

Nucleation-free 3D rigidity
and
Convex Cayley configuration space

Nucleation-free 3D rigidity

Meera Sitharam
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Contents

- 1 Implied non-edges and nucleation
- 2 The construction
- 3 Proofs

Implied non-edges

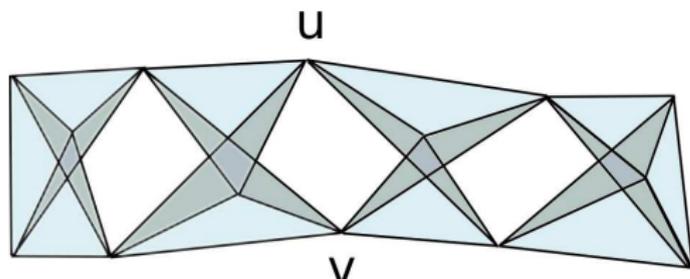
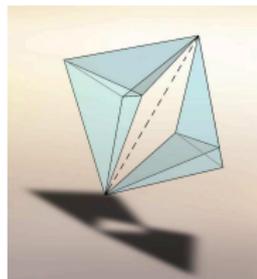
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Implied non-edges

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- Independence = independence in the 3D rigidity matroid.
- Rank = rank of the 3D rigidity matroid.

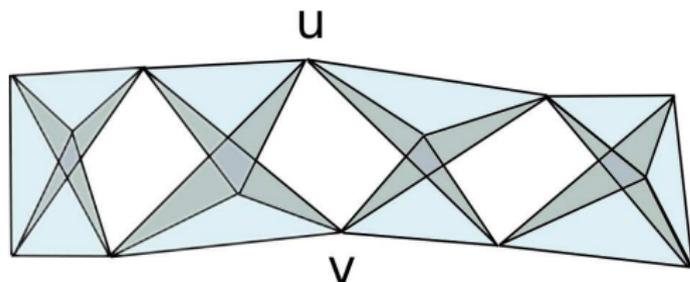
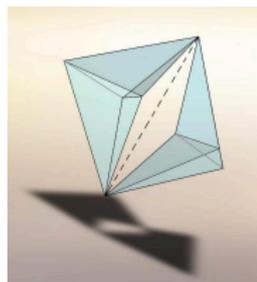


Nucleation property:

Nucleation property. A graph G has the nucleation property if it contains a non-trivial rigid induced subgraph, i.e., a rigid nucleus. Trivial means a complete graph on 4 or fewer vertices.

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Two natural questions in 3D

- Question 1 **Nucleation-free Graphs with implied non-edges:** Do all graphs with implied non-edges have the nucleation property?

Two natural questions in 3D

- Question 1 **Nucleation-free Graphs with implied non-edges:** Do all graphs with implied non-edges have the nucleation property?
- Question 2 : **Nucleation-free, rigidity circuits** Does every rigidity circuit automatically have the nucleation property?

Answering the two questions in the negative

- In order to answer Question 1, we construct an infinite family of flexible 3D graphs which have no proper rigid nuclei besides trivial ones (triangles), yet have implied edges.

Answering the two questions in the negative

- In order to answer Question 1, we construct an infinite family of flexible 3D graphs which have no proper rigid nuclei besides trivial ones (triangles), yet have implied edges.
- We also settle Question 2 in the negative by giving a family of arbitrarily large examples that follow directly from the examples constructed for Question 1.

The construction: a ring of k roofs

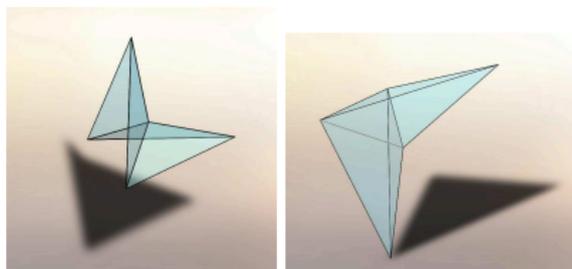
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The construction: a ring of k roofs

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- A roof together with (either) one of its two non-edges forms a banana.

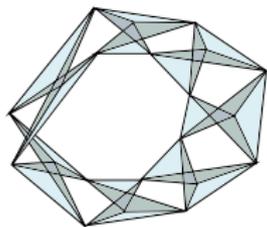
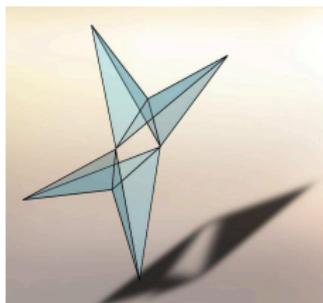
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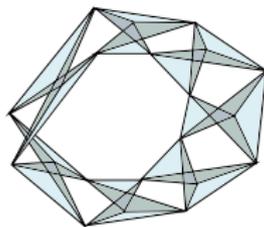
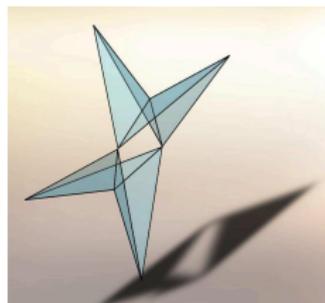
Ring graph

A ring graph \mathcal{R}_k of $k \geq 7$ roofs is constructed as follows. Two roofs are connected along a non-edge. We refer to these two non-edges within each roof as hinges. Such a chain of seven or more roofs is closed back into a ring.



Ring graph

A ring graph \mathcal{R}_k of $k \geq 7$ roofs is constructed as follows. Two roofs are connected along a non-edge. We refer to these two non-edges within each roof as hinges. Such a chain of seven or more roofs is closed back into a ring.



This example graph appears often in the literature.

Main theorem

Theorem

In a ring of roofs, the hinge non-edges are implied.

A proof of the main theorem

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This follows immediately from either one of two existing theorems.

Option 1

Theorem (Tay and White and Whiteley)

If $\forall i \leq k$, the i^{th} banana $B_i(p_i)$ is rigid, then the bar framework $\mathcal{B}_k(p)$ is equivalent to a body-hinge framework and is guaranteed to have at least $k - 6$ independent infinitesimal motions.

Observation

If $\mathcal{B}_k(p)$ is generic, then for all i , the rigidity matrix given by the banana framework $B_i(p_i)$ is independent, which in this case implies rigidity. Here p_i is the restriction of p to the vertices in the i^{th} roof R_i .

Option 2

A cover of a graph $G = (V, E)$ is a collection \mathcal{X} of pairwise incomparable subsets of V , each of size at least two, such that $\cup_{X \in \mathcal{X}} E(X) = E$. A cover $\mathcal{X} = \{X_1, X_2, \dots, X_n\}$ of G is 2-thin if $|X_i \cap X_j| \leq 2$ for all $1 \leq i < j \leq n$.

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Let $H(\mathcal{X})$ be the set of shared vertices. For each $(u, v) \in H(\mathcal{X})$, let $d(u, v)$ be the number of sets X_i in \mathcal{X} such that $\{u, v\} \subseteq X_i$.

