Two-Orbit Polyhedra in Ordinary Space

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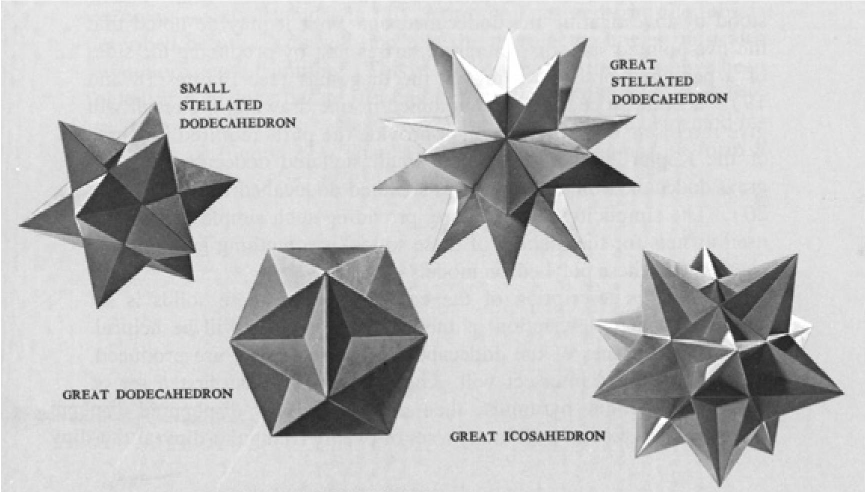
History

- Symmetry of figures studied since the early days of geometry.

- The regular solids occur from very early times and are attributed to Plato (427-347bce). Euclid (300bc).

dodecahedron, icosahedron $\{3, 5\}$, tetrahedron, octahedron, cube

• Regular star-polyhedra — Kepler-Poinsot polyhedra (Kepler 1619, Poinsot 1809). Cauchy (1813).

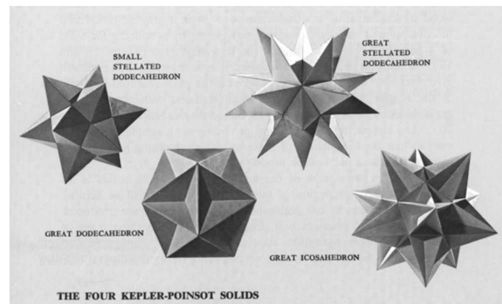


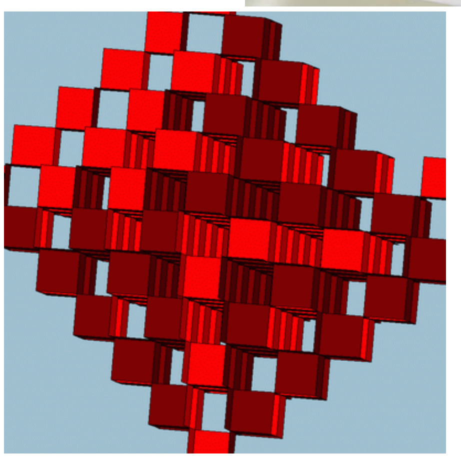
THE FOUR KEPLER-POINSOT SOLIDS

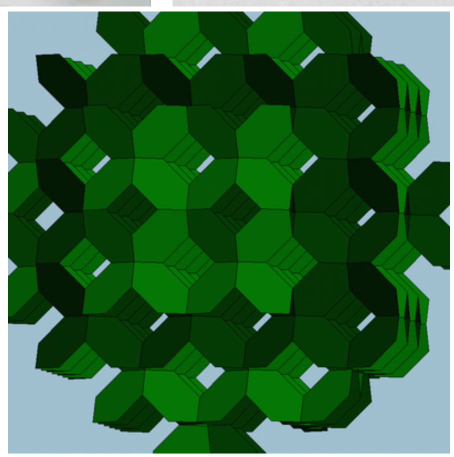
• Higher-dimensional geometry and group theory in the 19th century. Schläfli's work.

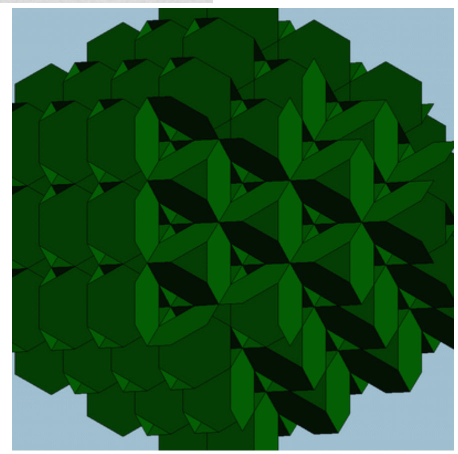
• Influential work of Coxeter. Unified approach based on a powerful interplay of geometry and algebra.

Polyhedra With the passage of time, many changes in point of view about polyhedra or complexes, and their symmetry: Platonic (solids, convexity), Kepler-Poinsot (star polygons), Petrie-Coxeter (convex faces, infinite), ...









Skeletal approach to polyhedra and symmetry!

• Impetus by Grünbaum (1970's) in two ways — geometrically and combinatorially.

Basic question: what are the regular polyhedra in ordinary space? Answer: Grünbaum-Dress Polyhedra.

• Rid the theory of the psychologically motivated block that membranes must be spanning the faces! Allow skew faces! Restore the symmetry in the definition of "polyhedron"! Graph-theoretical approach!

• Later: the group theory forces skew faces and vertex-figures! General reflection groups.

