Tear film dynamics with evaporation and osmolarity

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- Motivation from experimental results
- Some past results
- Thoughts on Leveling (OSU, NSF/NIH)
- Surfactant dependent evaporation (PSU York, NSF)
- First thoughts for two layer film (OCCAM, OSU, NSF/NIH/KAUST)
- Summary

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the NSF, NIH

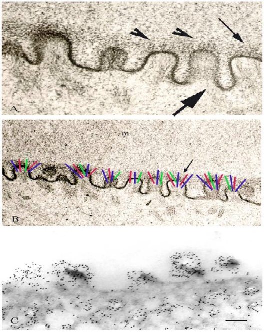
23 February 2012

What is Human Tear Film?

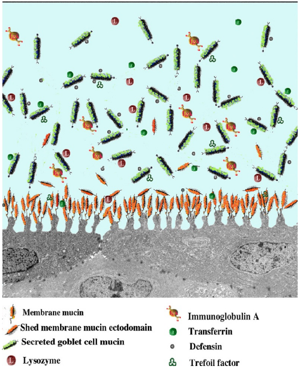
Lipid layer floating fatty/oil slick at interface with air

Aqueous mostly water between lipid and ocular surface

Ocular surface Mucus-rich region and microplicae at epithelium



Gipson (rabbit)

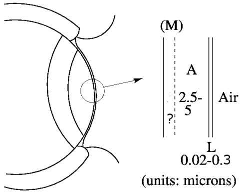


Govindarajan

Idealizing the Tear Film?

Tear film
A multilayer structure playing a vital role in health and function of the eye.
Millions affected by problems with tear film: dry eye.

Precorneal tear film breakup DEWS 07: Important for dry eye Osmolarity (salt concentration) increased from evaporative thinning
Osmosis from cornea possible



Typical thickness of each layer in microns.

- M: Mucus-rich region, glycocalix and microplicae
- A: Aqueous layer, primarily water (est. up to 98%). Salts/sugars in A important: osmolarity.
- L: Lipid layer, polar surfactants at the A/L interface.

Important ingredients

- Leveling due to surface tension
  Orchard, Schwartz, Roy,others

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  Gaffney et al (2009); Bron et al (02,...), Zubkov et al in progress
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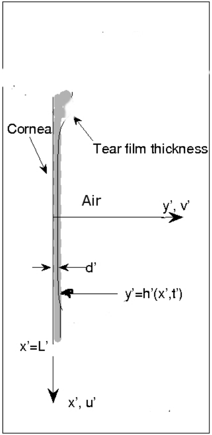
- Fluorescein for visualizing thickness
  e.g. OSU, IU, others.

Modeling Choices

Idealized domain



Aqueous fluid: Newtonian (water)

Mucus/cornea: wetting and osmosis BC

Lipid layer: BCs (Tangentially immobile; slows evaporation)

Rate of evaporation for flat surface fit to OSU thinning rates

Characteristic length scales

For x' direction:

$$L' = 5mm,$$ half width of palp. fissure.

For y' direction:

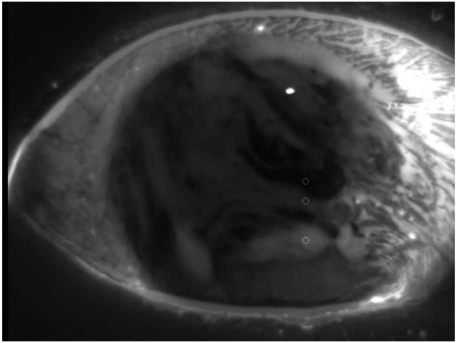
$$d' = 5\mu\text{m},$$ thickness of film.

The ratio of length scales $\epsilon = d'/L' \approx 10^{-3} \Rightarrow$ lubrication.

Prior Work: Measurements

Evidence of hydraulic connectivity

- Lampblack moving between menisci after blink (Maurice 73)
- Fluorescein moves more slowly superiorly and more rapidly inferiorly (Harrison *et al* 08)
- King-Smith imaging of fluorescein (09)



Tear Film Evolution Model

The evolution of the free surface is given by

$$h_t + \nabla \cdot \left[-\frac{h^3}{12} \nabla (p + Gy) \right] = 0, \quad p + S\Delta h = 0.$$

$$S = \frac{\epsilon^3 \sigma}{\mu U} = 10^{-5}, \quad G = \frac{\rho g d^2}{\mu U} = 0.025.$$

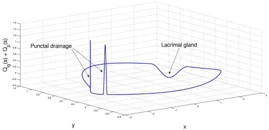
Boundary conditions

Fix TMW:

$h|_{\partial\Omega} = h_0$, where $h_0 = 13$.

