi^2}\nabla_x \cdot \int_{|y|=1} \left(\nu(y)\frac{1}{t}\frac{d}{dt}\frac{g(y,t)}{t}\right)\Bigg|_{t=|y-x|} \mathrm{d}S_y.$$

The latter is a partial case of an explicit formula in any dimension (Kunyansky).

$T = \infty$: a backward Cauchy problem with zero initial data.

$T < \infty$: time reversal

$$\begin{cases} (\partial_t^2 - c^2\Delta)v_0 &= 0 \quad \text{in } (0,T)\times\Omega, \\ v_0\big|_{[0,T]\times\partial\Omega} &= \chi h, \\ v_0\big|_{t=T} &= 0, \\ \partial_t v_0\big|_{t=T} &= 0, \end{cases}$$

where $h = \Lambda f$; χ: cuts off smoothly near $t = T$.

Time Reversal

$$f \approx A_0 h := v_0(0,\cdot) \quad \text{in } \bar{\Omega}, \text{ where } h = \Lambda f.$$

Uniqueness

Underlying metric: $c^{-2}\mathrm{d}x^2$. Set

$$T_0 = \max_{x\in\bar\Omega} \operatorname{dist}(x, \partial\Omega).$$

Theorem (Stefanov–U)

$T \geq T_0 \implies$ *uniqueness.*
$T < T_0 \implies$ *no uniqueness. We can recover $f(x)$ for* $\operatorname{dist}(x, \partial\Omega) \leq T$ *and nothing else.*

The proof is based on the unique continuation theorem by Tataru.

Stability

$T_1 \leq \infty$: length of the longest (maximal) geodesic through $\bar{\Omega}$.

The "stability time": $T_1/2$. If $T_1 = \infty$, we say that the speed is **trapping** in Ω.

Theorem (Stefanov–U)

$T > T_1/2 \implies$ *stability.*
$T < T_1/2 \implies$ *no stability, in any Sobolev norms.*

The second part follows from the fact that Λ is a smoothing FIO on an open conic subset of $T^*\Omega$. In particular, if the speed is trapping, there is no stability, whatever T.

Reconstruction. Modified time reversal

A modified time reversal, harmonic extension

Given h (that eventually will be replaced by Λf), solve

$$\begin{cases} (\partial_t^2 - c^2\Delta)v &= 0 \quad \text{in } (0,T)\times\Omega, \\ v\big|_{[0,T]\times\partial\Omega} &= h, \\ v\big|_{t=T} &= \phi, \\ \partial_t v\big|_{t=T} &= 0, \end{cases}$$

where ϕ is the harmonic extension of $h(T,\cdot)$:

$$\Delta\phi = 0, \quad \phi|_{\partial\Omega} = h(T,\cdot).$$

Note that the initial data at $t = T$ satisfies compatibility conditions of first order (no jump at $\{T\}\times\partial\Omega$). Then we define the following pseudo-inverse

$$Ah := v(0,\cdot) \quad \text{in } \bar{\Omega}.$$

We are missing the Cauchy data at $t = T$; the only thing we know there is its value on $\partial\Omega$. The time reversal methods just replace it by zero. We replace it by that data (namely, by $(\phi, 0)$), having the same trace on the boundary, that minimizes the energy. Given $U \subset \mathbf{R}^n$, the energy in U is given by

$$E_U(t,u) = \int_U \left(|\nabla u|^2 + c^{-2}|u_t|^2 \right) \mathrm{d}x.$$

We define the space $H_D(U)$ to be the completion of $C_0^\infty(U)$ under the Dirichlet norm

$$\|f\|_{H_D}^2 = \int_U |\nabla u|^2 \, \mathrm{d}x.$$

The norms in $H_D(\Omega)$ and $H^1(\Omega)$ are equivalent, so

$$H_D(\Omega) \cong H_0^1(\Omega).$$

The energy norm of a pair $[f,g]$ is given by

$$\|[f,g]\|_{\mathcal{H}(\Omega)}^2 = \|f\|_{H_D(\Omega)}^2 + \|g\|_{L^2(\Omega, c^{-2}\mathrm{d}x)}^2$$

$$A\Lambda f = f - Kf$$

$$Kf = w(0,.)$$

where w solves

$$\begin{cases} \left(\partial_t^2 - c^2(x)\,\Delta\right) w = 0 & \text{in } (0,T)\times\Omega, \\ w\big|_{[0,T]\times\partial\Omega} = 0, \\ w\big|_{t=T} = u\big|_{t=T} - \phi, \\ w_t\big|_{t=T} = u\big|_{t=T}\,. \end{cases}$$

