Contact lines with a 180° contact angle

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The background

- The Navier–Stokes equations with the standard set of boundary conditions cannot be used if some of the rigid and free boundaries of the flow intersect, resulting in contact lines.
- One way to show the above is to treat the shape of the free boundary as given, then demonstrate that the pressure at the contact liner is infinite.
- If, however, the contact angle is $180°$, the pressure singularity simply does not arise.
- Two papers explored flows with a $180°$ contact angle: Benney & Timson (1980) and Nir & Pismen (1982). Both were shown to contain errors – actually, the *same* error – by Ngan & Dussan V. (1982).

- In the paper by Nir & Pismen the error turned out to be "fatal"...
- ...but the "amended" result of Benney & Timson seems to be physically meaningful.
- In particular, the amended shape of the free boundary in BT80 is given by

$$y = a\,x^{q} + \mathrm{O}(x^{q+1}) \quad \text{as} \quad x \to 0^{+},$$

where $a > 0$, and $q > 1$ is one of the roots of the equation

$$\tan q\pi = -2\,\frac{\mu V}{\sigma}.$$

The velocity V of the contact line in the analysis of BT80 remains undetermined.

- Despite the apparent self-consistency of the amended version of the BT80 result, Ngan & Dussan V. still insisted that...

> *"...it is our belief that there is something inherently wrong with this* [considered by BT80] *boundary-value problem. We strongly expect that the solution for the interface shape is completely determined upon specifying the contact angle. However, the approach of BT80 does not have this basic characteristic because of the fact that, regardless of the choice of q, the value of the constant a remains undetermined."*

The aim of the present work

In this work, the disagreement between BT80 and Ngan & Dussan V. (1984) is resolved.

We consider an example tractable both locally and globally and demonstrate that a is uniquely determined by matching the former solution to the latter. We also show that matching determines the choice of the root for q. Finally, the global solution yields the velocity of the contact line.

Overall, the local analysis of BT80 and our global analysis provide a fully consistent description of a flow with a $180°$ contact angle and no singularity.

Why is this problem important?

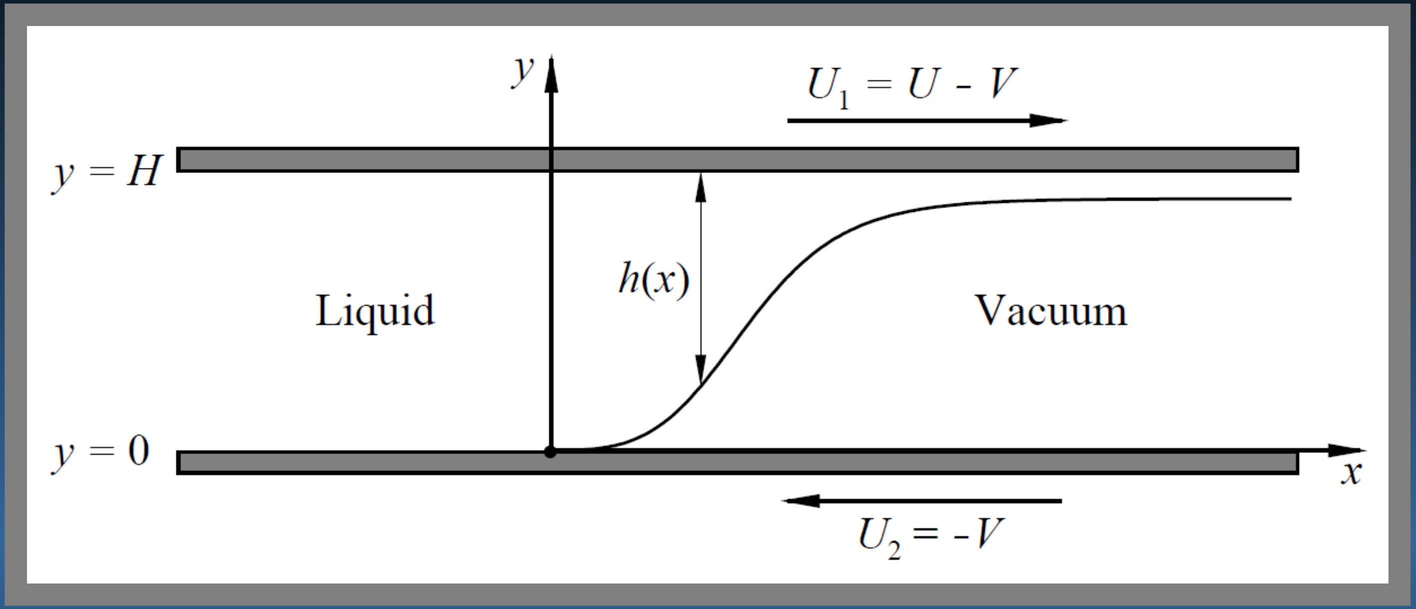
1. The result of BT80 appeals to the scientist's general philosophy, according to which a new concept should not be introduced in a problems where the old one (in this case, the no-slip boundary condition) is not exhausted completely.