Observation

If $\mathcal{X} = \{X_1, X_2, \dots, X_m\}$ is a 2-thin cover of graph $G = (V, E)$ and subgraph $(V, H(\mathcal{X}))$ is independent, then in 3D, the rank of the rigidity matrix of a generic framework $G(p)$, denoted as $\text{rank}(G)$, satisfies the following

$$\text{rank}(G) \leq \sum_{X_i \in \mathcal{X}} \text{rank}(G_1[X_i]) - \sum_{(u,v) \in H(\mathcal{X})} (d(u,v) - 1), \quad (1)$$

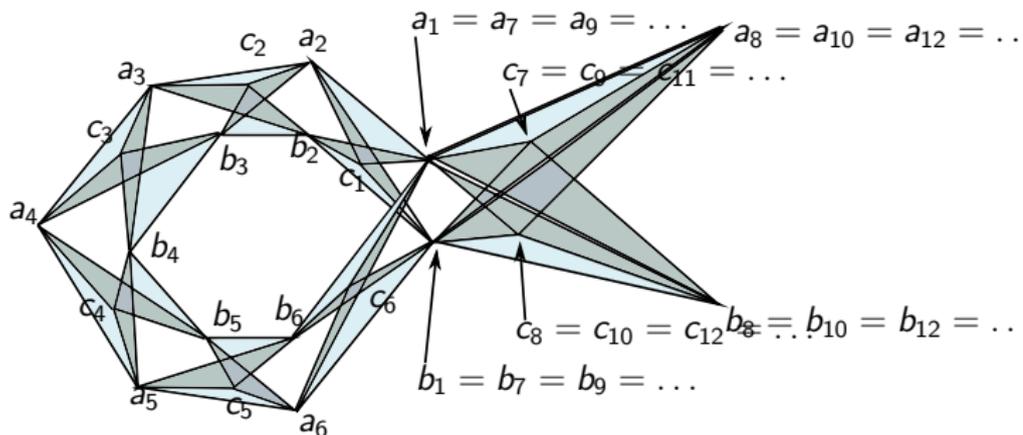
where $G_1 = G \cup H(\mathcal{X})$.

Proof of independence of ring

- We will show a specific framework $\mathcal{R}_k(p)$ is independent, thus the generic frameworks must also be independent.

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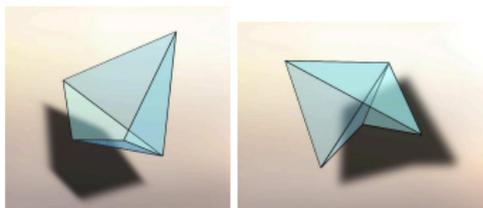


The repeated roofs have some symmetries that are utilized in the proof.

Proof of independence of ring

- Use induction: two base cases, according to the parity of number of roofs.
- Induction step is proved by contradiction and inspection of the rigidity matrix of $\mathcal{R}_{k+2}(p)$ and of $\mathcal{R}_k(p)$: after adding 2 new roofs to the current ring, if the new ring does not have full row rank, then the original one does not have full row rank either.
- The k^{th} roof is identical to the $k + 2^{\text{nd}}$ roof. This is true for both even and odd k 's and hence the induction step is the same.

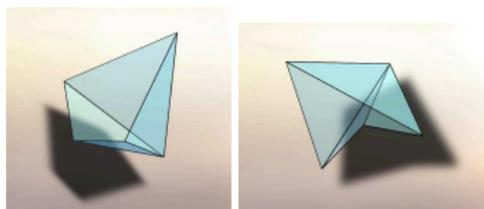
A special generic framework



Lemma

The hinge non-edges are implied, for all rings $\mathcal{R}_k(p)$ of $k - 1$, pointed pseudo-triangular roofs and one convex roof.

A special generic framework

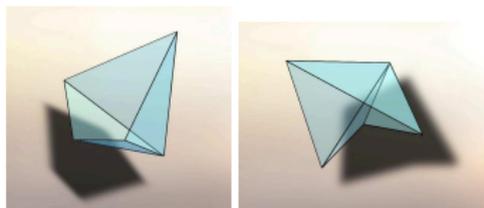


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This uses previous results by Connelly, Streinu and Whiteley about expansion/contraction properties of convex polygons, the infinitesimal properties of single-vertex origamis and pointed pseudo-triangulations.

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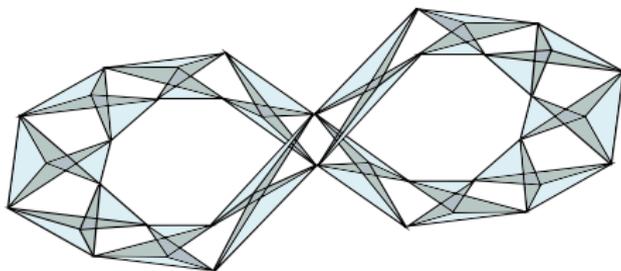
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Lemma

There are generic frameworks $\mathcal{R}_k(p)$ as in the previous Lemma.

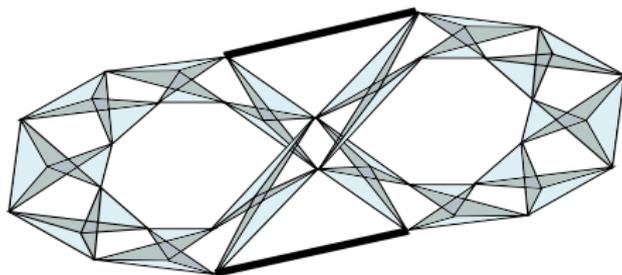
Nucleation-free dependent graph

Question 1 Do all graphs with implied non-edges have the nucleation property?



Nucleation-free non-rigid dense graph

Question 2 If a graph $G = (V, E)$ with at least $3|V| - 6$ edges is non-rigid, i.e, dependent, then does it automatically have the nucleation property?



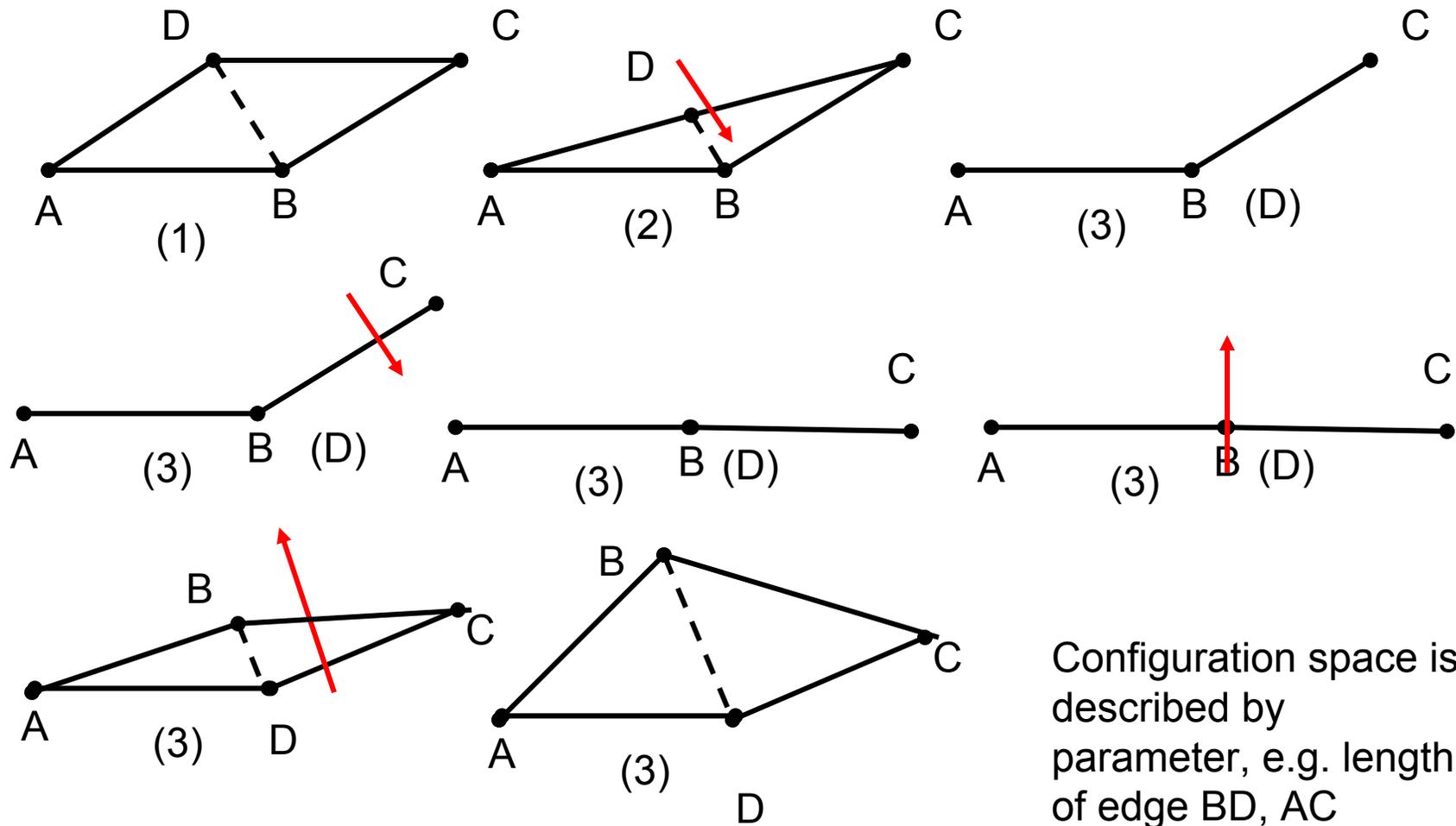
Graphs with Convex Cayley Configuration Spaces