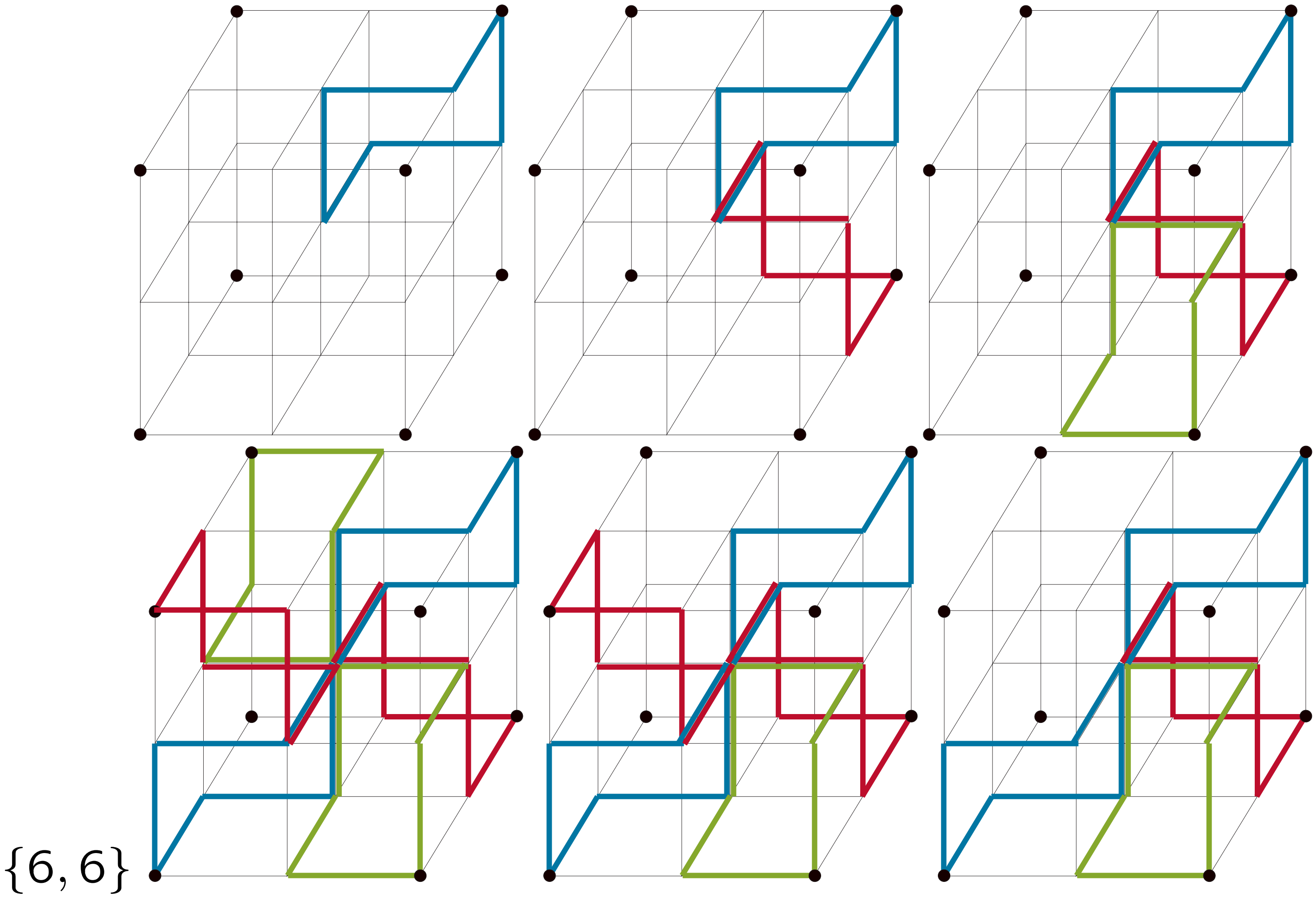
Polyhedron

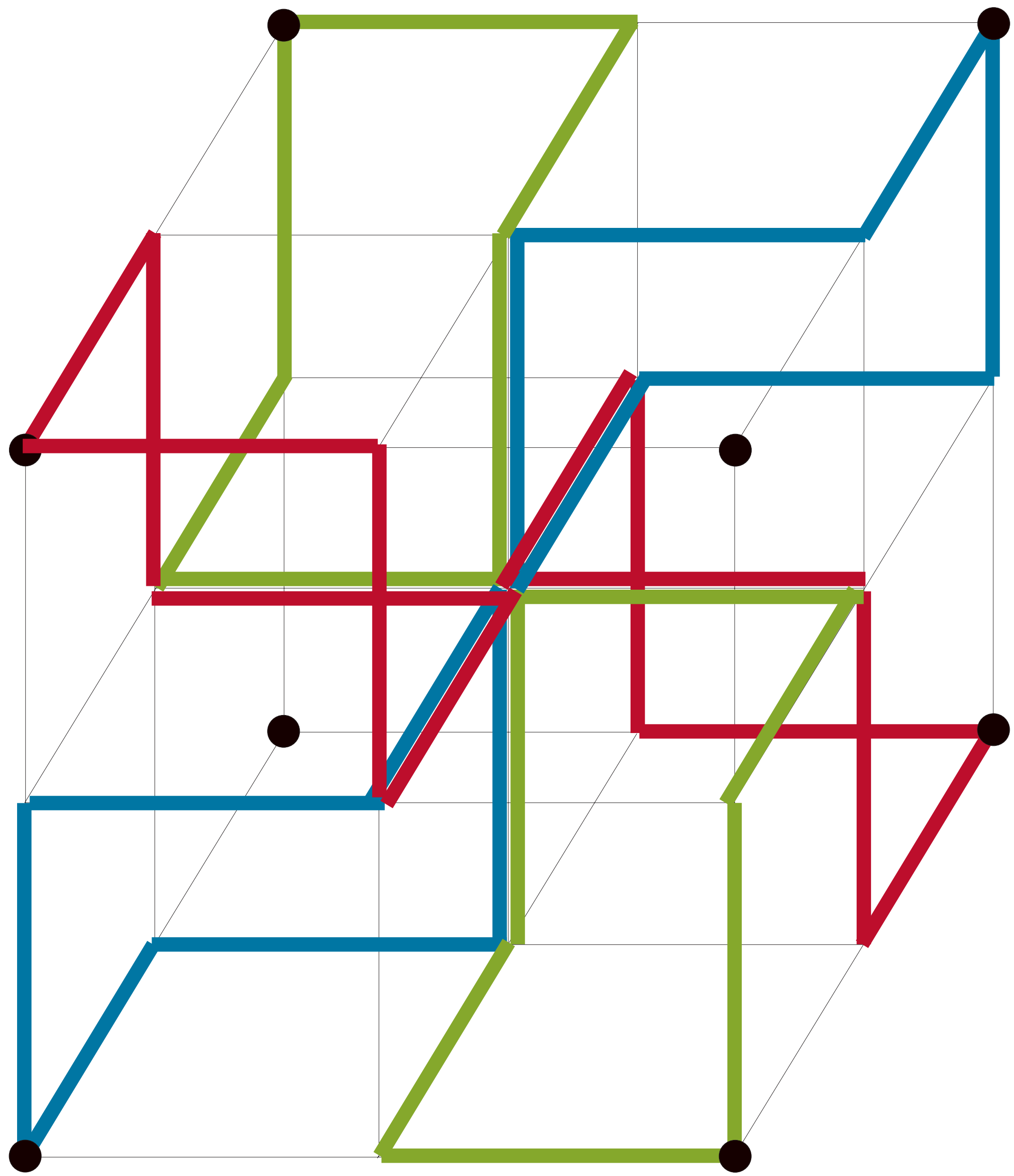
A polyhedron P in $\mathbb{E}^3$ is a family of simple polygons, called *faces*, such that

- each edge of a face is an edge of just one other face,
- all faces incident with a vertex form one circuit,
- P is connected,
- each compact set meets only finitely many faces (discreteness).

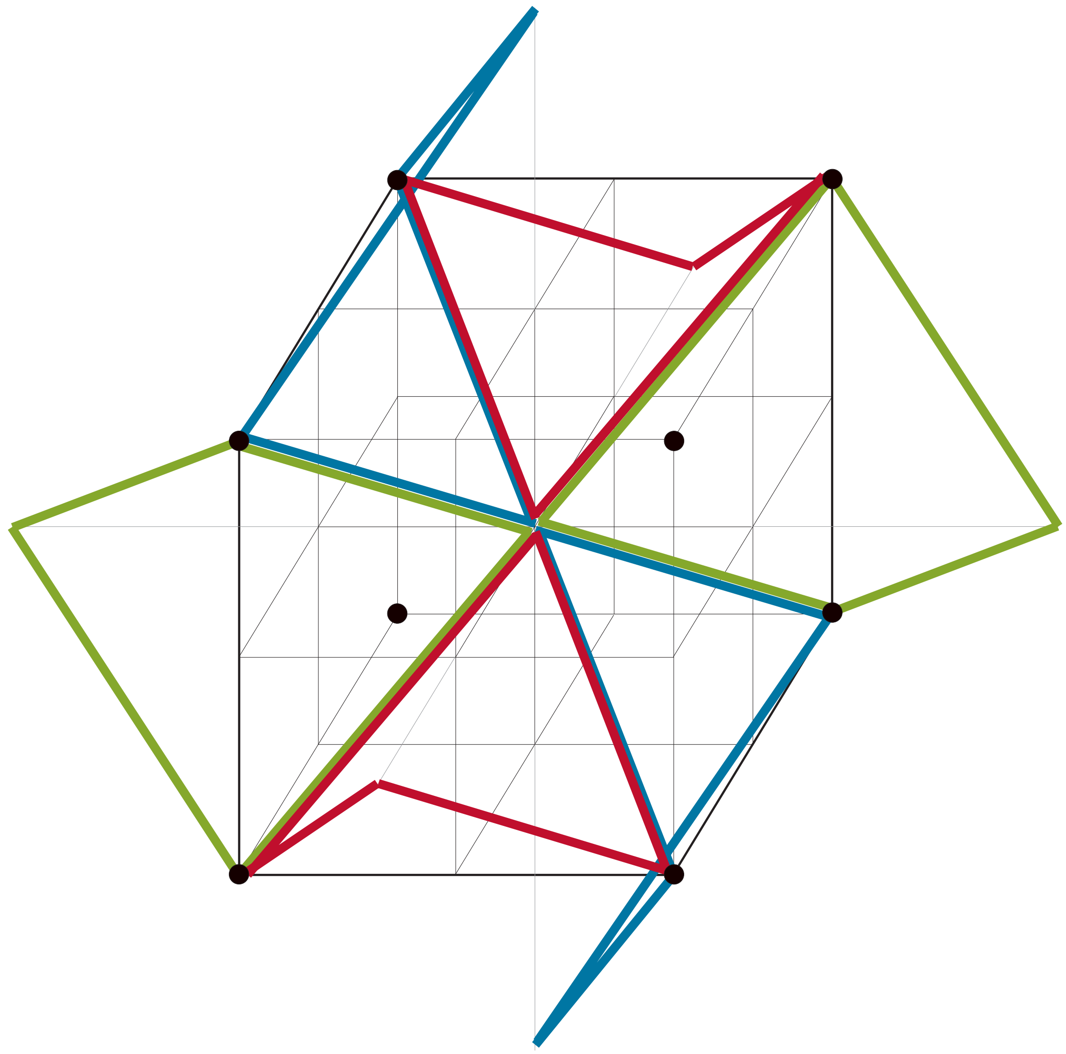
P is regular if its symmetry group is transitive on the flags.

(flag: incident triple of a vertex, an edge, and a face)


$\{6,6\}$

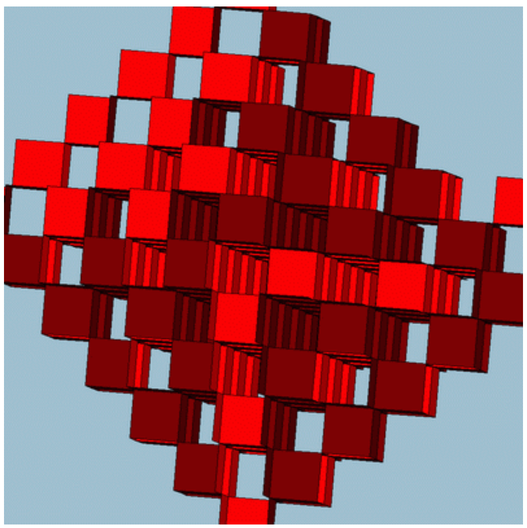


{6, 6}

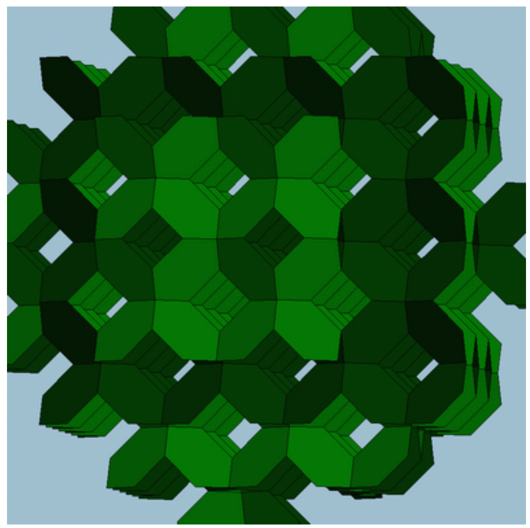


$\{4, 6\}$

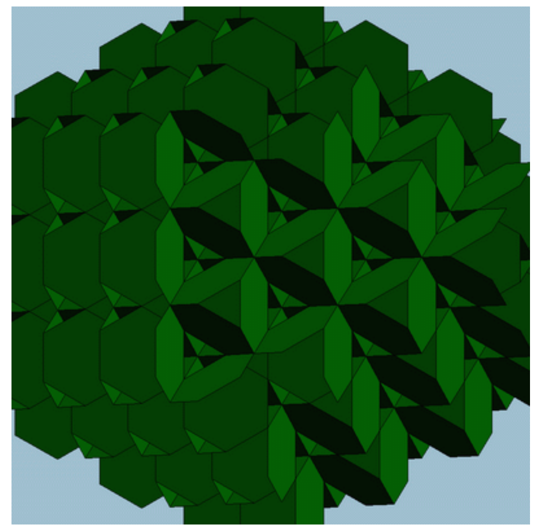
Petrie-Coxeter Polyhedra (1930's): convex faces, skew vertex-figures. Just three such polyhedra!



$\{4, 6|4\}$

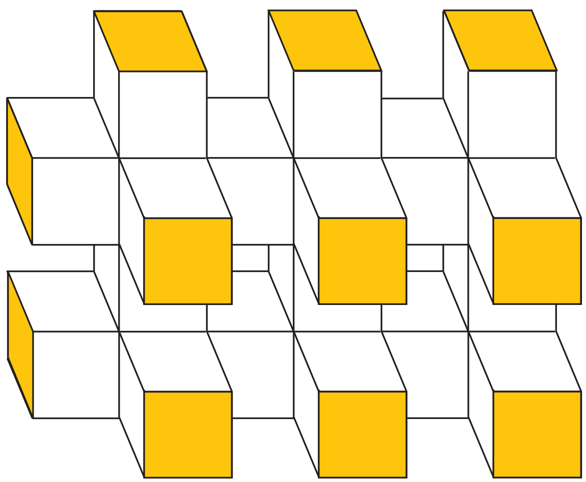


$\{6, 4|4\}$

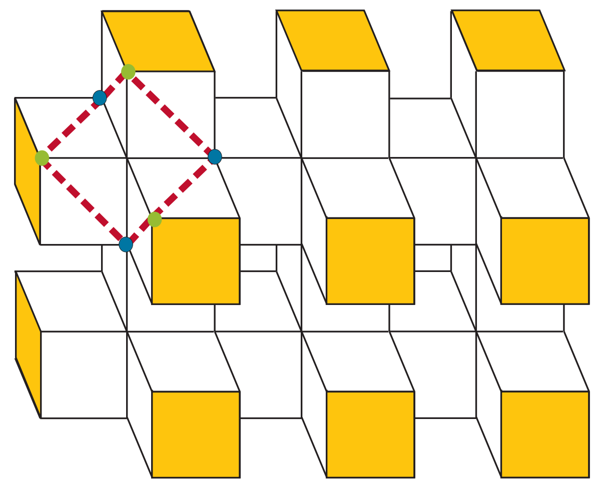


$\{6, 6|3\}$

Polyhedron $\{6,6\}_4$ derived from the Petrie-Coxeter polyhedron $\{4,6|4\}$



$\{4,6|4\}$



- Bicolor the vertices of $\{4,6|4\}$.
- Vertex-figures at vertices in one class give faces of $\{6,6\}_4$.
- New polyhedron $\{6,6\}_4$ has planar vertex-figures.