$A\Lambda f = f - Kf$

Consider the "error operator" K,

$$Kf = \text{first component of: } U_{\Omega,D}(-T)\Pi_\Omega U_{\mathbf{R}^n}(T)[f,0],$$

where

- $U_{\mathbf{R}^n}(t)$ is the dynamics in the whole $\mathbf{R}^n$,
- $U_{\Omega,D}(t)$ is the dynamics in Ω with Dirichlet BC,
- $\Pi_\Omega : \mathcal{H}(\mathbf{R}^n) \to \mathcal{H}(\Omega)$ is the orthogonal projection.

That projection is given by $\Pi_\Omega[f,g] = [f|_\Omega - \phi, g|_\Omega]$, where ϕ is the harmonic extension of $f|_{\partial\Omega}$.

Obviously,

$$\|Kf\|_{H_D} \leq \|f\|_{H_D}.$$

Reconstruction, whole boundary

Theorem (Stefanov–U)

Let $T > T_1/2$. Then $A\Lambda = I - K$, where $\|K\|_{\mathcal{L}(H_D(\Omega))} < 1$. In particular, $I - K$ is invertible on $H_D(\Omega)$, and the inverse thermoacoustic problem has an explicit solution of the form

$$f = \sum_{m=0}^{\infty} K^m A h, \quad h := \Lambda f.$$

If $T > T_1$, then K is compact.

Reconstruction, whole boundary

We have the following estimate on $\|K\|$:

Theorem (Stefanov–U)

$$\|Kf\|_{H_D(\Omega)} \leq \left(\frac{E_\Omega(u,T)}{E_\Omega(u,0)} \right)^{\frac{1}{2}} \|f\|_{H_D(\Omega)}, \quad \forall f \in H_{D(\Omega)},\ f \neq 0,$$

where u is the solution with Cauchy data $(f,0)$.

Summary: Dependence on T

(i) $T < T_0 \implies$ **no uniqueness**
Λf does not recover uniquely f. $\|K\| = 1$.

(ii) $T_0 < T < T_1/2 \implies$ **uniqueness, no stability**
We have uniqueness but not stability (there are invisible singularities). We do not know if the Neumann series converges. $\|Kf\| < \|f\|$ but $\|K\| = 1$.

(iii) $T_1/2 < T < T_1 \implies$ **stability and explicit reconstruction**
This assumes that c is non-trapping. The Neumann series converges exponentially but maybe not as fast as in the next case (K contraction but not compact). There is stability (we detect all singularities but some with $1/2$ amplitude). $\|K\| < 1$

(iv) $T_1 < T \implies$ **stability and explicit reconstruction**
The Neumann series converges exponentially, K is contraction and compact (all singularities have left $\bar{\Omega}$ by time $t = T$). There is stability. $\|K\| < 1$

If c is trapping ($T_1 = \infty$), then (iii) and (iv) cannot happen.

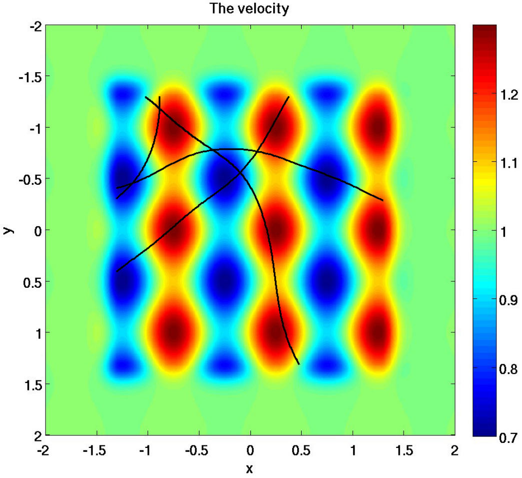


Figure: The speed, $T_0 \approx 1.15$. $\Omega = [-1.28, 1.28]^2$, computations are done in $[-2, 2]^2$