Meera Sitharam

Heping Gao Jialong Cheng

-
- Useful representation of configuration spaces of flexible linkages (machines, molecules) – important problems, many applications, little progress
 - Obstacles to progress so far
 - (a). Good formalization of “useful representation of configuration space”
 - (b). Which linkages have such a representation
 - Novel feature of our results - relate combinatorial properties of underlying graph (forbidden minors and other graph properties) with:
 - geometric properties (convexity) of configuration space and topological properties (connectedness, number of connected components) of configuration space
 - algebraic complexity of configuration space
 - Applications to molecular biology and chemistry
-

Representation of Configuration Space



Outline

- Definition and notation
- Cayley configuration space
- 2D connected/convex configuration space
- 3D connected configuration space
- Arbitrary dimensions
- Application: Helix packing configurations

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- **Definition and notation**
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Notation

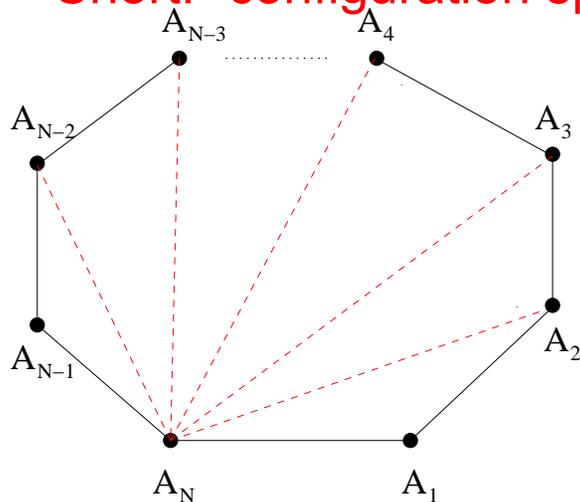
- *Graph*: $G=(V,E)$
- *non-edge* : f in \bar{E} (the complement of E)
- *non-edge set* : F , subset of \bar{E}
- *linkage*: (G, d_E)
- *realization of (G, d_E) in δ -dimensions*: a realization or coordinate values of all vertices in δ -dimension preserving distance constraints

Definition: Cayley configuration space

- Definition: given linkage (G, d_E) non-edge set F , the *Cayley configuration space on F* is

$$\Phi_F^\delta(G, d_E) := \{d_F \mid (G \cup F, d_E \cup d_F) \text{ has a solution in } \delta\text{-dimension}\}$$

Short: “configuration space of (G, d_E) on F ”



non-edges: dashed line

The projection on the non-edges is described by triangle inequalities

Schoenberg's Theorem (1935):

Given an $n \times n$ matrix $\Delta = (d_{ij})_{n \times n}$, there exists a Euclidean realization in \mathbb{R}^δ , i.e., a set of points $p_1, p_2, \dots, p_n \in \mathbb{R}^\delta$ s.t. $\forall i, j$, $\|p_i - p_j\|^2 = d_{ij}$ if and only if matrix Δ is negative semidefinite of rank δ .

- ▶ Negative semidefinite matrices form a convex cone.
- ▶ The rank- δ stratum of this cone may not be convex.
- ▶ A linkage (G, d_E) is a partially filled distance matrix: this is a section consisting of all possible negative semidefinite completions (of rank δ).
- ▶ (δ -dimensional)Cayley configuration space, $\Phi_F(\Delta(G, d_E))$, of the linkage (G, d_E) on non-edge set F is the projection of this section (completions) onto F .

Question: For which graphs $G \cup F$ is this projection “nice” for all d_E ?

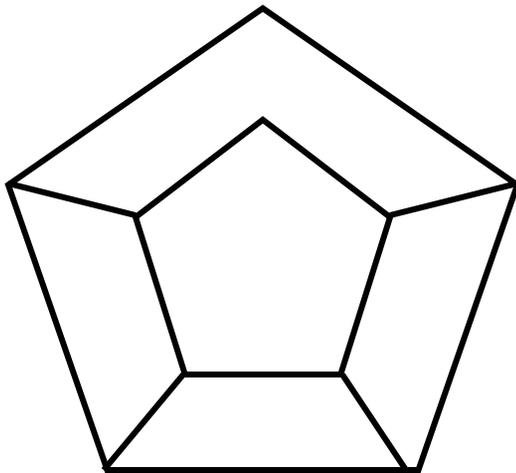
Easier to deal with δ -Projection on d^2

- Definition: given distance constraint system (G, d_E) and non-edge set F , the *squared-distance configuration space* of (G, d_E) on F is

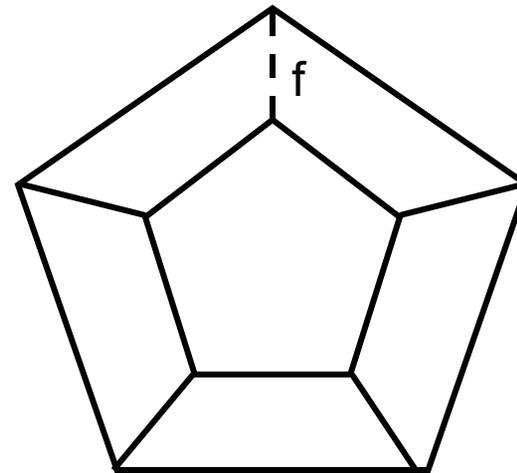
$$(\Phi_F^\delta)^2 = (G, d_E) := \{ d_F^2 \mid (G \cup F, d_E \cup d_F) \text{ has a solution in } \delta\text{-dimension} \}$$

Configuration Space Description

- Definition: constraint system (G, d_E) has *connected configuration space description (CCS)* in δ dimension if there **exists** a non-edge set F such that the Cayley space on F is **connected**. We say (G, d_E) has a *CCS on F* .