Symmetry group $G(P)$

- Generated by reflections R_0, R_1, R_2 in points, lines, or planes.

- Standard relations $(R_0R_1)^p = (R_1R_2)^q = (R_0R_2)^2 = I$, and in general more relations (geometry of the polyhedron).

- Wythoff's construction recovers polyhedron from its group.

Classification of triples of reflections R_0, R_1, R_2 such that R_0 and R_2 commute and R_1 and R_2 have a common fixed point.

Grünbaum (70's), Dress (1981); McMullen & S. (1997)

Enumeration of regular polyhedra

18 finite (5 Platonic, 4 Kepler-Poinsot, 9 Petrials)

$$
\begin{array}{lccccccc}
\textit{tetrahedral} & & & \{3,3\} & \stackrel{\pi}{\longleftrightarrow} & \{4,3\}_3 & & \\
\textit{octahedral} & \{6,4\}_3 & \stackrel{\pi}{\longleftrightarrow} & \{3,4\} & \stackrel{\delta}{\longleftrightarrow} & \{4,3\} & \stackrel{\pi}{\longleftrightarrow} & \{6,3\}_4 \\
\textit{icosahedral} & \{10,5\} & \stackrel{\pi}{\longleftrightarrow} & \{3,5\} & \stackrel{\delta}{\longleftrightarrow} & \{5,3\} & \stackrel{\pi}{\longleftrightarrow} & \{10,3\} \\
 & \updownarrow \varphi_2 & & \updownarrow \varphi_2 & & & & \\
 & \{6,\frac{5}{2}\} & \stackrel{\pi}{\longleftrightarrow} & \{5,\frac{5}{2}\} & \stackrel{\delta}{\longleftrightarrow} & \{\frac{5}{2},5\} & \stackrel{\pi}{\longleftrightarrow} & \{6,5\} \\
 & & & & & \updownarrow \varphi_2 & & \updownarrow \varphi_2 \\
 & \{\frac{10}{3},3\} & \stackrel{\pi}{\longleftrightarrow} & \{\frac{5}{2},3\} & \stackrel{\delta}{\longleftrightarrow} & \{3,\frac{5}{2}\} & \stackrel{\pi}{\longleftrightarrow} & \{\frac{10}{3},\frac{5}{2}\}
\end{array}
$$

duality $\delta : R_2, R_1, R_0$; Petrie $\pi : R_0R_2, R_1, R_0$; facetting $\varphi_2 : R_0, R_1R_2R_1, R_2$

Infinite polyhedra, or *apeirohedra*

Their symmetry groups are crystallographic groups (discrete groups of isometries with compact fundamental domain)!

6 planar (3 tessellations by squares, triangles, hexagons; and their Petrials)

24 apeirohedra (12 reducible, or *blends*; 12 irreducible)

- The 12 reducible polyhedra are obtained by blending a planar polyhedron and a linear polygon (line segment or tessellation).

- In a sense, the 12 irreducible polyhedra fall into a single family, derived from the cubical tessellation. Various relationships between them.

Irreducible polyhedra

$$
\begin{array}{ccccccc}
\{\infty,4\}_{6,4} & \stackrel{\pi}{\longleftrightarrow} & \{6,4|4\} & \stackrel{\delta}{\longleftrightarrow} & \{4,6|4\} & \stackrel{\pi}{\longleftrightarrow} & \{\infty,6\}_{4,4} \\
 & & \sigma \downarrow & & \downarrow \eta & & \\
 & & \{\infty,4\}_{\cdot,*3} & & \{6,6\}_4 & \stackrel{\varphi_2}{\longrightarrow} & \{\infty,3\}^{(a)} \\
 & & & & \pi \updownarrow & & \updownarrow \pi \\
 & & \{6,4\}_6 & \stackrel{\delta}{\longleftrightarrow} & \{4,6\}_6 & \stackrel{\varphi_2}{\longrightarrow} & \{\infty,3\}^{(b)} \\
 & & \sigma\delta \downarrow & & \downarrow \eta & & \\
 & & \{\infty,6\}_{6,3} & \stackrel{\pi}{\longleftrightarrow} & \{6,6|3\} & &
\end{array}
$$

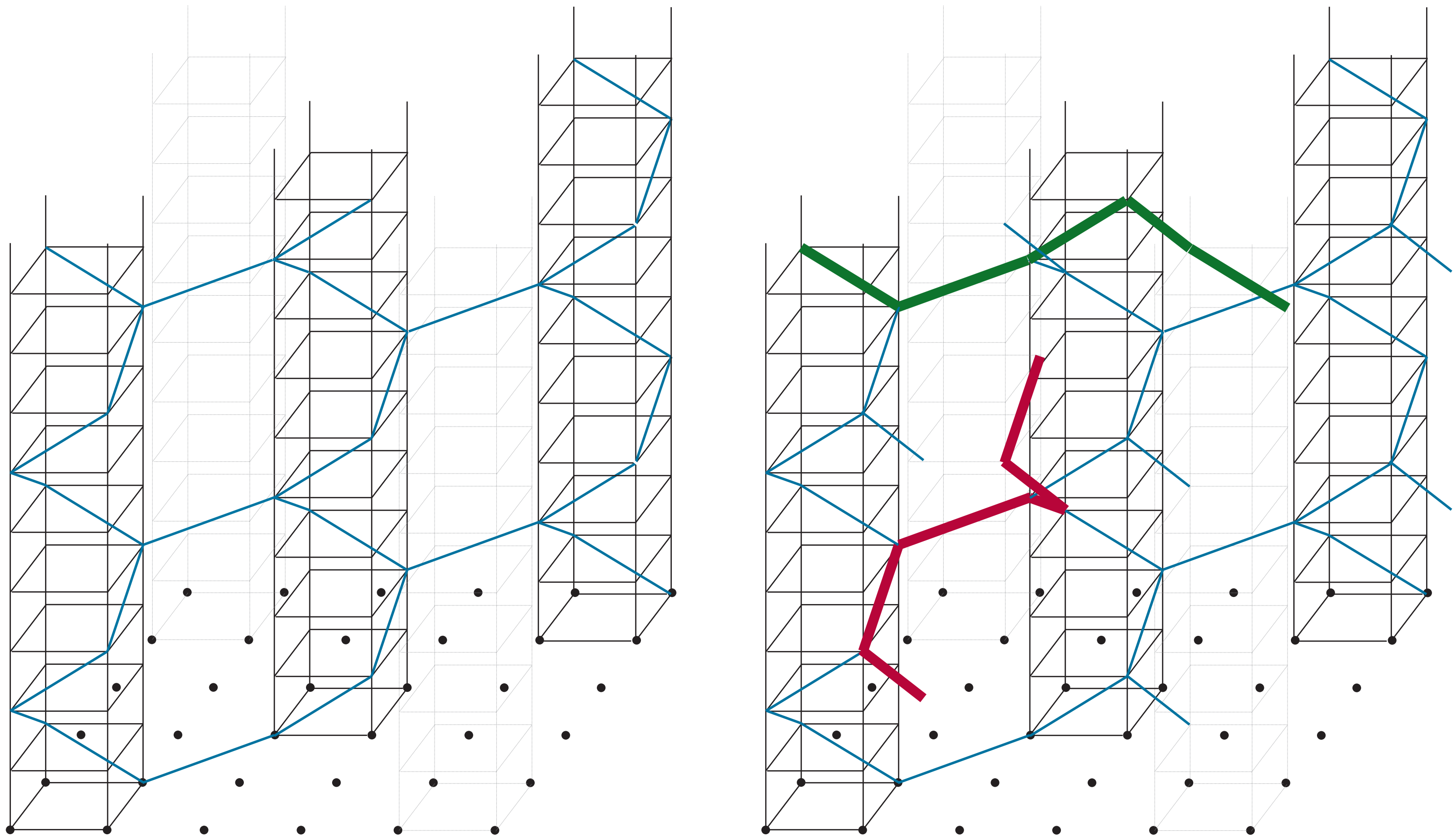
halving η: $R_0R_1R_0, R_2, R_1$; skewing $\sigma = \pi\delta\eta\pi\delta$: $R_1, R_0R_2, (R_1R_2)^2$

Breakdown by mirror vector (for R_0, R_1, R_2)