Example 1: Nontrapping speed

Figure: Original

Example 1: Nontrapping speed

Figure: Neumann Series reconstruction, $T = 4T_0 = 4.6$, error $= 3.45\%$

Example 1: Nontrapping speed

Figure: Time Reversal, $T = 4T_0 = 4.6$, error $= 23\%$

Example 2: Trapping speed

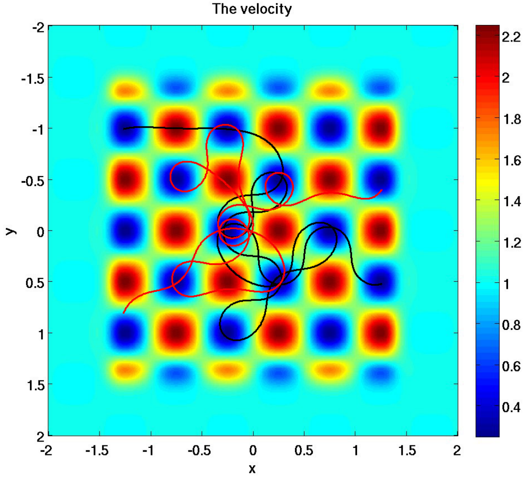


Figure: The speed, $T_0 \approx 1.18$

Example 2: Trapping speed

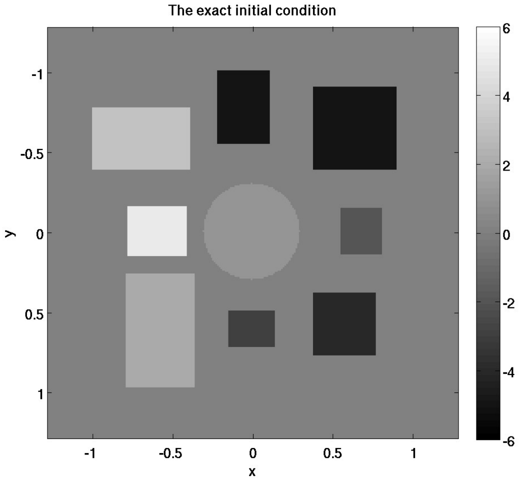


Figure: The original

Example 2: Trapping speed

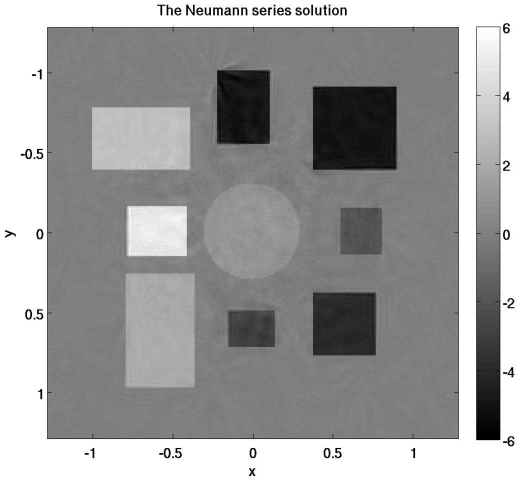


Figure: Neumann Series reconstruction, 10 steps, $T = 4T_0 = 4.7$, error $= 8.75\%$

Example 2: Trapping speed

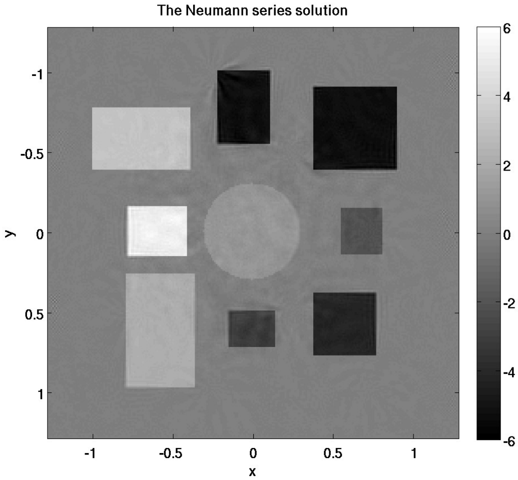


Figure: Neumann Series reconstruction with 10% noise, 15 steps, $T = 4T_0 = 4.7$, error $= 8.72\%$