No CCS in 2D

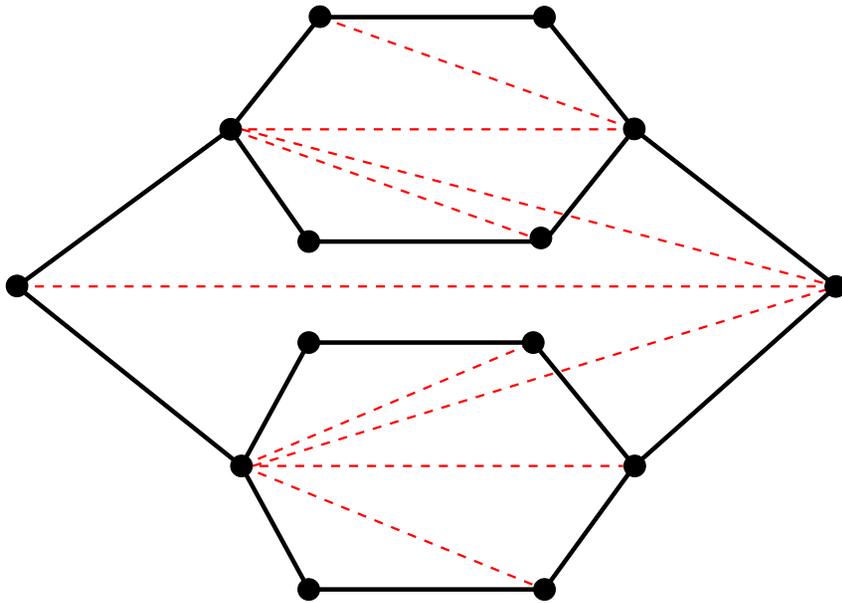


Has CCS on f in 3D

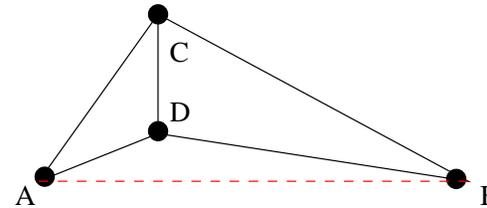
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2D Connected Configuration Space: Examples



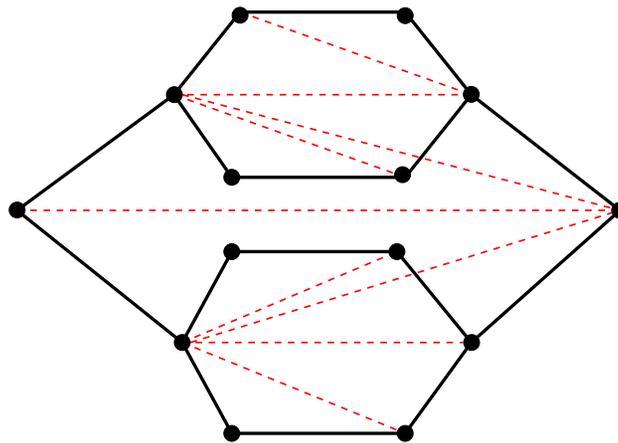
Projection on the non-edges is convex,
connected, and polytope



Projection on the non-edges
is not connected

Simple & Complete Configuration Space in 2D

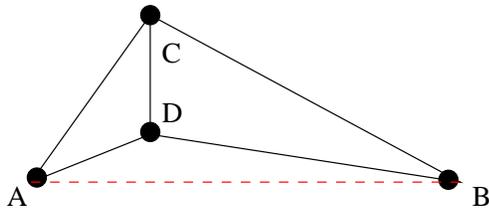
- **Theorem** : There exists connected & convex configuration description in 2D if and only if all the non-rigid 2-sum components are partial 2-trees.



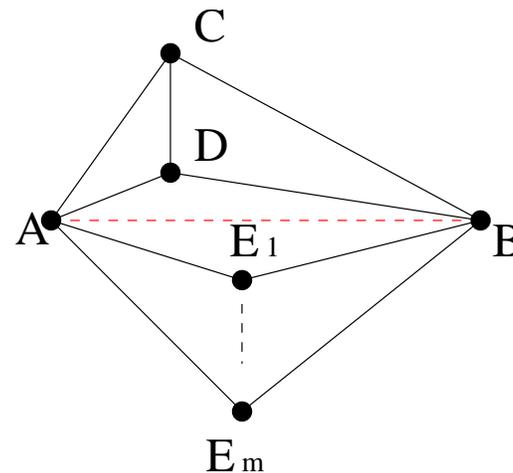
2D Connected configuration space:

Theorem

- **Lemma** : Given a graph $G=(V,E)$ and non-edge f , G can be reduced to base case 1 and base case 2 **only** by edge shrinking if and only if there **exists** one 2-Sum component of $G \cup f$ which **contains** f and is **not** a partial 2-Tree.



Base Case 1



Base Case 2

Proof

- Proof needs graph reduction technique different from minor: keep the non-edge.

2D Connected configuration space:

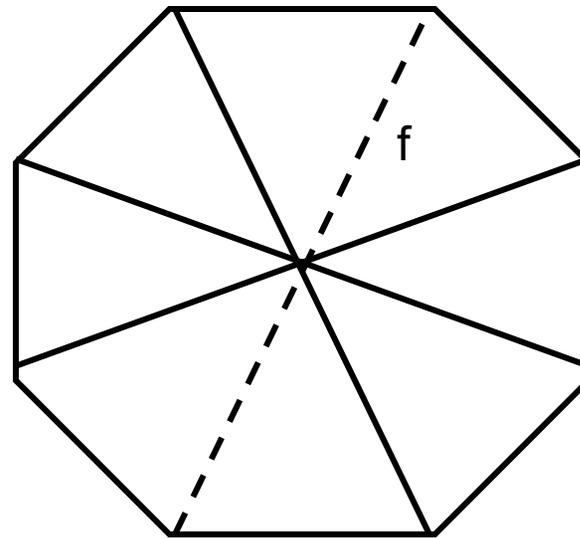
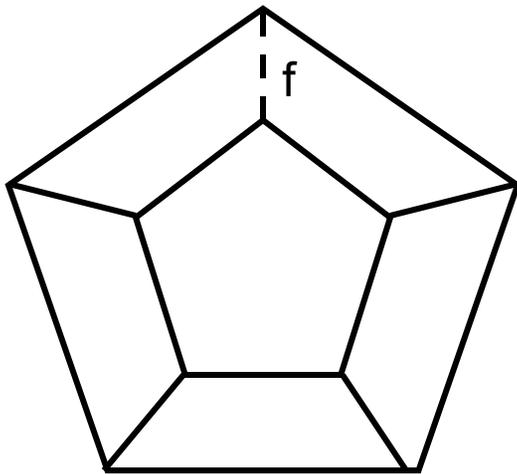
Theorem

- **Theorem** : Given graph G and non-edge $f=AB$, if G has 2D CCS on f (single interval) if and only **all** 2-Sum components of $G \cup AB$ containing both A and B are partial 2-trees.
- **Theorem** : Given a graph $G=(V,E)$ and non-edge set F , G has 2D CCS on F if and only if all 2-Sum components containing any subset of F are partial 2-trees.