2. Recent experiments show that, if surface tension is sufficiently strong, drops rolling on a rigid substrate can indeed exhibit contact angles close, or even equal, to $180°$.

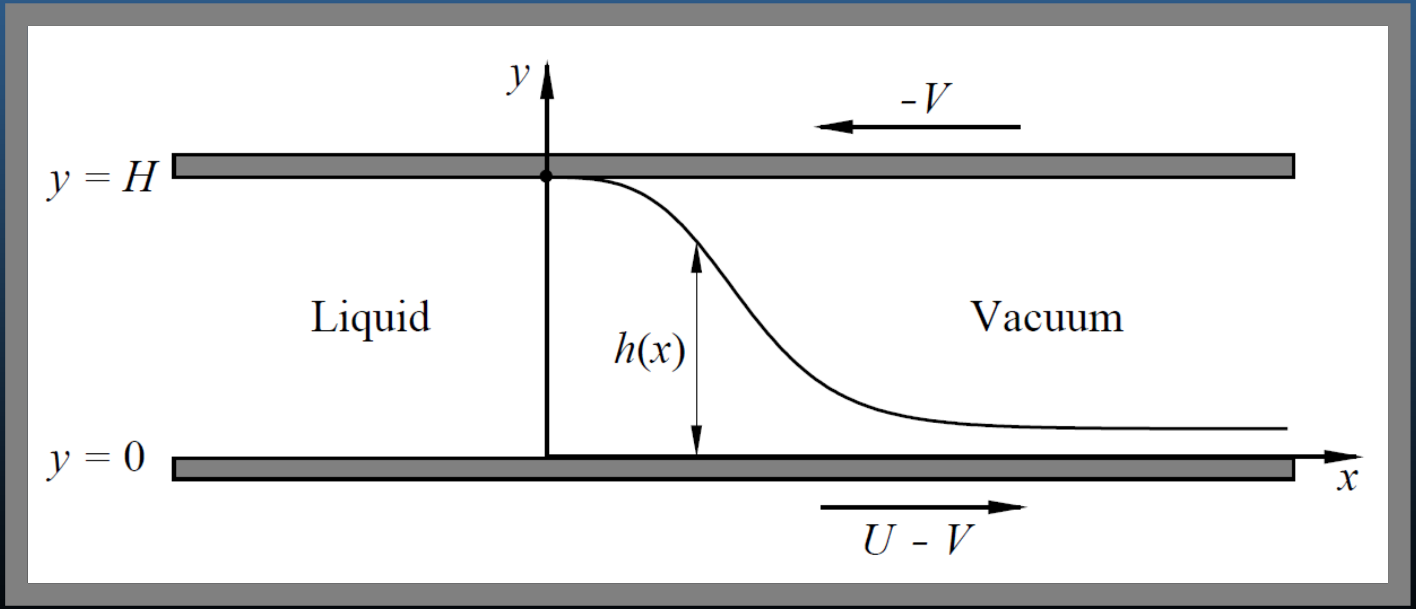
3. The above and some other examples suggest that, if there are strong forces in the problem that push the contact line "to the limit", they enforce a $180°$ contact angle .

Couette flows with a free boundary

Type I:



Type II:



The formulation

- Two spatial dimensions.
- The limit $\mathrm{Re} \to 0$ (hence, Stokes equations).
- Standard boundary conditions:
  - at the rigid boundaries: no flow/no slip,
  - at the free boundary: no tangential stress and a jump in the normal stress due to surface tension.