Example 2: Trapping speed

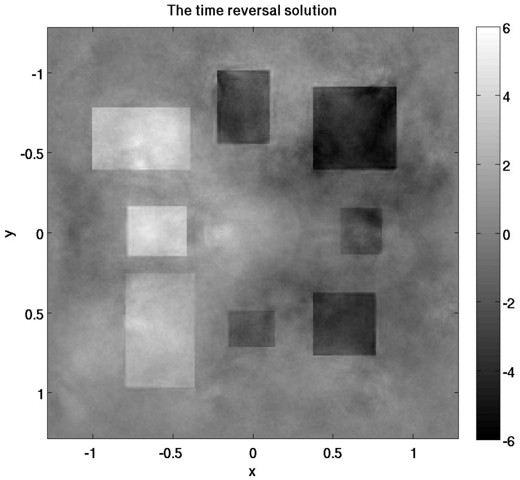


Figure: Time Reversal, $T = 4T_0 = 4.7$, error $= 55\%$

Example 2: Trapping speed

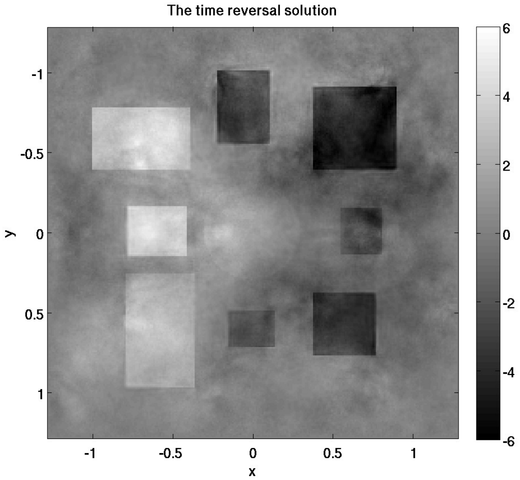


Figure: Time Reversal with 10% noise, $T = 4T_0 = 4.7$, error $= 54\%$

Example 3: The same trapping speed, Barbara

Figure: Original

Example 3: The same trapping speed, Barbara

Figure: Neumann series, $T = 4T_0 = 4.7$, error $= 7.5\%$, 10 steps

Example 3: The same trapping speed, Barbara

Figure: Time Reversal, $T = 4T_0 = 4.7$, error $= 27.7\%$

Example 3: The same trapping speed, Barbara

Figure: Time Reversal, $T = 12T_0 = 14.1$, error $= 99.67\%$

Example 4: a radial trapping speed

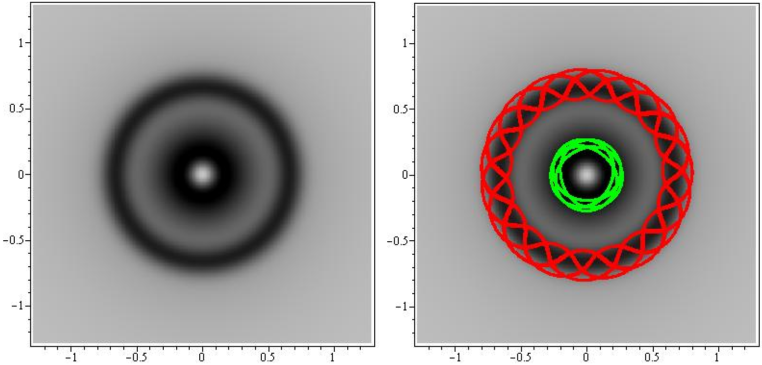


Figure: A trapping speed. Darker regions represent a slower speed. The circles of radii approximately 0.23 and 0.67 are stable periodic geodesics. Left: the speed. Right: the speed with two trapped geodesics

Example 4: a radial trapping speed

Figure: Original, lower resolution than before

Example 4: a radial trapping speed

Figure: Neumann series, 10 steps, $T = 8T_0 = 8.7$, error $= 9.7\%$

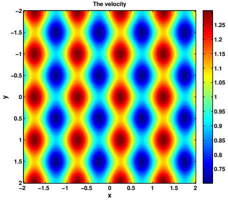
Example 4: a radial trapping speed

Figure: **Iterated** Time Reversal, 10 steps, $T = 8T_0 = 8.7$, error $= 12.1\%$

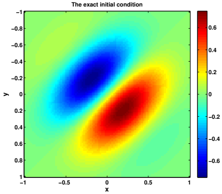
Example 4: a radial trapping speed

Figure: Time Reversal, $T = 8T_0 = 8.7$, error $= 21.7\%$

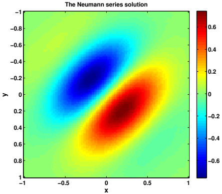
What if the waves can come back to Ω (reflectors)?