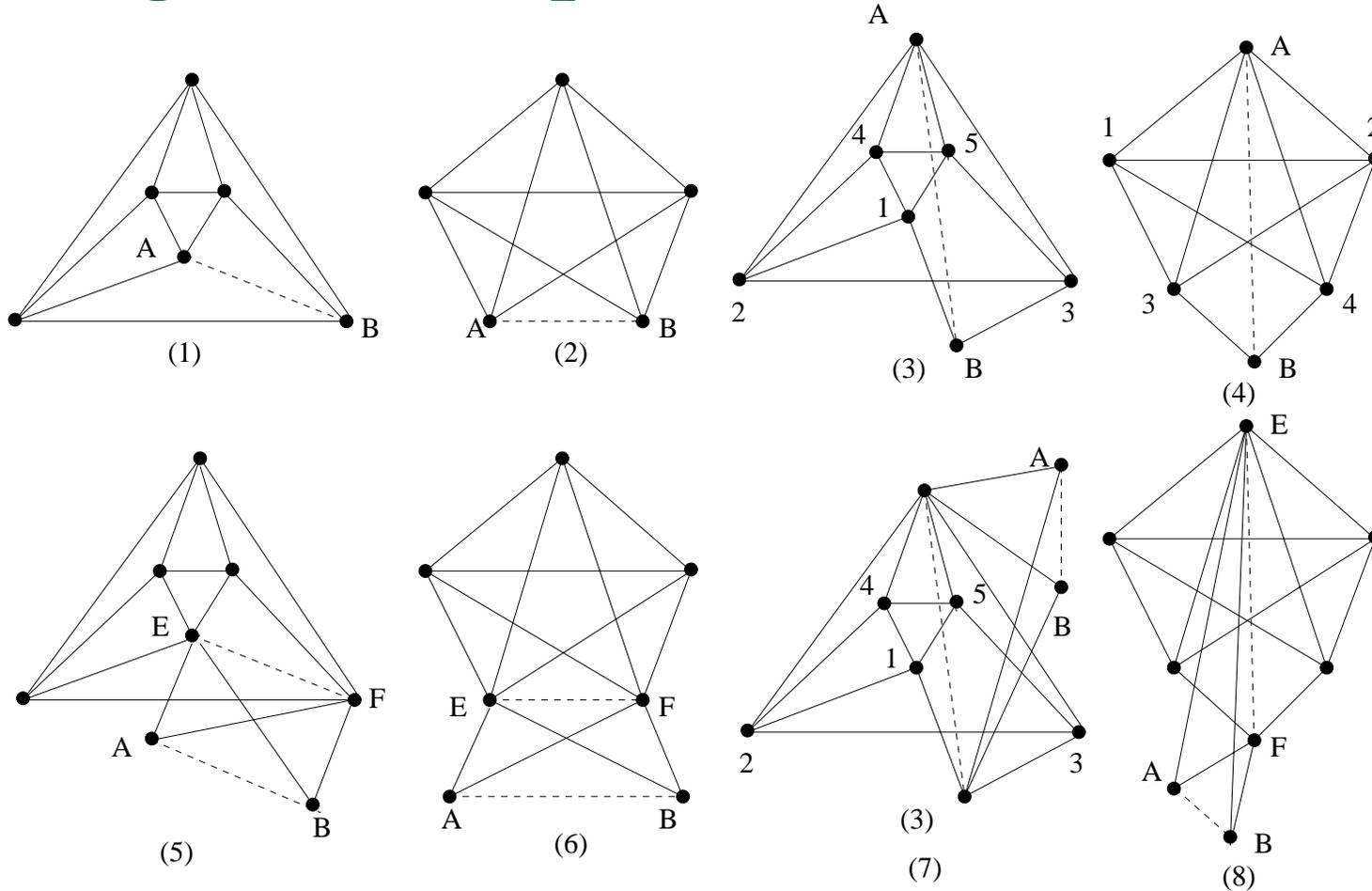
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3D Connected Configuration Space

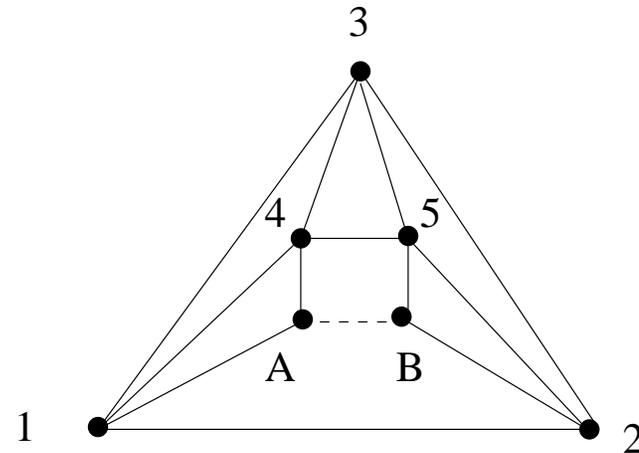
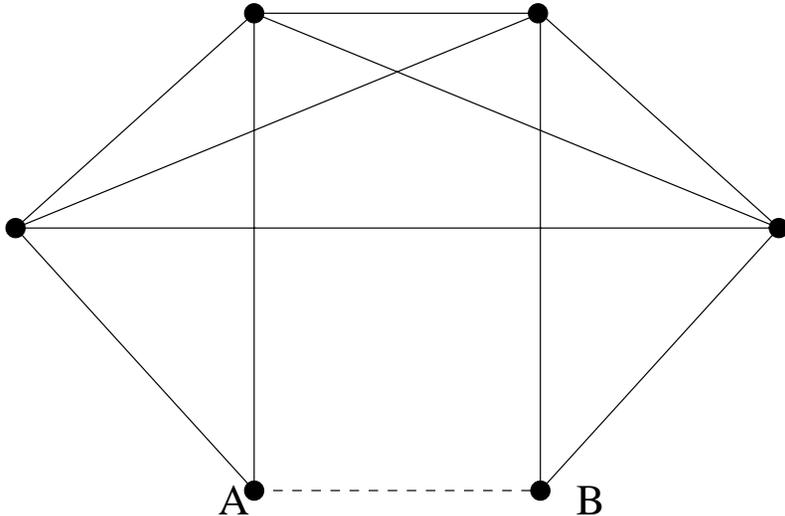


Examples without 3D Connected Configuration Space



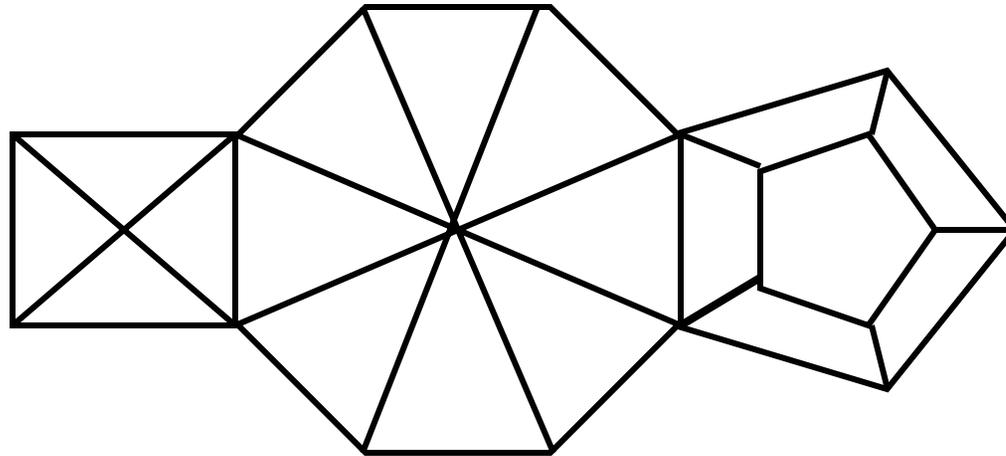
Examples with 3D Connected Configuration Space