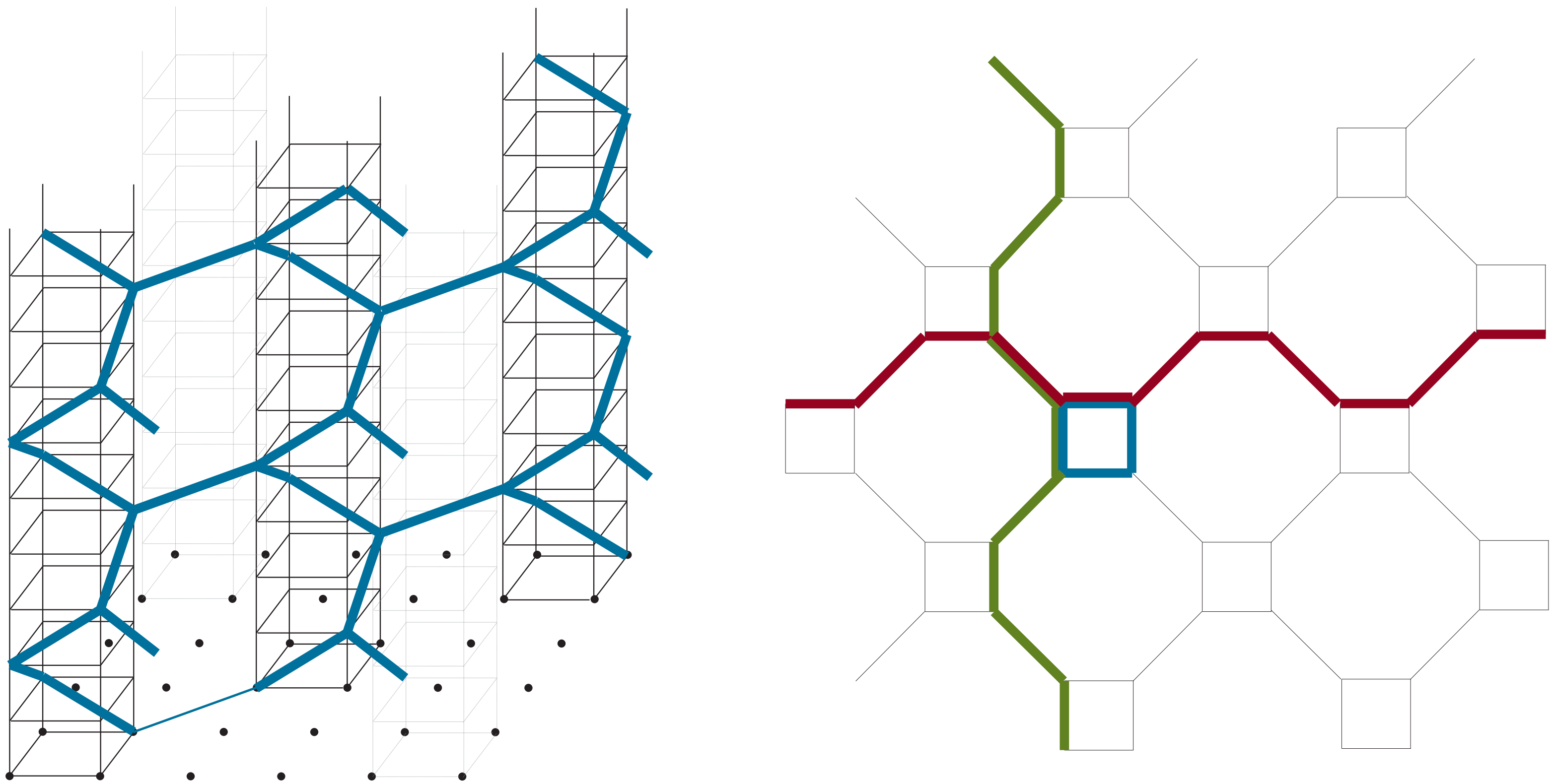
mirror vector	$\{3,3\}$	$\{3,4\}$	$\{4,3\}$	faces	vertex-figures
(2,1,2)	$\{6,6\|3\}$	$\{6,4\|4\}$	$\{4,6\|4\}$	planar	skew
(1,1,2)	$\{\infty,6\}_{4,4}$	$\{\infty,4\}_{6,4}$	$\{\infty,6\}_{6,3}$	helical	skew
(1,2,1)	$\{6,6\}_4$	$\{6,4\}_6$	$\{4,6\}_6$	skew	planar
(1,1,1)	$\{\infty,3\}^{(a)}$	$\{\infty,4\}_{\cdot,*3}$	$\{\infty,3\}^{(b)}$	helical	planar

The polyhedra in the last line occur in two enantiomorphic forms, yet they are geometrically regular!

Presentations for the symmetry group are known. The fine Schläfli symbol signifies defining relations. Extra relations specify order of $R_0R_1R_2$, $R_0R_1R_2R_1$, or $R_0(R_1R_2)^2$.



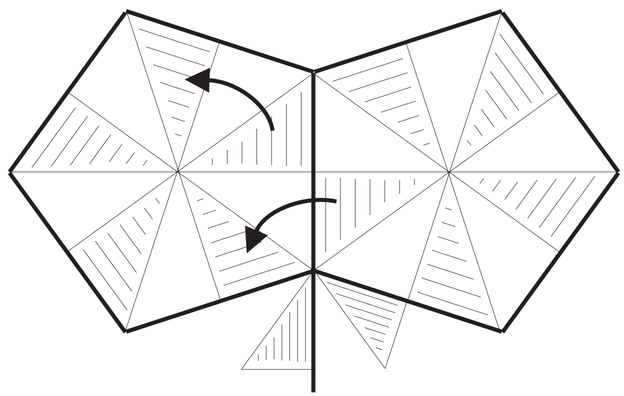
$\{\infty, 3\}^{(b)}$ $\qquad$ $(R_0R_1)^4(R_0R_1R_2)^3 = (R_0R_1R_2)^3(R_0R_1)^4$



Helix-faced polyhedron $\{\infty, 3\}^{(b)}$

Chiral Polyhedra in $\mathbb{E}^3$

- Two orbits on the flags under the geometric symmetry group, such that adjacent flags are always in different orbits.
- Local definition



Generators S_1, S_2 for type $\{p, q\}$

$$S_1^p = S_2^q = (S_1 S_2)^2 = 1 \text{ \& generally more relations}$$

- Maximal “rotational” symmetry but no “reflexive” symmetry! Irreflexible!

Observations

• No examples were known (to me). Convex polytopes cannot be chiral! (McMullen)

• Variant of Wythoff's construction (exploiting $S_1 S_2$)!

• There are no finite chiral polyhedra in $\mathbb{E}^3$!

• There are no planar or blended chiral polyhedra in $\mathbb{E}^3$.

• Classification breaks naturally into finite-faced and helix-faced polyhedra!

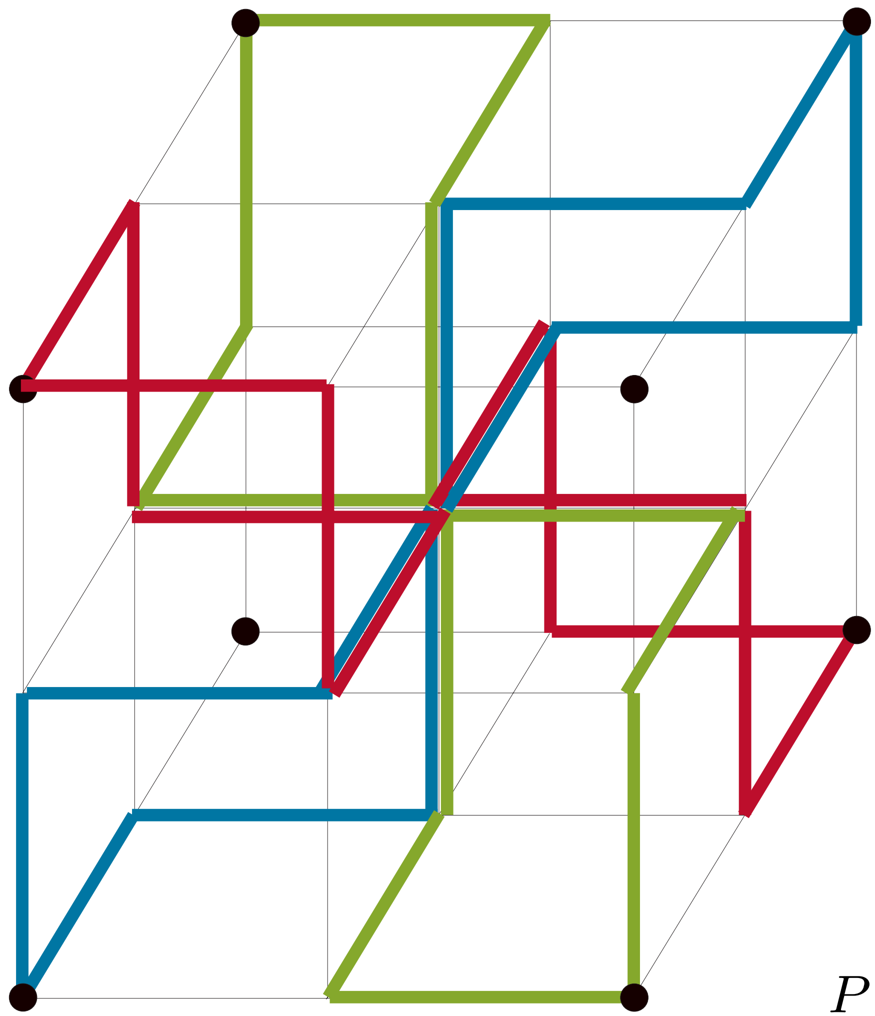
S. (2004/5)

Three Classes of Finite-Faced Chiral Polyhedra

(S_1, S_2 rotatory reflections, hence skew faces and skew vertex-figures.)

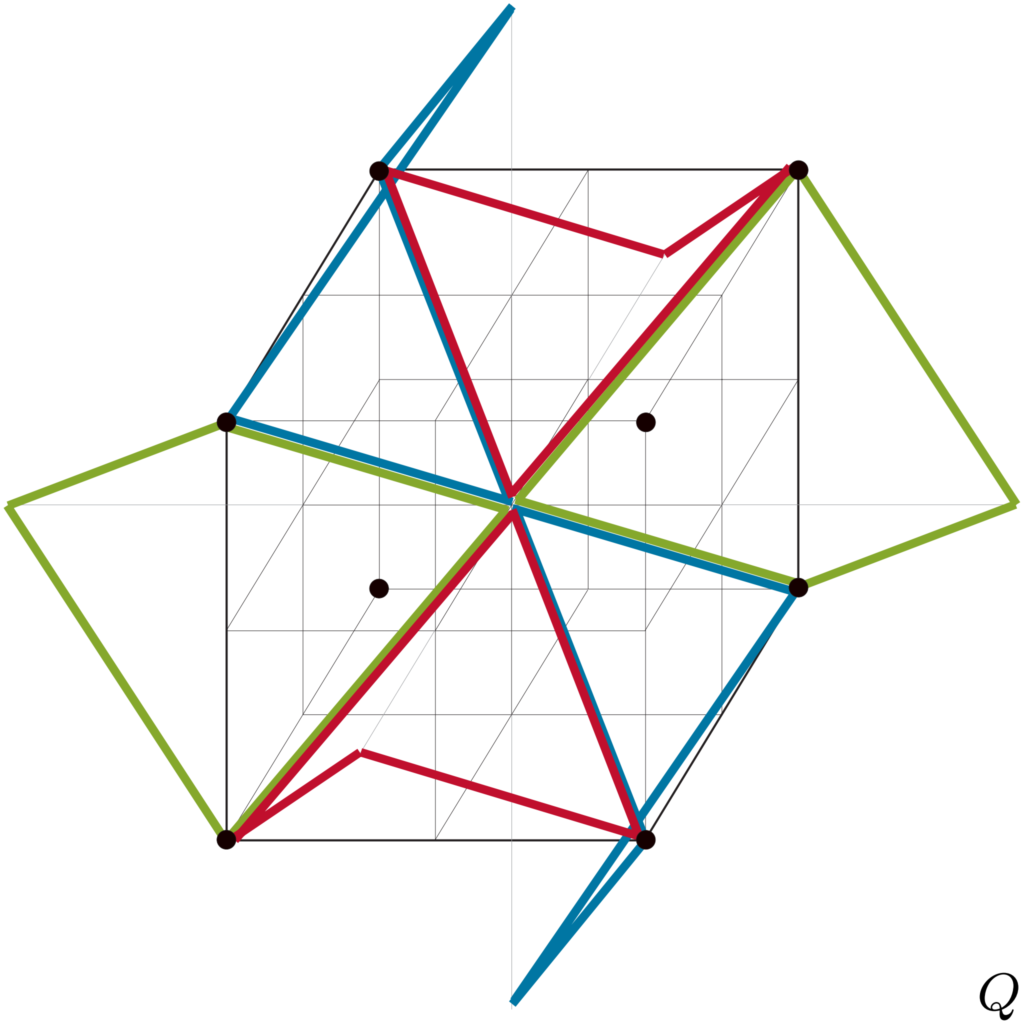
Schläfli	$\{6,6\}$	$\{4,6\}$	$\{6,4\}$
Notation	$P(a,b)$	$Q(c,d)$	$Q(c,d)^*$
Param.	$a,b \in \mathbb{Z}$, $(a,b)=1$	$c,d \in \mathbb{Z}$, $(c,d)=1$	$c,d \in \mathbb{Z}$, $(c,d)=1$
	geom. self-dual $P(a,b)^* \cong P(a,b)$		
Special gr	$[3,3]^+ \times \langle -I \rangle$	$[3,4]$	$[3,4]$
Regular cases	$P(a,-a)=\{6,6\}_4$ $P(a,a)=\{6,6\|3\}$	$Q(a,0)=\{4,6\}_6$ $Q(0,a)=\{4,6\|4\}$	$Q(a,0)^*=\{6,4\}_6$ $Q(0,a)^*=\{6,4\|4\}$