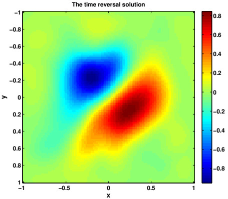
The speed



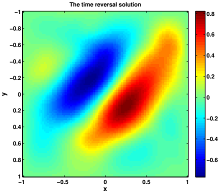
original



NS, $T = 6$



TR, $T = 6$



TR, $T = 26$

Figure: $T_0 \approx 1.2$, $2.9 < T_1 < 3.5$. There are Neumann BC here at the boundary of the larger square! Waves leaving Ω come back without any damping!

Discontinuous Speeds, Modeling Brain Imaging (Proposed by L. Wang)

Let c be piecewise smooth with a jump across a smooth closed surface Γ. The direct problem is a transmission problem, and there are reflected and refracted rays.

In brain imaging, the interface is the skull. The sound speed jumps by about a factor of 2 there. Experiments show that the ray that arrives first carries about 20% of the energy.

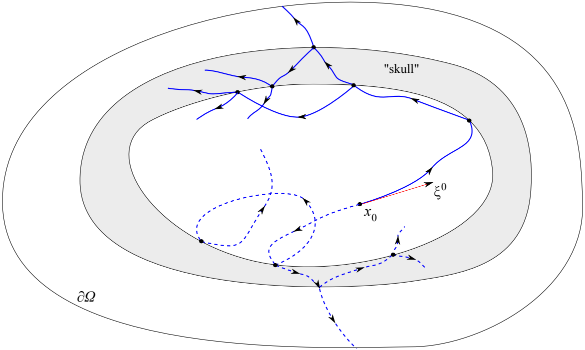


Figure: Propagation of singularities in the "skull" geometry

Propagation of singularities is the key again.
(Completely) trapped singularities are a problem, as before. Let $\mathcal{K} \subset \Omega$ be a compact set such that all rays originating from it are never tangent to Γ and non-trapping. For f satisfying

$$\operatorname{supp} f \subset \mathcal{K}$$

the Neumann series above still converges (uniformly to f).
We need a small modification to keep the support in $\mathcal{K}$ all the time. We use the projection

$$\Pi_{\mathcal{K}} : H_D(\Omega) \to H_D(\mathcal{K})$$

for that purpose.

Reconstruction

Theorem (Stefanov–U)

Let all rays from $\mathcal{K}$ have a path never tangent to Γ that reaches $\partial\Omega$ at time $|t| < T$. Then

$$\Pi_{\mathcal{K}} A \Lambda = I - K \text{ in } H_D(\mathcal{K}), \text{ with } \|K\|_{H_D(\mathcal{K})} < 1.$$

In particular, $I - K$ is invertible on $H_D(\mathcal{K})$, and Λ restricted to $H_D(\mathcal{K})$ has an explicit left inverse of the form

$$f = \sum_{m=0}^{\infty} K^m \Pi_{\mathcal{K}} A h, \quad h = \Lambda f.$$

The assumption $\operatorname{supp} f \subset \mathcal{K}$ means that we need to know f outside $\mathcal{K}$; then we can subtract the known part.
In the numerical experiments below, we do not restrict the support of f, and still get good reconstruction images but the invisible singularities remain invisible.

Brain imaging of square headed people

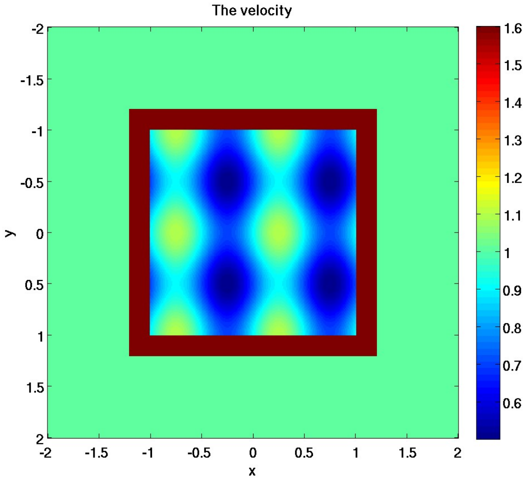


Figure: The speed jumps by a factor of 2 in average from the exterior of the "skull". The region Ω, as before, is smaller: $\Omega = [-1.28, 1.28]^2$.