- Case 1: $G \cup f$ has universally inherent CCS in 3D
- Case 2: $G \cup f$ doesn't have universally inherent CCS in 3D



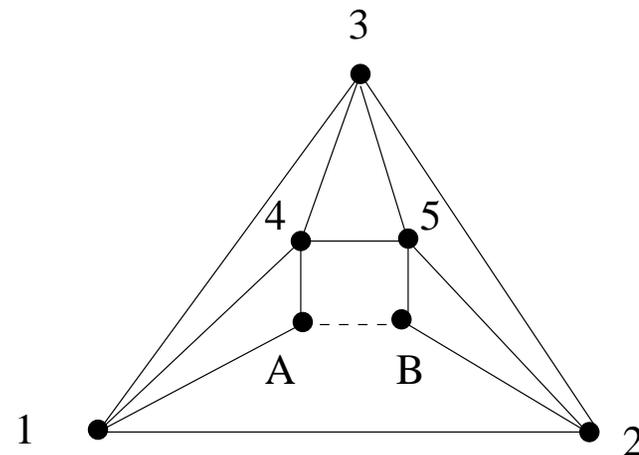
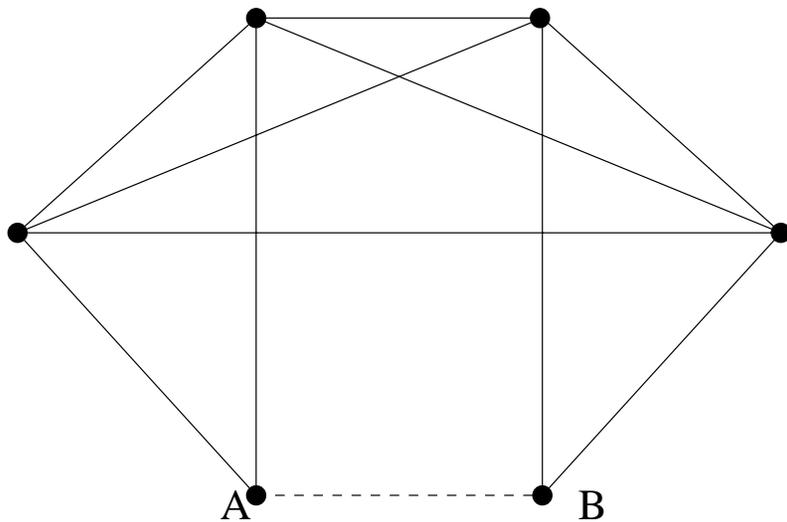
Theorem on Maximal 3-realizable Graphs

- **Theorem :** if a graph G is maximal 3-realizable, for any non-edge f , G doesn't have 3D connected configuration space on f .

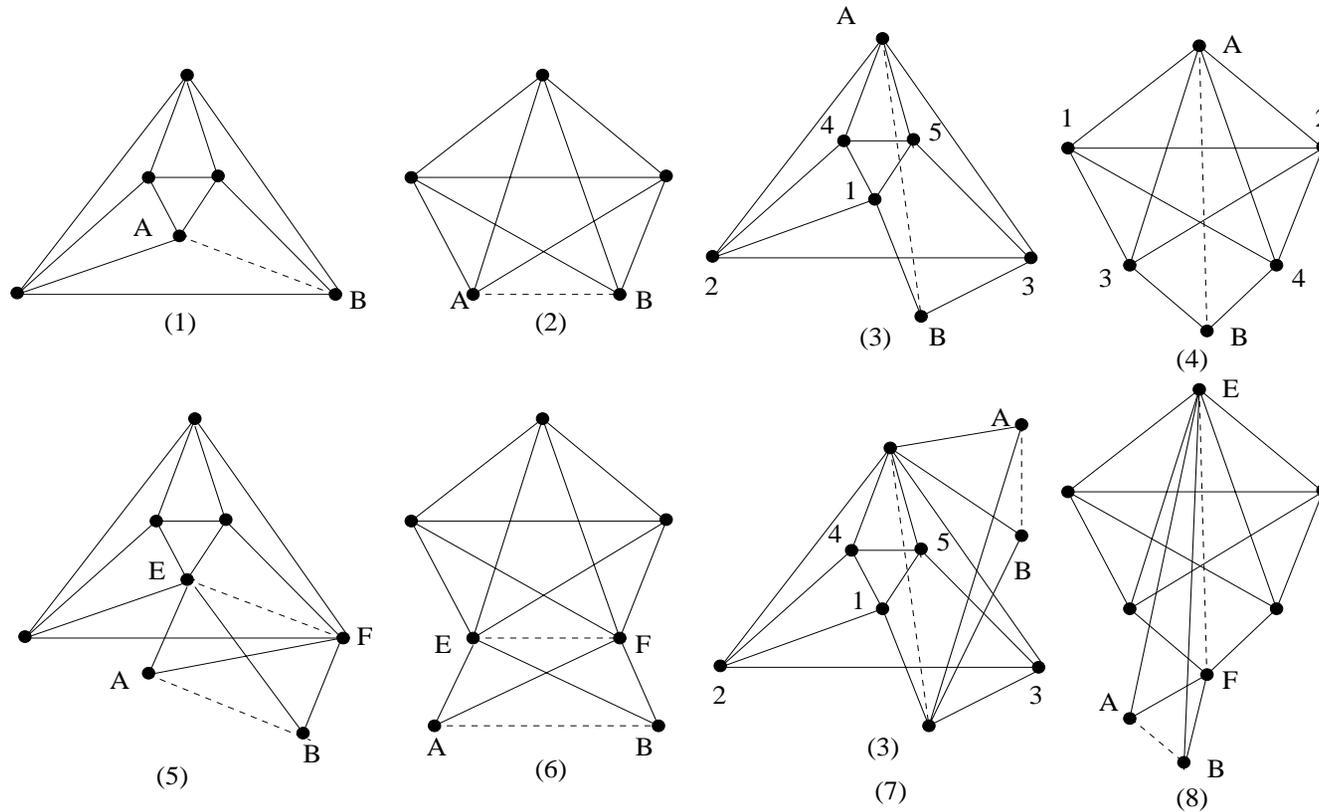


3D Connected Configuration Space : Conjectures

- Conjecture 1: Given partial 3-tree G and virtual edge AB , if A and B must be shrunk together in order to get a K_5 or K_{222} minor, then G has 3D connected configuration space on f .