Vertex-sets and translation groups are known!



$P(1,0)$, of type $\{6,6\}$

Neighborhood of a single vertex.



$Q(1,1)$, of type $\{4,6\}$

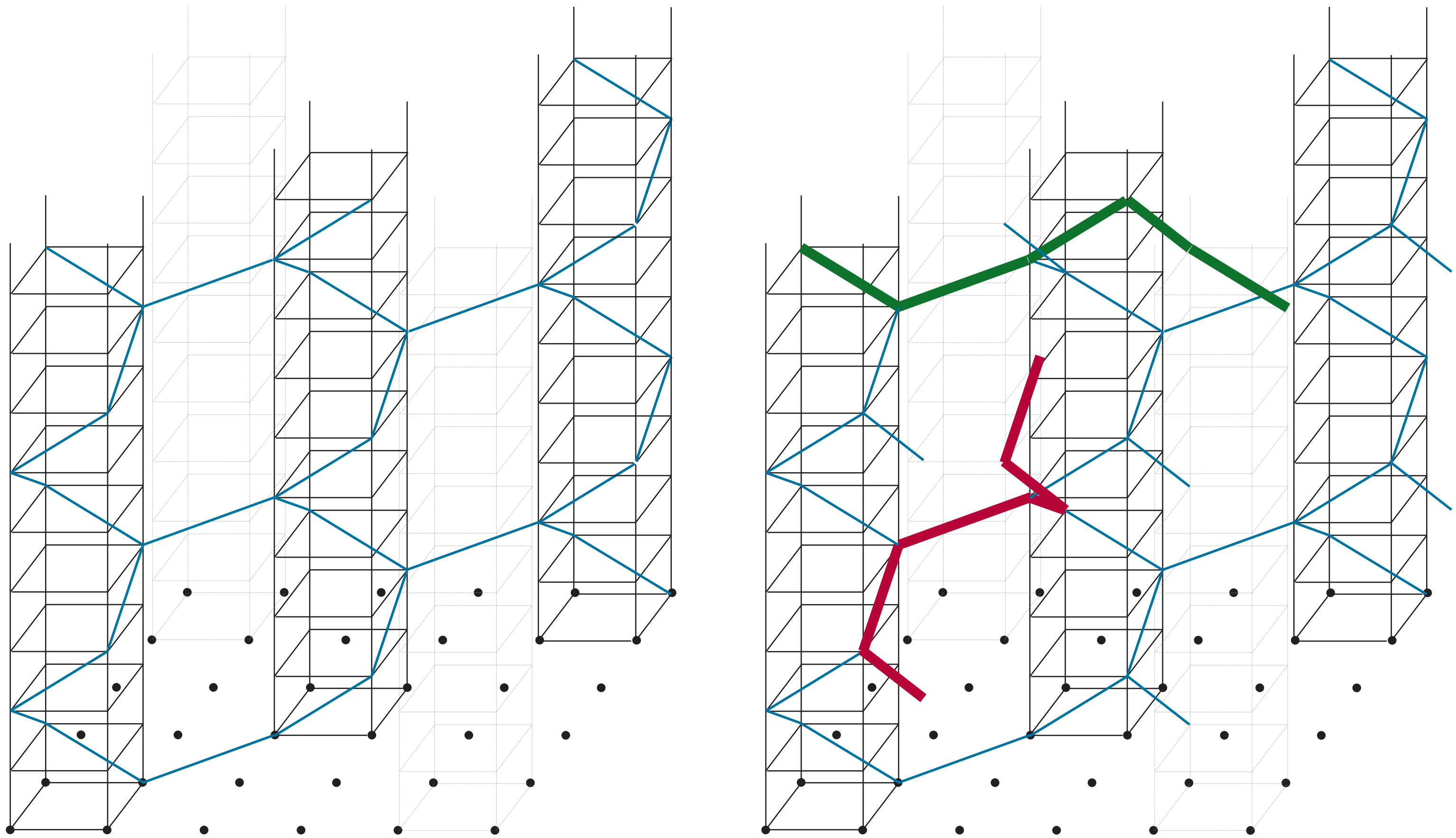
Neighborhood of a single vertex.

Three Classes of Helix-Faced Chiral Polyhedra

(S_1 screw motion, S_2 rotation; helical faces and planar vertex-figures.)

Schläfli symbol	$\{\infty, 3\}$	$\{\infty, 3\}$	$\{\infty, 4\}$
Helices over	triangles	squares	triangles
Special group	$[3, 3]^+$	$[3, 4]^+$	$[3, 4]^+$
Relationships	$P(a, b)^{\varphi 2}$ $a \neq b$ (reals)	$Q(c, d)^{\varphi 2}$ $c \neq 0$ (reals)	$Q^*(c, d)^{\kappa}$ c, d reals
Regular cases	$\{\infty, 3\}^{(a)}$ $= P(1, -1)^{\varphi 2}$ $= \{6, 6\}_4^{\varphi 2}$	$\{\infty, 3\}^{(b)}$ $= Q(1, 0)^{\varphi 2}$ $= \{4, 6\}_6^{\varphi 2}$	$\{\infty, 4\}_{\cdot, *3}$ self- Petrie

Vertex-sets and translation groups are known!



$$\{\infty, 3\}^{(b)}$$

Remarkable facts

• Essentially: any two finite-faced polyhedra of the same type are non-isomorphic.

$$P(a,b) \cong P(a',b') \text{ iff } (a',b') = \pm(a,b), \pm(b,a).$$

$$Q(c,d) \cong Q(c',d') \text{ iff } (c',d') = \pm(c,d), \pm(-c,d).$$

• The finite-faced polyhedra are intrinsically (combinatorially) chiral! [Pellicer & Weiss 2009]

• The helix-faced polyhedra are combinatorially regular! Combinatorially only three polyhedra! Chiral helix-faced polyhedra are "chiral deformations" of regular helix-faced polyhedra! [Pellicer & Weiss 2009]

- Chiral helix-faced polyhedra unravel Platonic solids!

Coverings

$$\{\infty,3\}\mapsto\{3,3\},\ \ \{\infty,3\}\mapsto\{4,3\},\ \ \{\infty,4\}\mapsto\{3,4\}.$$

- Relationships between the classes of chiral polyhedra

$$\begin{array}{ccccc} Q^* & \overset{\delta}{\longleftrightarrow} & Q & \xrightarrow{\varphi_2} & P_2 \\ \updownarrow\kappa & & \downarrow\eta & & \\ P_3 & & P & \xrightarrow{\varphi_2} & P_1 \\ & \delta\circlearrowleft & & & \end{array}$$

$$\delta=(S_2^{-1},S_1^{-1}),\ \ \eta=(S_1^2S_2,S_2^{-1}),\ \ \varphi_2=(S_1S_2^{-1},S_2^2),\ \ \kappa=(-S_1,-S_2)$$

Finite Regular Polyhedra of Index 2 in $\mathbb{E}^3$

(joint with A.Cutler)

• P is combinatorially regular. Combinatorial automorphism group $\Gamma(P)$ is flag-transitive!

• Geometric symmetry group $G(P)$ is of index 2 in the combinatorial automorphism group $\Gamma(P)$.

Combinatorially regular but "fail geometric regularity by a factor of 2". Hidden combinatorial symmetries!

Orientable finite regular polyhedra of index 2 with *planar* faces (Wills, 1987).

Five polyhedra

- dual maps $\{4,5\}_6$, $\{5,4\}_6$ of genus 4;
- dual maps $\{6,5\}_4$, $\{5,6\}_4$ of genus 9;
- self-dual map of type $\{6,6\}_6$ (not universal) of genus 11.

General case was open!

Models by David Richter.

The Regular Polyhedra
(of index two)

Click here to see these in stereo.

Here are all five regular polyhedra (of index two). By name, going from left to right, they are the ditrigonal

Full classification of *finite* polyhedra: 32 regular polyhedra of index 2.

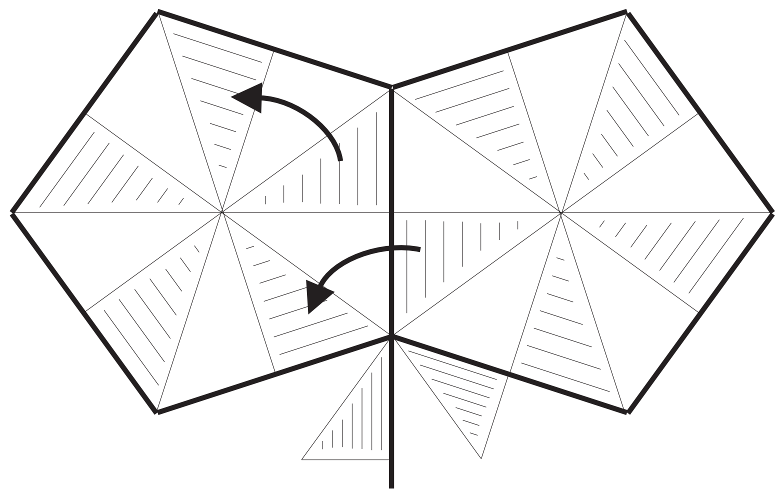
- Exactly two flag orbits under $G(P)$; and at most two orbits under $G(P)$ on the vertices, edges, and faces.