A "skull" speed, Neumann series

original

$T = 2T_0$, error $= 15\%$

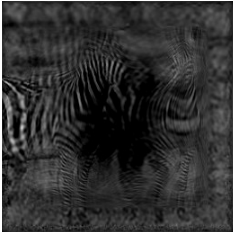
$T = 4T_0$, error $= 9.75\%$

$T = 8T_0$, error $= 7.55\%$

Figure: Neumann Series, 15 steps

A “skull” speed, Time Reversal

original



$T = 2T_0$, error $= 68\%$

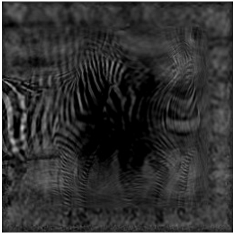
$T = 4T_0$, error $= 23.7\%$

$T = 8T_0$, error $= 78.5\%$

Figure: Time Reversal. There is a lot of “white clipping” in the last image, many values in $[1, 1.6]$

A “skull” speed, Time Reversal

original



$T = 2T_0$, error $= 68\%$

$T = 4T_0$, error $= 23.7\%$

$T = 8T_0$, error $= 78.5\%$

Figure: Time Reversal. The values in last image are compressed from $[0, 1]$ to $[-0.05, 1.6]$

Original vs. Neumann Series vs. Time Reversal

original NS, error = 7.55% TR, error = 78.5%

Figure: $T = 8T_0$. Original vs. Neumann Series vs. Time Reversal (the latter compressed from $[0, 1]$ to $[-0.05, 1.6]$)

Measurements on a part of the boundary

Assume that $c = 1$ outside Ω. Let $\Gamma \subset \partial\Omega$ be a relatively open subset of $\partial\Omega$. Assume now that the observations are made on $[0, T] \times \Gamma$ only, i.e., we assume we are given

$$\Lambda f|_{[0,T]\times\Gamma}.$$

We consider f's with

$$\operatorname{supp} f \subset \mathcal{K},$$

where $\mathcal{K} \subset \Omega$ is a fixed compact.

Uniqueness

Heuristic arguments for uniqueness: To recover f from Λf on $[0, T] \times \Gamma$, we must at least be able to get a signal from any point, i.e., we want for any $x \in \mathcal{K}$, at least one "signal" from x to reach some Γ for $t < T$. Set

$$T_0(\mathcal{K}) = \max_{x \in \mathcal{K}} \operatorname{dist}(x, \Gamma).$$

The uniqueness condition then should be

$$T \geq T_0(\mathcal{K}). \qquad (*)$$

Theorem (Stefanov–U)

Let $c = 1$ outside Ω, and let $\partial\Omega$ be strictly convex. Then if $T \geq T_0(\mathcal{K})$, if $\Lambda f = 0$ on $[0, T] \times \Gamma$ and $\operatorname{supp} f \subset \mathcal{K}$, then $f = 0$.

Proof based on Tataru's uniqueness continuation results. Generalizes a similar result for constant speed by Finch, Patch and Rakesh.
As before, without (*), one can recover f on the reachable part of $\mathcal{K}$. Of course, one cannot recover anything outside it, by finite speed of propagation. Therefore,

(*) is an "if and only if" condition for uniqueness with partial data.

Stability

Heuristic arguments for stability: To be able to recover f from Λf on $[0, T] \times \Gamma$ *in a stable way*, we need to recover all singularities. In other words, we should require that

$$\forall (x, \xi) \in \mathcal{K} \times S^{n-1}, \text{ the ray (geodesic) through it reaches } \Gamma \text{ at time } |t| < T.$$

We show next that this is an "if and only if" condition (up to replacing an open set by a closed one) for stability. Actually, we show a bit more.

Proposition (Stefanov–U)

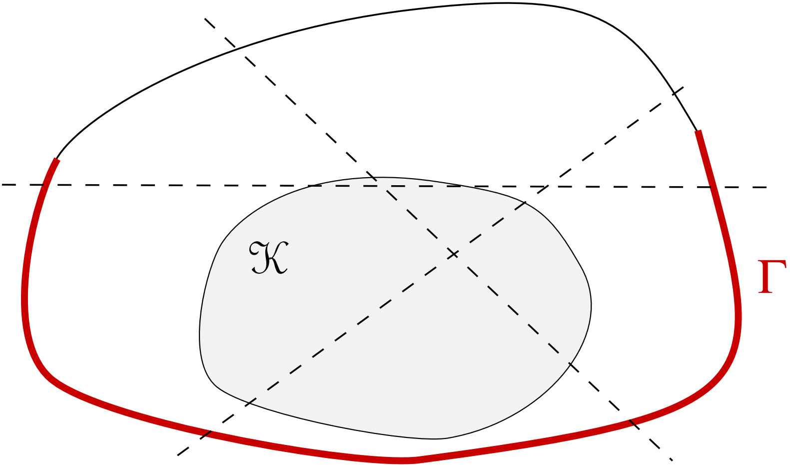
If the stability condition is not satisfied on $[0, T] \times \bar{\Gamma}$, *then there is no stability, in any Sobolev norms.*

