A new class of low discrepancy sequences of partitions and points

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In 1976 Kakutani introduced the well-known splitting procedure of the unit interval. Fix a number $\alpha \in]0,1[$. If π is any partition of $[0,1[$, its *α-refinement*, denoted by $\alpha\pi$, is obtained subdividing the longest interval(s) into two parts proportional to α and $(1-\alpha)$. By $\alpha^n\pi$ we denote the α-refinement of $\alpha^{n-1}\pi$ and by $\{\alpha^n\pi\}$ the sequences of successive α-refinement of π. The sequence of successive α-refinement of the trivial partition of $[0,1[$, denoted by $\{\kappa_n\}$, is known as the *Kakutani α-sequence*. Kakutani proved that $\{\kappa_n\}$ is *uniformly distributed*.

In 2011 Volčič generalized Kakutani's splitting procedure as follows: if ρ is a non trivial finite partition of $[0,1[$, the *ρ-refinement* of a partition π of $[0,1[$, denoted by $\rho\pi$, is obtained by subdividing all the intervals of π having maximal length positively (or directly) homothetically to ρ. $\{\rho^n\pi\}$ denotes the sequence of successive ρ-refinements of π. Volčič proved that the sequence of ρ-refinements $\{\rho^n\}$ of the trivial partition of $[0,1[$ is *uniformly distributed*.

Soon afterwards, we introduced and studied a countable class of sequences of ρ- refinements of the trivial partition of $[0,1[$, denoted by $\{\rho^n_{LS}\}$ and called *LS-sequences of partitions*, where ρ is the partition made by L intervals of length β and S intervals of length β^2. We proved that $\{\rho^n_{LS}\}$ is *uniformly distributed* and we gave estimates from above and from below for their discrepancy. In the case $L \geq S$ the sequence of partitions $\{\rho^n_{LS}\}$ has *low discrepancy*. If $L = S = 1$ we obtain a Kakutani sequence.

In the same article we also presented an explicit algorithm á la van der Corput which associates to each *LS*-sequence of partitions a sequence of points we called *LS-sequence of points* and denoted by $\{\xi^n_{LS}\}$. We estimated the discrepancy of these sequences of points and we proved that whenever $L \geq S$ the sequence of points $\{\xi^n_{LS}\}$ has *low discrepancy*.

Furthermore, we introduced two new algorithms to construct *LS*-sequences of points, one of which uses the $(L+S)$-radix notation of integer numbers, and does not need the concept of *LS*-sequence of partitions.

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Due to the important role of low discrepancy sequences of points in Quasi-Monte Carlo Methods, it is interesting to see what happens in higher dimension. A first step in the multidimensional direction has been done very recently by introducing the two following extensions of *LS*-sequences of points to the unit square.

Definition 0.1. For each *LS*-sequence of points $\{\xi_{LS}^n\}$, the sequence $\{(\xi_{LS}^n, \frac{n}{N})\}$, where $n = 1, \ldots, N$, is called *LS-sequence of points à la van der Corput - Hammersley of order N* in the unit square.

Definition 0.2. For each pair of *LS*-sequences of points $\{\xi_{L_1 S_1}^n\}$ and $\{\xi_{L_2 S_2}^n\}$, the sequence $\{(\xi_{L_1 S_1}^n, \xi_{L_2 S_2}^n)\}$ is called *LS-sequence of points à la Halton* in the unit square.

An important result concerning the *LS*-sequence of points à la van der Corput-Hammersely is given by the following

Theorem 0.3. *The discrepancy of* $\{(\xi_{LS}^n, \frac{n}{N})\}$ *coincides with the discrepancy of* $\{\xi_{LS}^n\}$.

A consequence of the previous theorem is the following

Corollary 0.4. $\{(\xi_{LS}^n, \frac{n}{N})\}$ *has low discrepancy whenever* $L \geq S$.

In the case of the Halton-type sequences $\{(\xi_{L_1 S_1}^n, \xi_{L_2 S_2}^n)\}$ a lot of work has still to be done. We have at our disposal several graphical examples which indicate that the situation varies very much: some sequences seem to be uniformly distributed, while other show an unexpected bad behavior.