Conjecture 2: given graph G and non-edge AB , G doesn't have 3D connected configuration space on f if and only if G can be reduced to one of the eight cases by edge shrinking while preserving AB as non-edge



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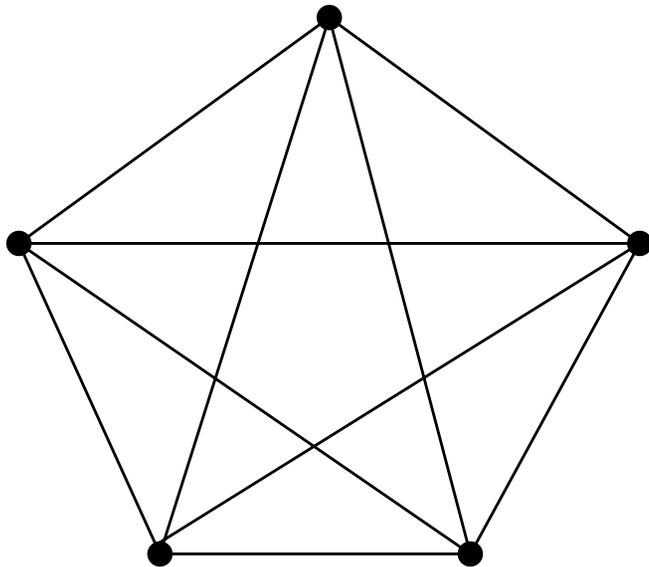
Universally Inherent CCS

We obtain strong results in arbitrary dimension for more restrictive class of graphs

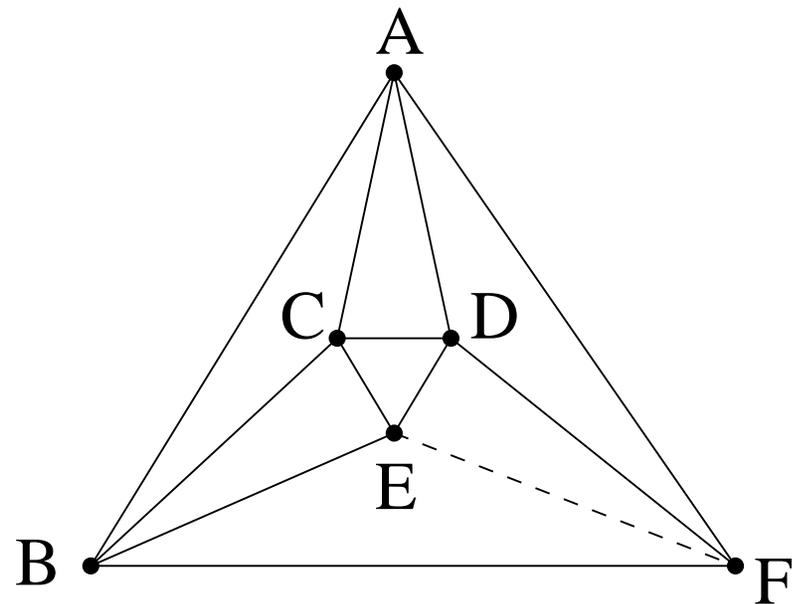
- Definition: H has an *universally inherent CCS* in δ -dimension if for **every** partition of H as $G \cup F$ where G has a CCS on F .

Universally Inherent CCS : Examples

- K_5 and K_{222} doesn't have universally inherent CCS in 3D.
- Any proper subgraph of K_5 or K_{222} has universally inherent CCS in 3D.



K_5



K_{222}

Universally Inherent CCS: Results

- Definition: a graph G is δ -realizable if, for any d_E , (G, d_E) has a solution in **some** dimension implies that (G, d_E) has a solution in **δ -Dimension** [Belk & Connelly].
- **Theorem 1:** A graph G is δ -realizable iff G has universally inherent CCS in δ -dimension. In fact, G has a universally inherent **convex squared-distance** configuration space in δ -dimension.

Previous results on δ -realizability

- Previous Theorem: a graph G is 2-realizable if and only if G has no K_4 minor; a graph G is 3-realizable if and only if it has no K_5 or K_{222} minor [Belk, Connelly].

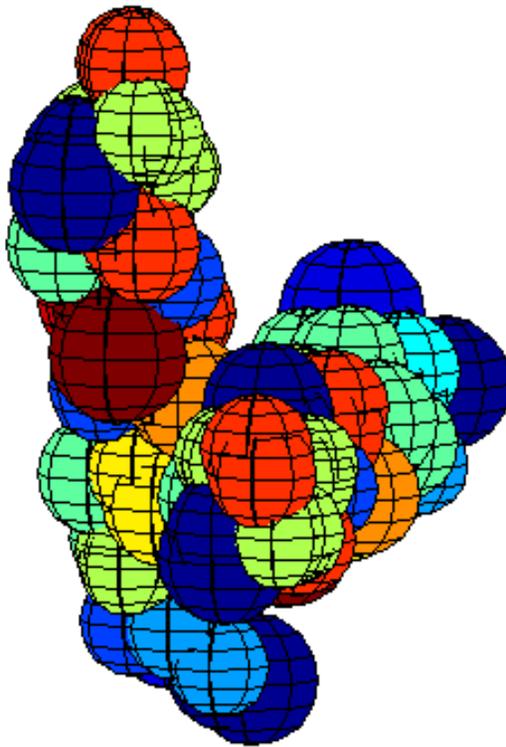
Graph Characterization for Universally Inherent CCS

Theorem : a graph G has universally inherent CCS in 2D if and only if it has no K_4 minor; in 3D if and only if it has no K_5 or K_{222} minor.

Outline

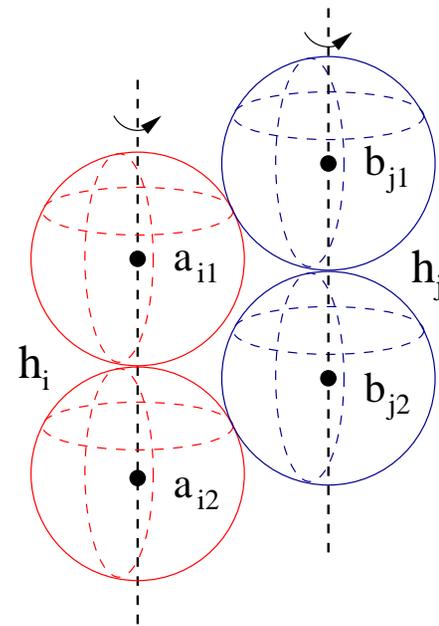
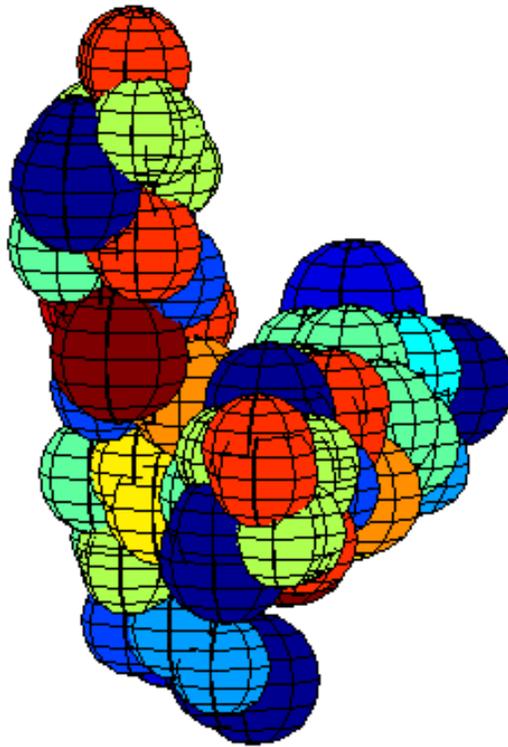
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Helix Packing: Problem