- $G(P)$ is a finite subgroups of $O(3)$. Rule out reducible groups and rotation subgroups of Platonic solids.

Only possibilities: full symmetry groups of Platonic solids.

Platonic solids provide reference figures!

• Combinatorial regularity means $\Gamma(P) = \langle\rho_0, \rho_1, \rho_2\rangle$ and $\Gamma^+(P) = \langle\sigma_1, \sigma_2\rangle$, with $\sigma_1 := \rho_0\rho_1$ and $\sigma_1 := \rho_1\rho_2$.



Exploit index 2 property! Squaring ends up in G(P).

• Face stabilizers $G_F(P)$ are of index 1 or 2 in the (dihedral) face stabilizer $\Gamma_F(P)$.

• Class of regular polyhedra of index 2 invariant under Petrie duality.

Classification splits naturally

- two vertex orbits ($18 + 4 = 22$ polyhedra).
- one vertex orbit (10 polyhedra).

Case of two vertex orbits

- families of polyhedra rather than individual polyhedra, depending on relative sizes of the circumspheres of their vertex orbits.
- vertices of P located at those of a pair of similar, aligned or opposed, Platonic solids, S and $S^{\diamond}$, with $G(S) = G(P)$.

Cutler & S. (2011), Cutler (2011).

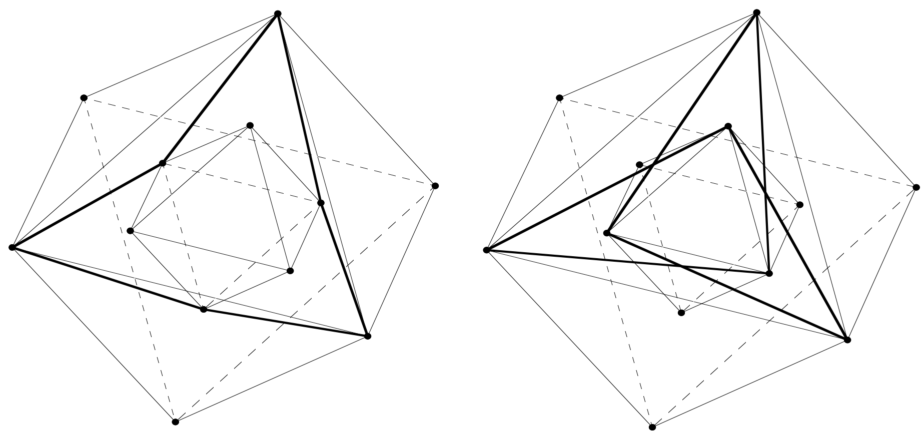
The 22 families of polyhedra with two vertex-orbits

- 18 are related to the ordinary finite regular polyhedra (of index 1).

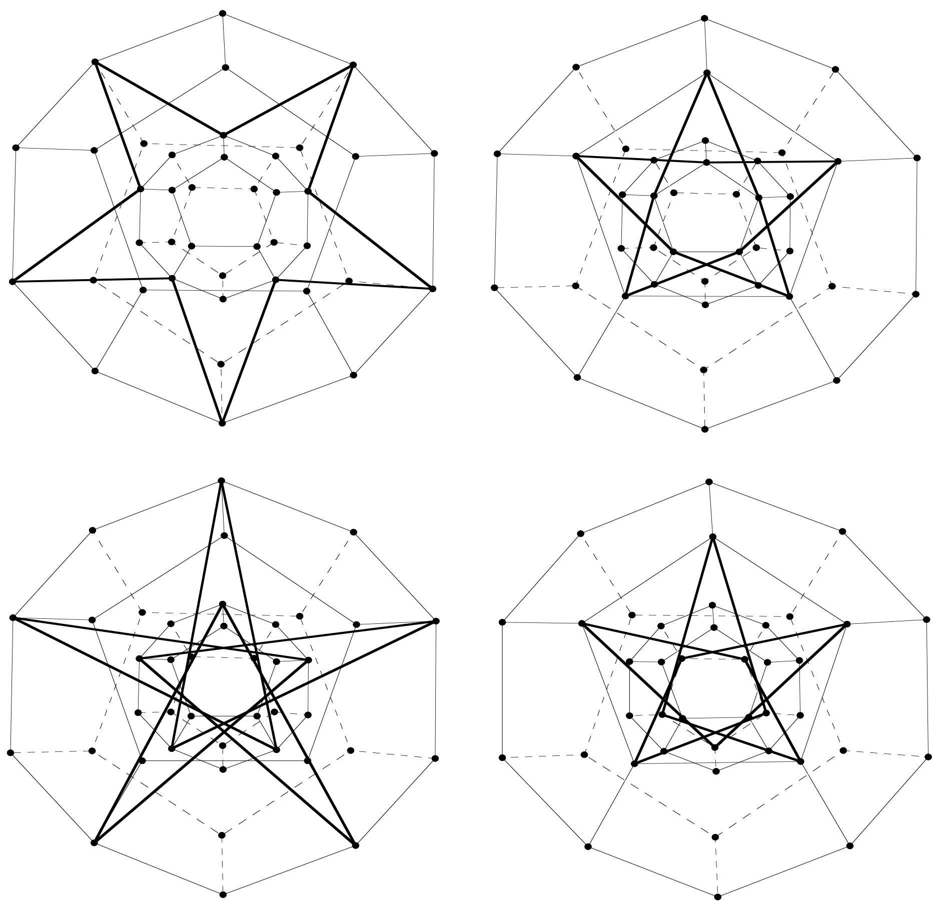
- 4 have full tetrahedral symmetry; 2 have full octahedral symmetry; and 16 have full icosahedral symmetry.

- All polyhedra are orientable and face-transitive. All, but two, individual polyhedra have non-planar faces.

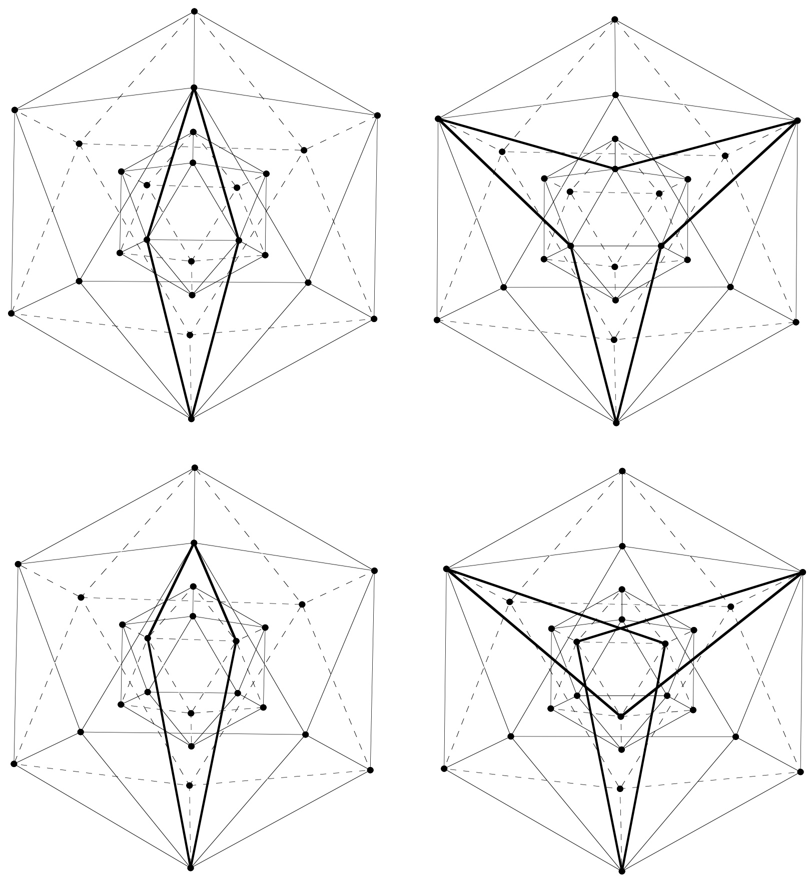
Type $\{p,q\}_r$	Generated from	Face Vector (f_0,f_1,f_2)	Edge Length	Face Shape	Map
$\{4,3\}_6$	$\{4,3\}$	$(8,12,6)$	1	$[r,r]$	—
$\{6,3\}_4$	Petrial of $\{4,3\}$	$(8,12,4)$	1	$[r,l]$	—
$\{4,3\}_6$	Petrial of $\{3,3\}$	$(8,12,6)$	1	$[r,l]$	—
$\{6,3\}_4$	$\{3,3\}$	$(8,12,4)$	1	$[r,r]$	—
$\{6,4\}_6$	Petrial of $\{3,4\}$	$(12,24,8)$	1	$[r,l]$	$R3.4^*$
$\{6,4\}_6$	$\{3,4\}$	$(12,24,8)$	1	$[r,r]$	$R3.4^*$
$\{10,3\}_{10}$	Petrial of $\{5,3\}$	$(40,60,12)$	1	$[r,l]$	$R5.2^*$
$\{10,3\}_{10}$	$\{5,3\}$	$(40,60,12)$	1	$[r,r]$	$R5.2^*$
$\{10,3\}_{10}$	$\{\frac{5}{2},3\}$	$(40,60,12)$	4	$[r,r]$	$R5.2^*$
$\{10,3\}_{10}$	Petrial of $\{\frac{5}{2},3\}$	$(40,60,12)$	4	$[r,l]$	$R5.2^*$
$\{4,5\}_6$	—	$(24,60,30)$	1	$[hr,sr]$	$R4.2$
$\{6,5\}_4$	—	$(24,60,20)$	1	$[hr,sl]$	$R9.16^*$
$\{4,5\}_6$	—	$(24,60,30)$	2	$[hr,sl]$	$R4.2$
$\{6,5\}_4$	—	$(24,60,20)$	2	$[hr,sr]$	$R9.16^*$
$\{6,5\}_{10}$	Petrial of $\{\frac{5}{2},5\}$	$(24,60,20)$	2	$[hr,hl]$	$R9.15^*$
$\{10,5\}_6$	$\{\frac{5}{2},3\}$	$(24,60,12)$	2	$[hr,hr]$	$R13.8^*$
$\{6,5\}_{10}$	$\{3,5\}$	$(24,60,20)$	1	$[hr,hr]$	$R9.15^*$
$\{10,5\}_6$	Petrial of $\{3,5\}$	$(24,60,12)$	1	$[hr,hl]$	$R13.8^*$
$\{6,5\}_{10}$	Petrial of $\{5,\frac{5}{2}\}$	$(24,60,20)$	1	$[sr,sl]$	$R9.15^*$
$\{10,5\}_6$	$\{5,\frac{5}{2}\}$	$(24,60,12)$	1	$[sr,sr]$	$R13.8^*$
$\{6,5\}_{10}$	$\{3,\frac{5}{2}\}$	$(24,60,20)$	2	$[sr,sr]$	$R9.15^*$
$\{10,5\}_6$	Petrial of $\{5,\frac{5}{2}\}$	$(24,60,12)$	2	$[sr,sl]$	$R13.8^*$



Octahedral symmetry.
From $\{3,4\}^{\pi}$, $\{3,4\}$.