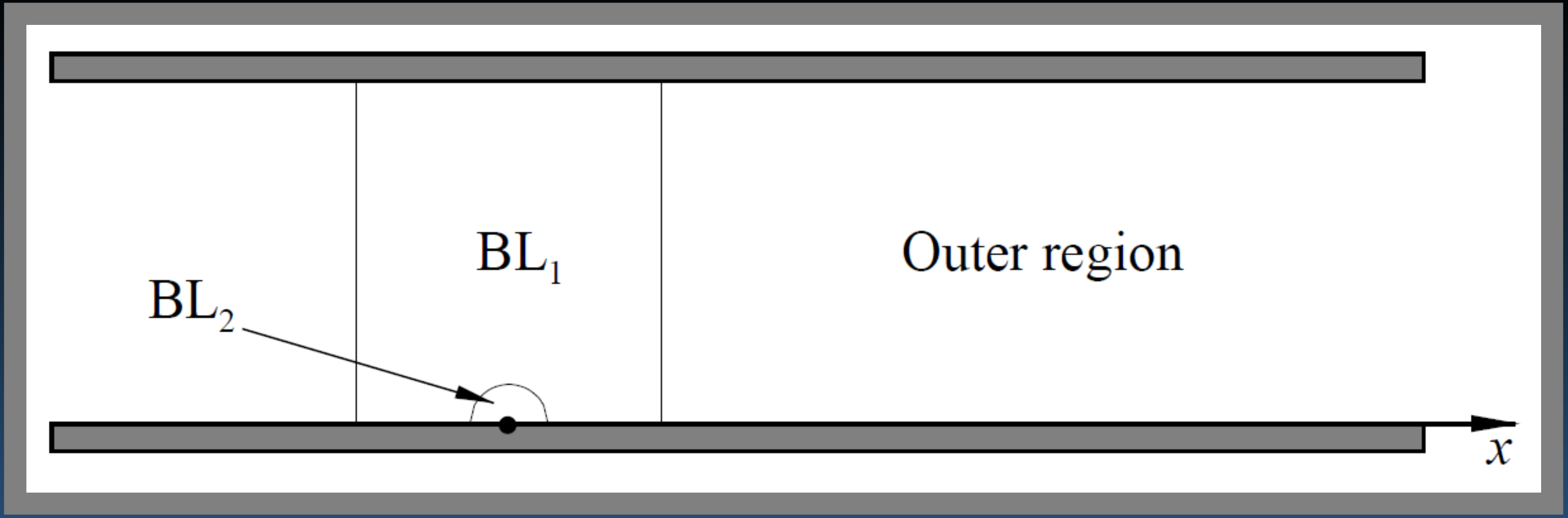
The asymptotic analysis

2 non-dimensional parameters:

$$\varepsilon = \frac{\mu U}{\rho g H^2}, \quad \alpha = \left(\frac{\mu U}{\rho g H^2}\right)\left(\frac{\sigma}{\mu U}\right)^{1/3}.$$

ε Ca^{-1}

We shall assume $\varepsilon \ll 1$, $\alpha \sim 1$ (which implies that $Ca \ll 1$).



The outer region : $x^2 + y^2 >> H^2$.

Governed by the lubrication approximation (if $\varepsilon << 1$, the slope of the free boundary is small).

Boundary layer BL$_1$: $x^2 + y^2 \sim H^2$.

Governed by a mixed BVP for the biharmonic equation.

Boundary layer BL$_2$: $x^2 + y^2 << H^2$.

Governed by the BT80 theory, “truncated” at the leading order in Ca (in particular, it yields $q \approx 2, 3, 4...$).

The outer region

Introduce the following non-dimensional variables:

$$x_* = \frac{\varepsilon x}{H}, \quad t_* = \frac{\varepsilon U t}{H}, \quad h_* = \frac{h}{H}, \quad V_* = \frac{V}{U}.$$

Then the outer solution is governed by (asterisks omitted)

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left[\frac{h^3}{3}\left(\alpha^3 \frac{\partial^3 h}{\partial x^3} \pm \frac{\partial h}{\partial x}\right) + (1 - V)\frac{\partial h}{\partial x}\right] = 0.$$

surface tension

hydrostatic pressure

advection by the plate

The boundary conditions:

$$h = 1, \quad \frac{\partial h}{\partial x} = 0, \quad \frac{h^3}{3}\left(\alpha^3 \frac{\partial^3 h}{\partial x^3} \pm \frac{\partial h}{\partial x}\right) + (1 - V)\,h = \frac{1}{2} - V \quad \text{at} \quad x = 0,$$