Here, $\tau_{\pm}(x, \xi)$ is the time needed to reach $\partial\Omega$ starting from $(x, \pm\xi)$.

A reformulation of the stability condition

- Every geodesic through $\mathcal{K}$ intersects Γ.
- $\forall (x, \xi) \in \mathcal{K} \times S^{n-1}$, the travel time along the geodesic through it satisfies $|t| < T$.

Let us call the least such time $T_1/2$, then $T > T_1/2$ as before.
In contrast, any small open Γ suffices for uniqueness.



Let A be the "modified time reversal" operator as before. Actually, ϕ will be 0 because of χ below. Let $\chi \in C_0^\infty([0, T] \times \partial\Omega)$ be a cutoff (supported where we have data).

Theorem

$A\chi\Lambda$ is a zero order classical ΨDO in some neighborhood of $\mathcal{K}$ with principal symbol

$$\frac{1}{2}\chi(\gamma_{x,\xi}(\tau_+(x,\xi))) + \frac{1}{2}\chi(\gamma_{x,\xi}(\tau_-(x,\xi))).$$

If $[0, T] \times \Gamma$ satisfies the stability condition, and $|\chi| > 1/C > 0$ there, then
(a) $A\chi\Lambda$ is elliptic,
(b) $A\chi\Lambda$ is a Fredholm operator on $H_D(\mathcal{K})$,
(c) there exists a constant $C > 0$ so that

$$\|f\|_{H_D(\mathcal{K})} \le C\|\Lambda f\|_{H^1([0,T]\times\Gamma)}.$$

(b) follows by building a parametrix, and (c) follows from (b) and from the uniqueness result.
In particular, we get that for a fixed $T > T_1$, the classical Time Reversal is a parametrix (of infinite order, actually).

Reconstruction

One can constructively write the problem in the form

Reducing the problem to a Fredholm one

$$(I - K)f = BA\chi\Lambda f \quad \text{with the r.h.s. given,}$$

i.e., B is an explicit operator (a parametrix), where K is compact with 1 not an eigenvalue.

Constructing a parametrix without the ΨDO calculus.

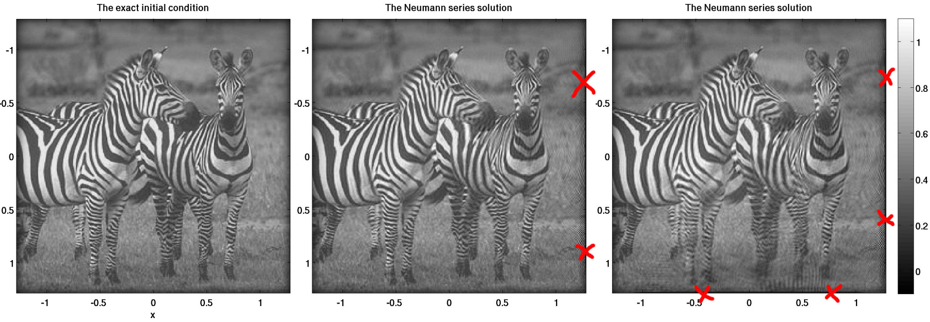
Assume that the stability condition is satisfied in the interior of $\operatorname{supp}\chi$. Then

$$A\chi\Lambda f = (I - K)f,$$

where $I - K$ is an elliptic ΨDO with $0 \le \sigma_p(K) < 1$. Apply the formal Neumann series of $I - K$ (in Borel sense) to the l.h.s. to get

$$f = (I + K + K^2 + \dots)A\chi\Lambda f \quad \text{mod } C^\infty.$$

Examples: Non-trapping speed, 1 and 2 sides missing



original NS, 3 sides, error = 7.99% NS, 2 sides, error = 12.2%

Figure: Partial data reconstruction, non-trapping speed, $T = 4T_0$.