Icosahedral. Type $\{10,3\}_{10}$. From $\{5,3\}^{\pi}$, $\{5,3\}$, $\{\frac{5}{2},3\}$, $\{\frac{5}{2},3\}^{\pi}$.



Icosahedral. Types $\{4,5\}_6$ or $\{6,5\}_4$.
Not derived. At top, planar faces poss.

The 10 families of polyhedra with one vertex-orbit

- All 10 have full icosahedral symmetry.

Type $\{p,q\}_r$	Face Vector (f_0,f_1,f_2)	Edge Length	Shape	Map	
$\{6,6\}_6$	$(20,60,20)$	$1,4$	$[r,r]$	$R11.5$	planar faces self-dual map
$\{6,6\}_6$	$(20,60,20)$	$1,4$	$[r,l]\&[l,r]$	$N22.3$	face trans.
$\{4,6\}_5$	$(20,60,30)$	2	$[hl,f]$	$N12.1$	
$\{5,6\}_4$	$(20,60,24)$	2	$[f,f]\&[hl,hl]$	$R9.16$	planar faces
$\{6,4\}_5$	$(30,60,20)$	d	$[r,l]$	$N12.1^*$	
$\{5,4\}_6$	$(30,60,24)$	d	$[r,r]\&[l,l]$	$R4.2^*$	planar faces
$\{4,6\}_{10}$	$(20,60,30)$	3	$[hl,f]$	$R6.2$	
$\{10,6\}_4$	$(20,60,12)$	3	$[f,f]\&[hl,hr]$	$N30.11^*$	
$\{6,4\}_{10}$	$(30,60,20)$	$2d$	$[r,r]$	$R6.2^*$	
$\{10,4\}_6$	$(30,60,12)$	$2d$	$[r,l]\&[l,r]$	$N20.1^*$	

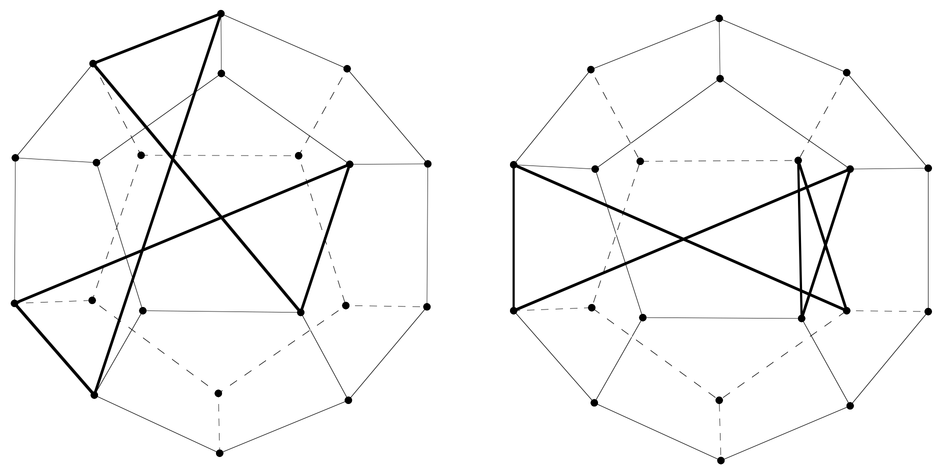


Figure 3: The two polyhedra with edges of unequal length. They have type $\{6,6\}_6$ and the vertices coincide with those of a dodecahedron. They are Petrie-dual and C(P)-dual to each other. The left one has shape [r,r] and is orientable; the right one has shape [r,l]&[l,r] and is non-orientable, with one face orbit under G(P). Shown is one face.

..... The End .....

Thank you

Abstract Two-Orbit Polyhedra in Ordinary Space

In the past few years, there has been a lot of progress in the classification of highly-symmetric discrete polyhedral structures in Euclidean space by distinguished transitivity properties of the geometric symmetry groups. We discuss recent results about two particularly interesting classes of polyhedra in ordinary 3-space, each described by a "two-flag orbits" condition. First we review the chiral polyhedra, which have two flag orbits under the symmetry group such that adjacent flags are in distinct orbits. They occur in six very large 2-parameter families of infinite polyhedra, three consisting of finite-faced polyhedra and three of helix-faced polyhedra. Second, we describe a complete classification of finite "regular polyhedra of index 2", a joint effort with Anthony Cutler.

These polyhedra are combinatorially regular but "fail geometric regularity by a factor of 2"; in other words, the combinatorial automorphism group is flag-transitive but their geometric symmetry group has two flag orbits. There are 32 such polyhedra.

POLYGONS

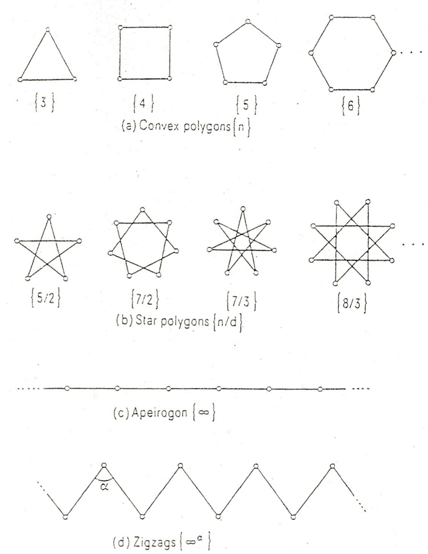


Figure 1

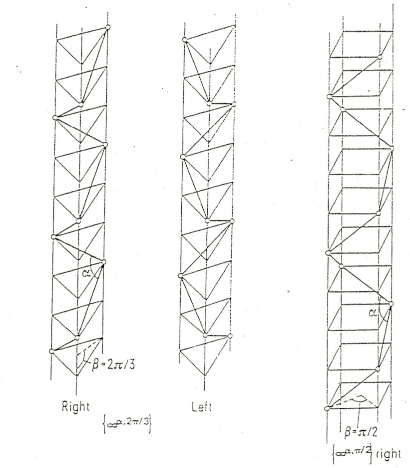
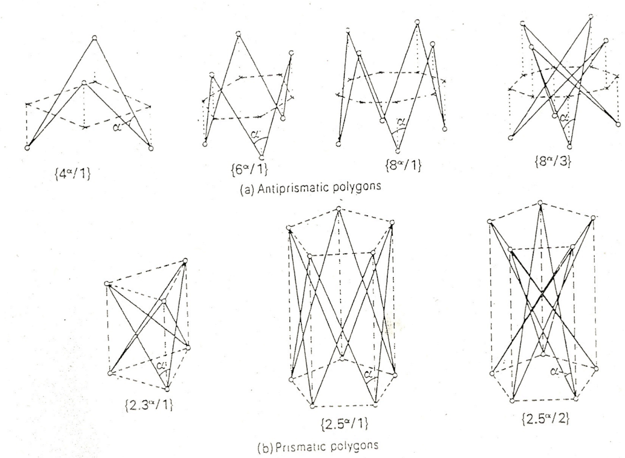
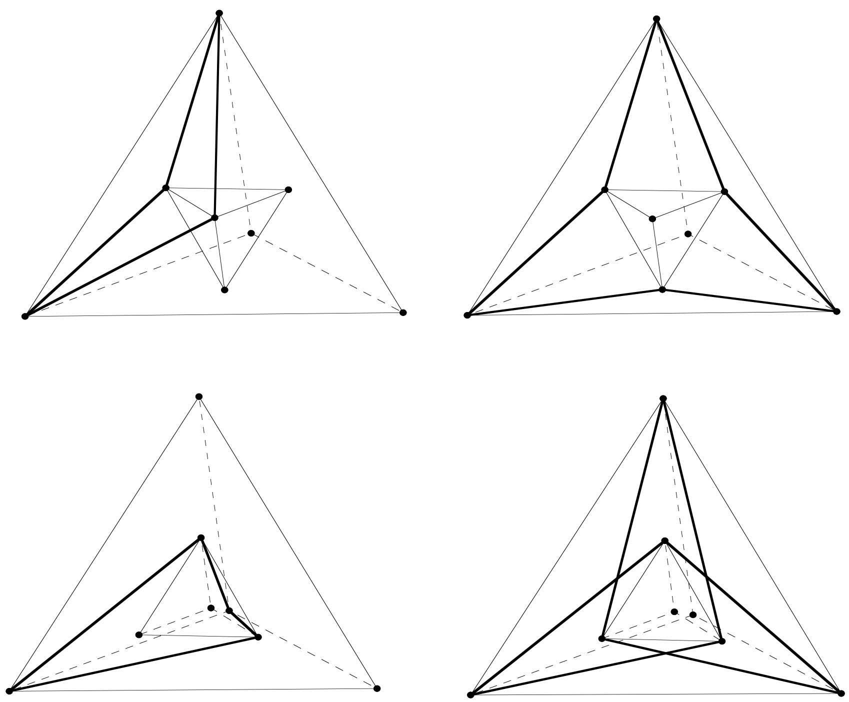
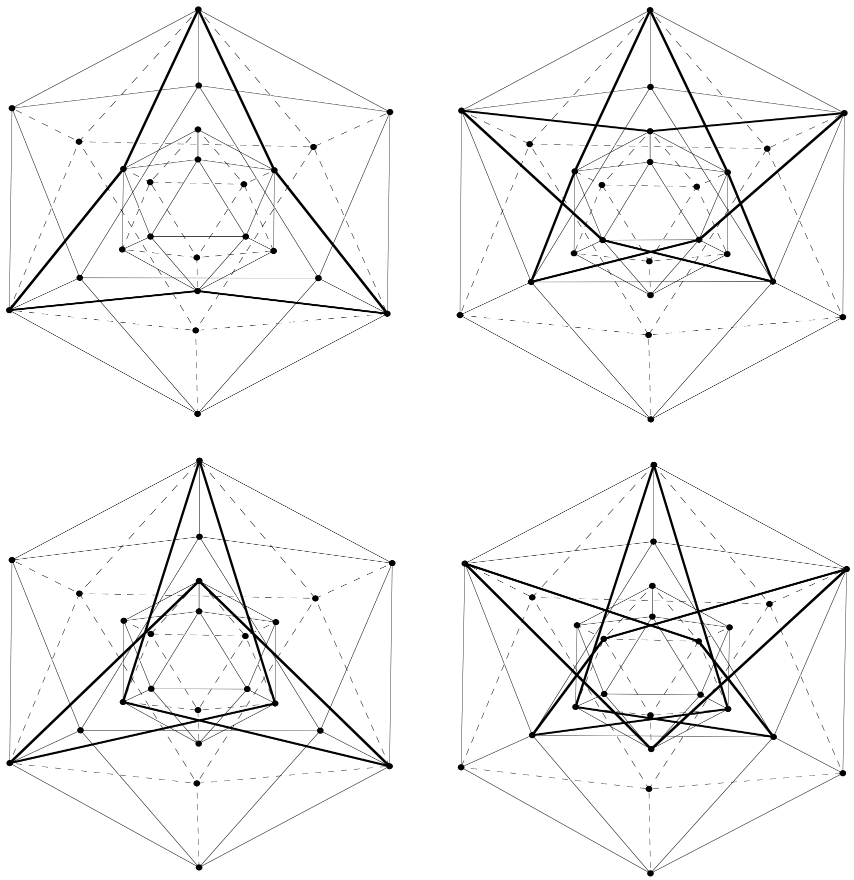


Figure 3
Helical polygons.