$$\frac{\partial h}{\partial x} \to 0 \quad \text{as} \quad x \to +\infty .$$

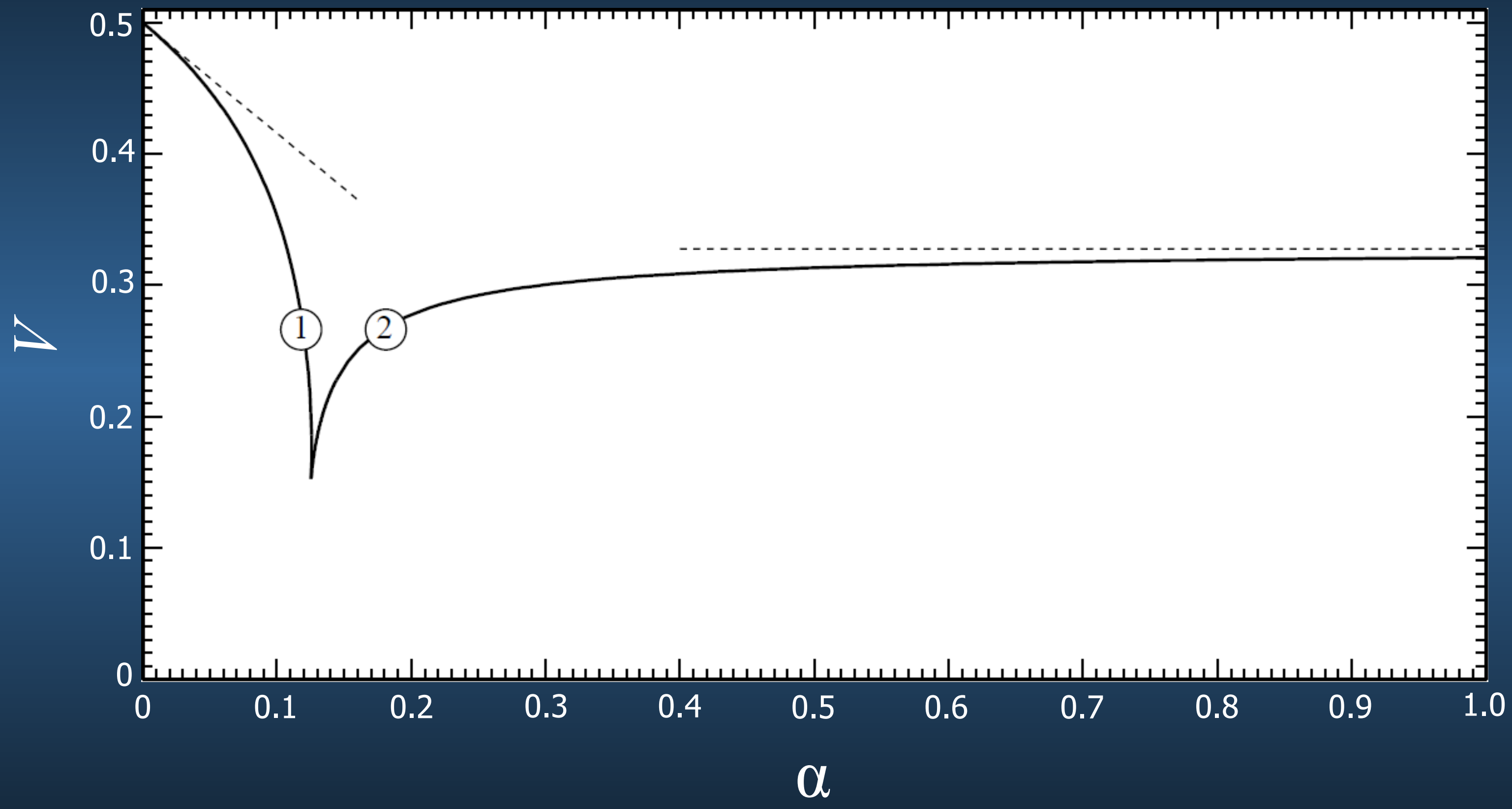
Due to the extra boundary condition, our boundary-value problem determines both $h(x, t)$ and $V(t)$.

In fact, one can show that

$$V = -\frac{1}{2} + \left(\frac{\alpha^3 \dfrac{\partial^5 h}{\partial x^5} \pm \dfrac{\partial^3 h}{\partial x^3}}{3\dfrac{\partial^2 h}{\partial x^2}} \right)_{x = 0} .$$