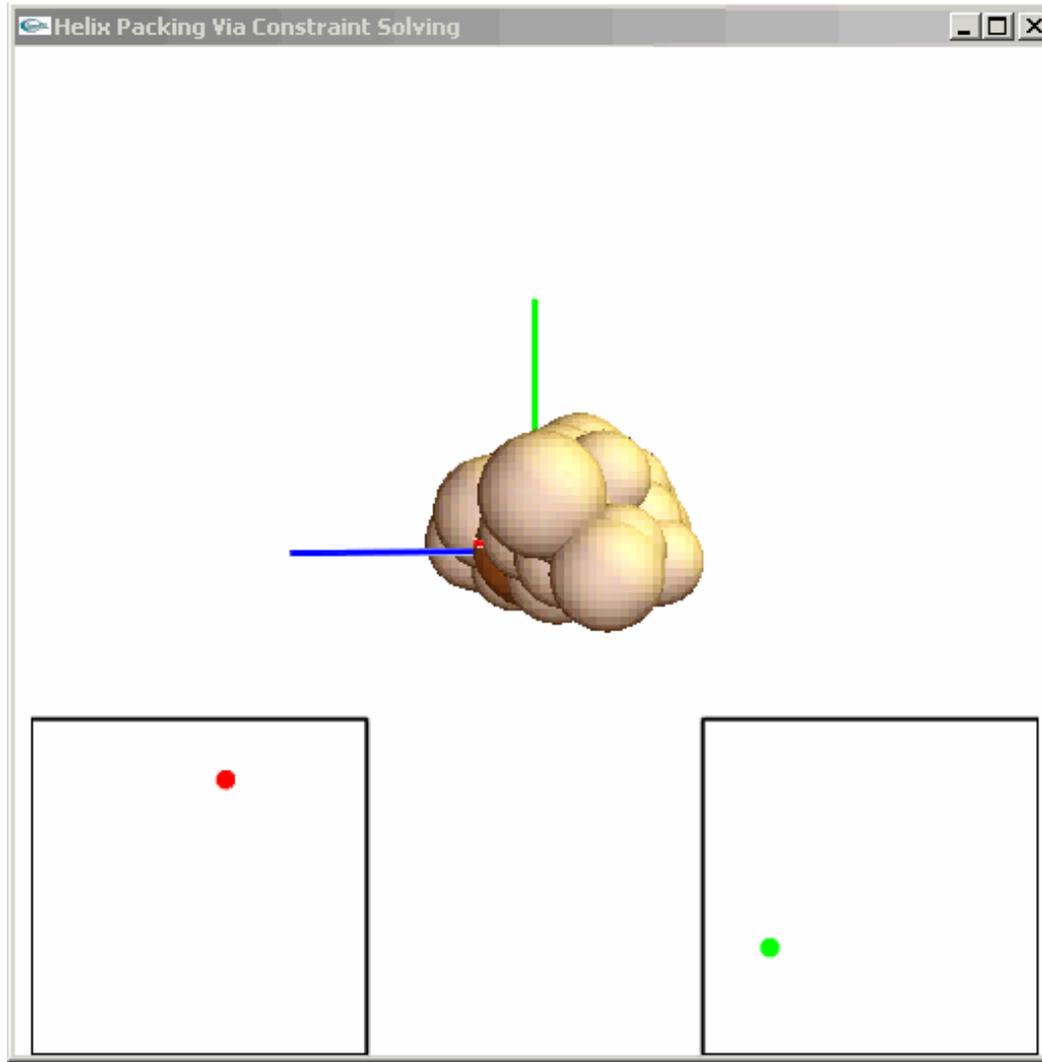


- Simulate and sample the configuration space of helices
- Focused on two helices in the current stage
- Helix is modeled as a collection of rigid balls; collision should be avoided between two balls from two different helices
- “Critical” configurations should be captured

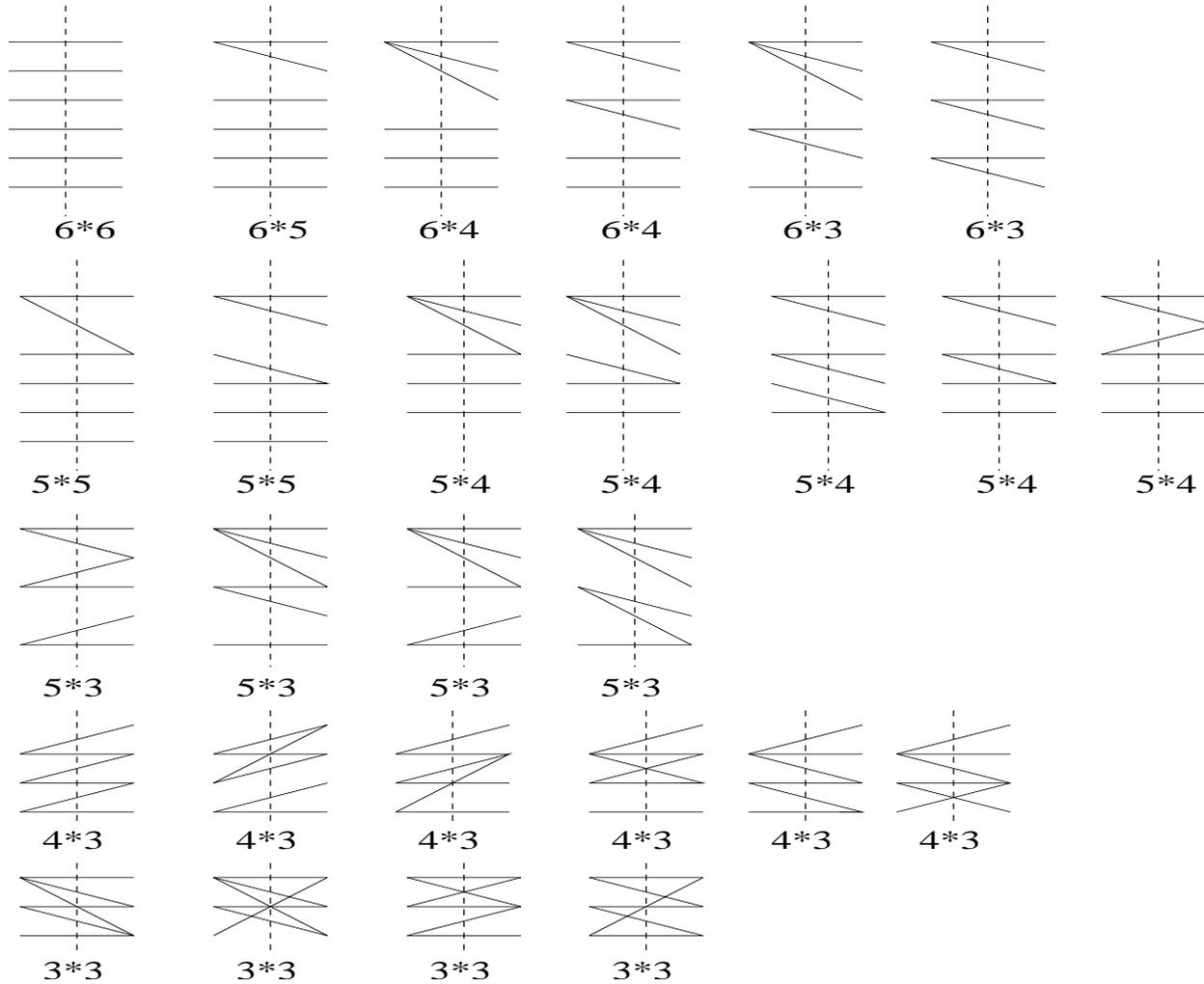
Helix Packing: Bi-Incidence



Bi-Incidence



Helix Packing: Graphs for Which Configuration Space is Sought for all possible edge subgraphs



Thanks!