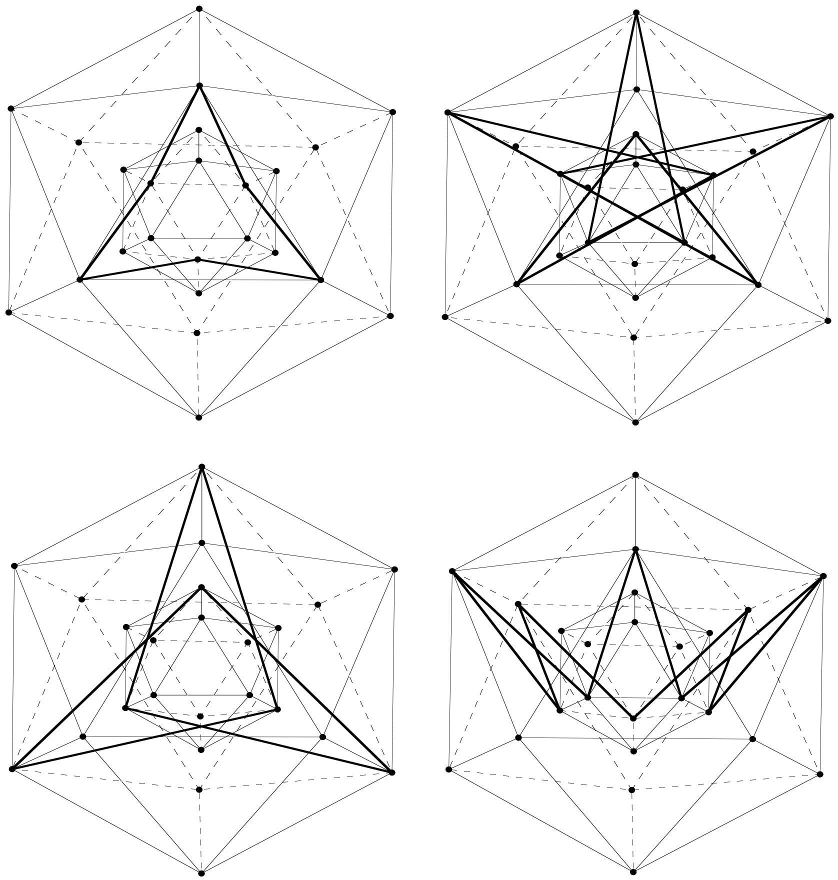




Tetrahedral symmetry. From $\{4,3\}$, $\{4,3\}^{\pi}$, $\{3,3\}^{\pi}$, $\{3,3\}$.



Icosahedral. Types $\{6,5\}_{10}$,$\{10,5\}_6$
(second set of four).
From $\{5,\frac{5}{2}\}^{\pi}$, $\{5,\frac{5}{2}\}$, $\{3,\frac{5}{2}\}$, $\{3,\frac{5}{2}\}^{\pi}$.



Icosahedral. Types $\{6,5\}_{10}$,$\{10,5\}_6$
(first set of four).
From $\{\frac{5}{2},5\}^{\pi}$, $\{\frac{5}{2},5\}$, $\{3,5\}$, $\{3,5\}^{\pi}$.

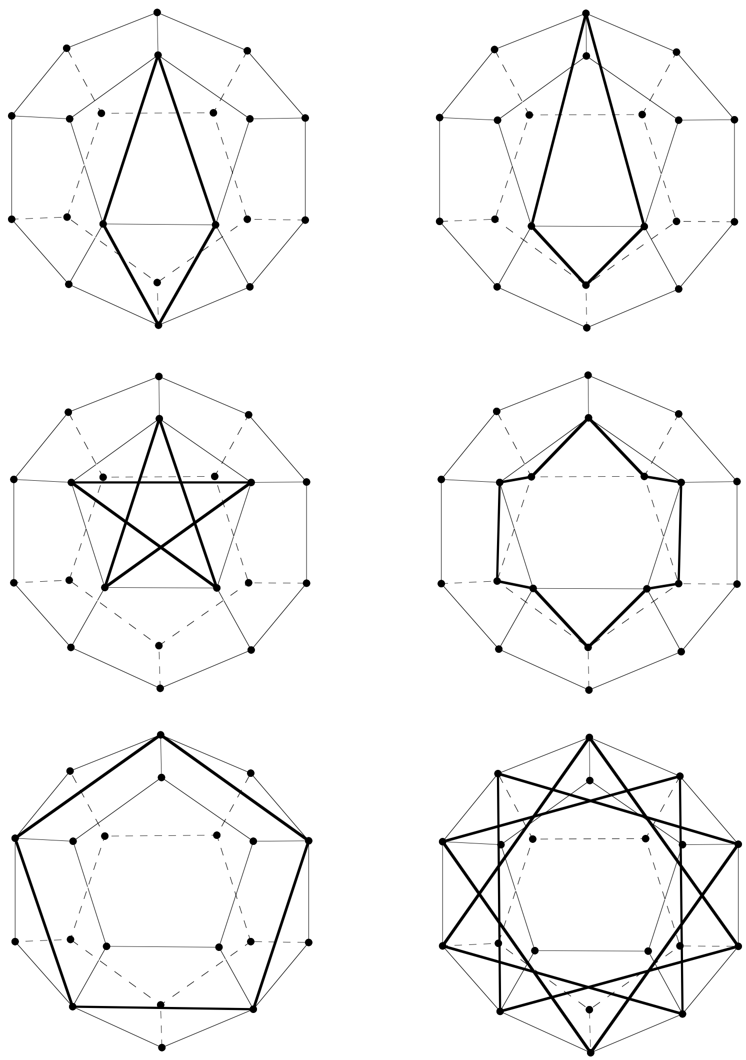


Figure 4: The four polyhedra with edges of equal length and vertices coinciding with those of a dodecahedron. The top left has type $\{4,6\}_5$ and shape [hl,f] and is non-orientable. Below it are shown the two face orbits of its Petrie-dual of type $\{5,6\}_4$ and shape [hl,hl]&[f,f], which is orientable. In the right column are the C(P)-dual polyhedra.The top one has type $\{4,6\}_{10}$ and shape [hl,f] and is orientable. Below it are the two face orbits of its Petrie-dual of type $\{10,6\}_4$ and shape [hl,hr]&[f,f], which is non-orientable.

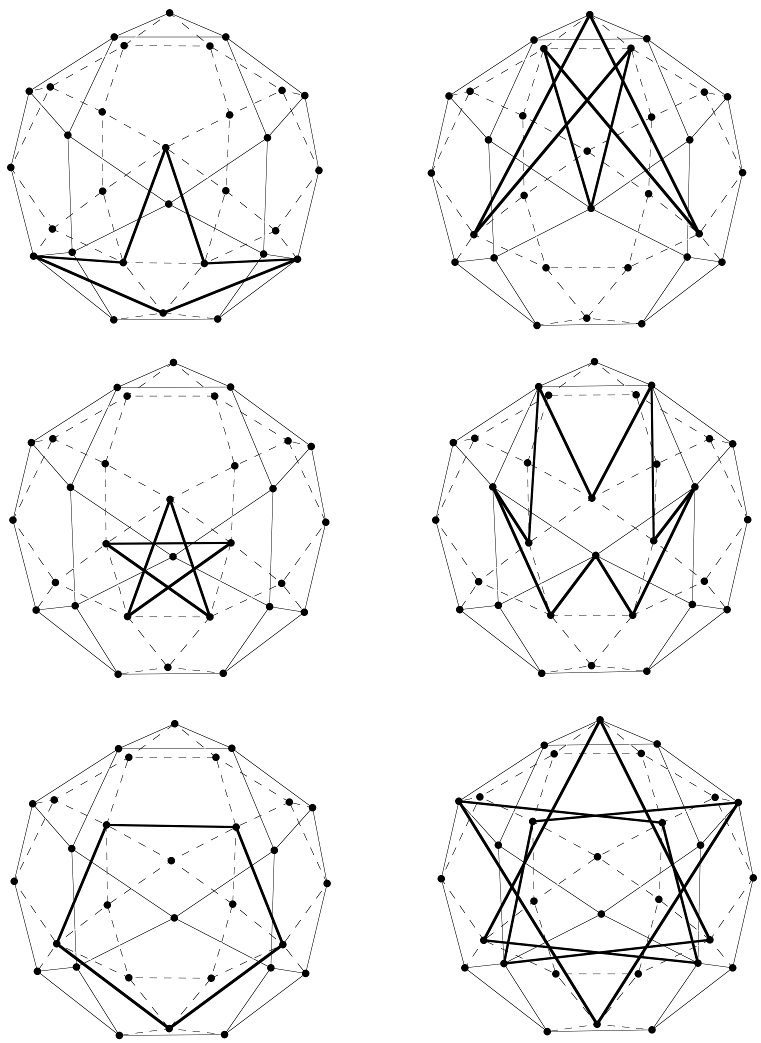


Figure 5: The four polyhedra with edges of equal length and vertices coinciding with those of an icosidodecahedron. The top left has type $\{6,4\}_5$ and shape [r,l] and is non-orientable. Below it are shown the two face orbits of its Petrie-dual of type $\{5,4\}_6$ and shape [r,r]&[l,l], which is orientable. In the right column are the C(P)-dual polyhedra.The top one has type $\{6,4\}_{10}$ and shape [r,r] and is orientable. Below it are the two face orbits of its Petrie-dual of type $\{10,4\}_6$ and shape [r,l]&[l,r], which is non-orientable.