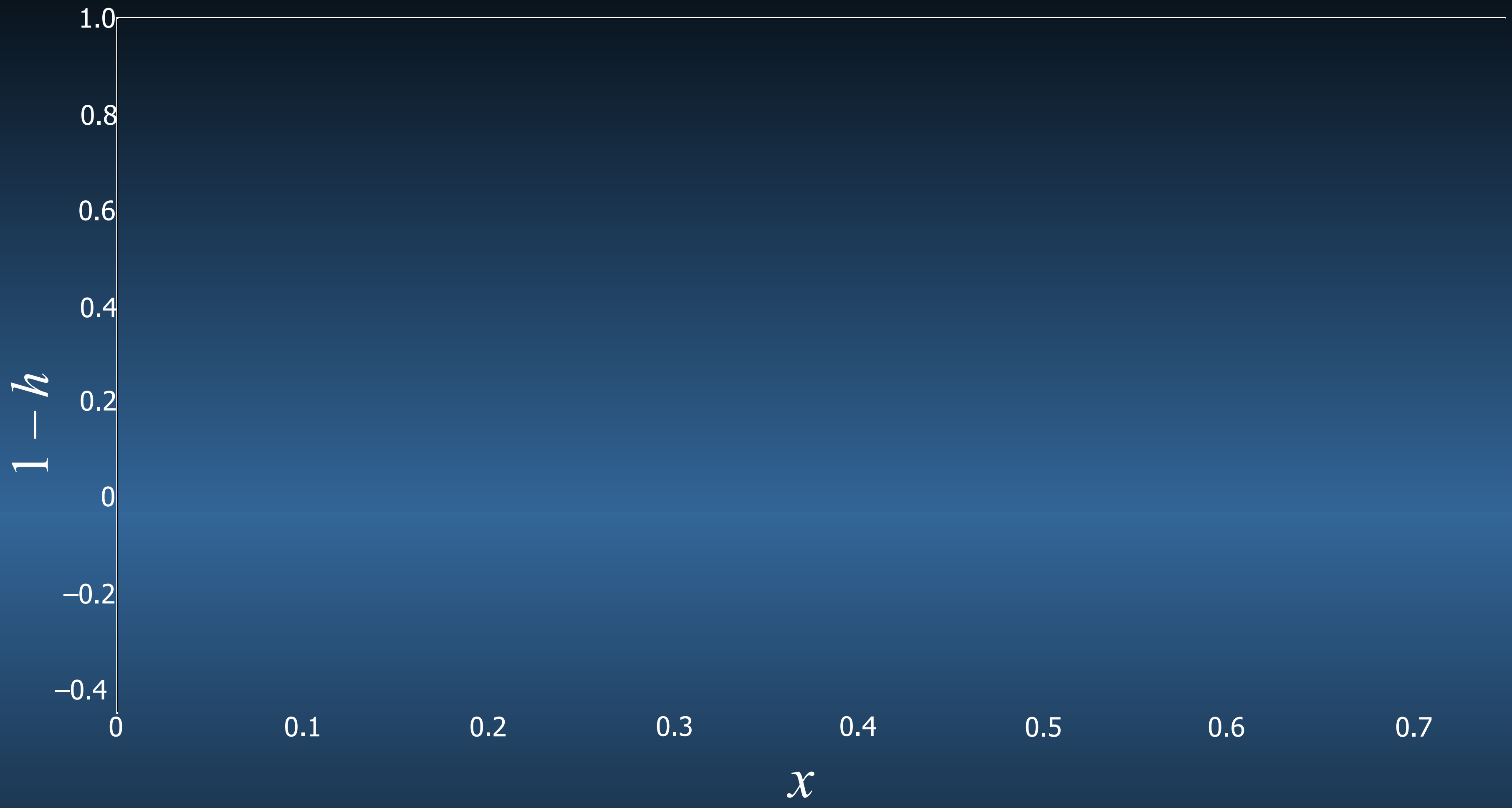
Steady solutions for the outer region

Couette flows of type I



The dotted lines show the asymptotics as $\alpha \longrightarrow 0$ and $\alpha \longrightarrow \infty$.

Solutions (1) and (2) are illustrated on the next slide.



The region inside the channel is shaded, which shows that solution (2) is meaningless physically.

Thus, if $\alpha > \alpha_c$, where $\alpha_c \approx 0.1255$, a Couette flow with a free boundary cannot be steady.

To understand whether α_c is large or small physically, let the liquid under consideration be water at $20°\mathrm{C}$.

Let also $H > 1\,\mathrm{mm}$, $U < 15\,\mathrm{cm/s}$.

For these parameter values,

$$\varepsilon < 0.016, \quad \alpha < 0.012,$$

i.e., a steady solution exists ($\alpha < \alpha_c$), and it is physically meaningful ($\varepsilon << 1$).

Thus, the results obtained are experimentally verifiable.

Couette flows of type II

No steady solutions exist for any values of α.

To understand why, recall that, for flows of type I, hydrostatic pressure causes *negative* diffusion, which balances the *positive* diffusion due to surface tension.

For flows of type II, however, both effects cause *positive* diffusion – hence, cannot hold the solution together.

Concluding remarks

1. Time-dependent solutions.

2. Modelling of “drops without a contact line” rolling down a sloping substrate (Richard & Quere 1999, Reznik & Yarin 2002).