The number of complex realisations of a rigid graphs

Bill Jackson
School of Mathematical Sciences
Queen Mary, University of London, England
and
John C. Owen
Siemans, Cambridge, England.

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- Jackson, Jordán, Szabadka (2006) showed that r(G,p) is the same for all generic rigid (G,p) when the rigidity matroid of G is connected and gave a formula for r(G,p) in this case. This implies that $r(G,p) \leq 2^{n/2} \approx 1.14^n$ when G has a connected rigidity matroid.

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- Two realisations (G, p) and (G, q) of a graph G in \mathbb{C}^2 are **equivalent** if d(p(u) p(v)) = d(q(u) q(v)) for all $e = uv \in E$.
- Given a realisation (G, p) of a graph G in \mathbb{C}^2 , let c(G, p) denote the number of distinct equivalent realisations.



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Problem

Can we determine c(G) for a given rigid graph G?

Henneberg moves

Lemma

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Suppose G is obtained from H by a type two Henneberg move performed on a redundant edge of H. Then $c(G) \leq c(H)$.

Conjecture

If G is obtained from H by a type two Henneberg move performed on a non-redundant edge of H then c(G) > c(H).

Global Rigidity

Theorem

A graph G has c(G) = 1 if and only if either G is 3-connected and redundantly rigid or $G \in \{K_2, K_3\}$.

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Corollary

A graph G has r(G, p) = 1 for some generic real p if and only if c(G) = 1.

Separable graphs

Theorem

Suppose $G = G_1 \cup G_2$ for two edge-disjoint subgraphs G_1 , G_2 with $V(G_1) \cap V(G_2) = \{u, v\}$. Let $H_i = G_i + uv$ for i = 1, 2.

- If G_1 , G_2 are both rigid, then $c(G) = 2c(H_1)c(H_2)$.
- If G_1 is rigid and G_2 is not rigid, then $c(G) = 2c(G_1)c(H_2)$.

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Theorem

Suppose $G = G_1 \cup G_2 \cup \{e_1, e_2, e_3\}$ for two disjoint subgraphs G_1 , G_2 and three disjoint edges e_1 , e_2 , e_3 . Then $c(G) = 12c(G_1)c(G_2)$.

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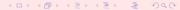
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It follows that we can reduce the problem of determining c(G) to the case when G is 3-connected and all 3-edge-cuts are 'trivial'.



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Problem (Thurston)

Does every rigid graph G have a generic realisation (G, p) in \mathbb{R}^2 such that r(G, p) = c(G)?