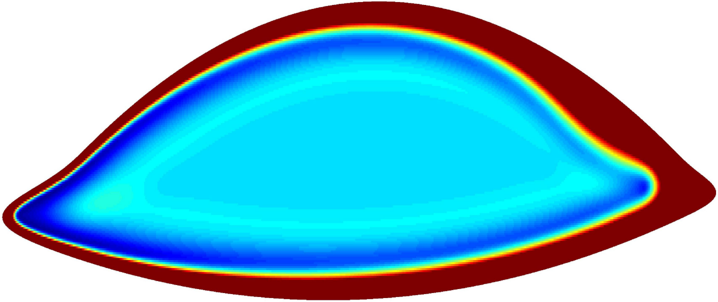
Specify flux at boundary:

$\mathbf{n} \cdot \left[-\frac{h^3}{12} \nabla (p + Gy) \right] = 0$
or a specified function of position only



Tear Flux: nonzero flux bc ($G = 0$)

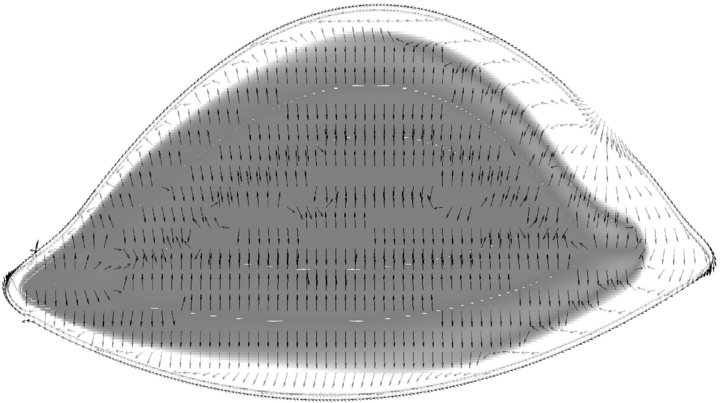
Tear film thickness at 10 seconds:



Flux from upper lid splits.
Some hydraulic connectivity.
Maki et al JFM 647, 2010.

Tear Flux: nonzero flux bc ($G = 0$)

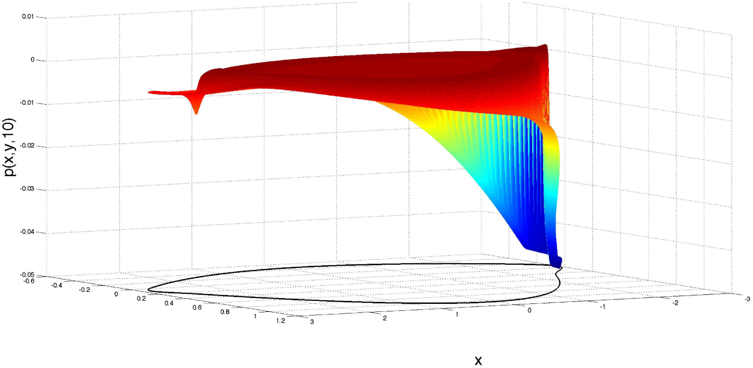
Flux vector field at 10 seconds:



Black line being pushed out of way.
Some hydraulic connectivity.

Tear Flux: nonzero flux bc ($G = 0$)

Pressure field at 10 seconds:



Dramatic steepening near puncta limits calculation.

Other work

- Comparison with partial blink thickness data
  Heryudono et al, Math Med Biol 2007
- Comparison with thickness measurements with reflex tearing
  Maki et al, Math Med Biol 2008
- Thermal modeling to capture cooling of ocular surface
  Li and Braun (11, submitted)

Part I: Thoughts on leveling

(Braun (UD), King-Smith (OSU))

Linear stability

- Dimensional lubrication theory: $\partial_{t'} h' + \partial_{x'} \left[\frac{(h')^3}{3} \frac{\sigma}{\mu} \partial^3_{x'} h' \right] = 0.$

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$$\tilde{h}(x,t) = A_1 \exp\left[-\frac{d^3 \sigma}{3\mu} \left(\frac{2\pi}{\lambda} \right)^4 t \right] \cos\left(\frac{2\pi}{\lambda} x \right)$$

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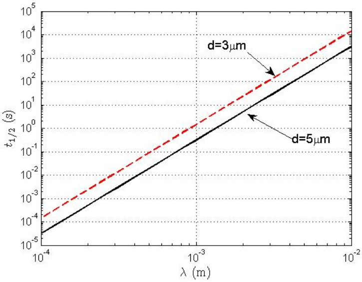
- Half life for decay is

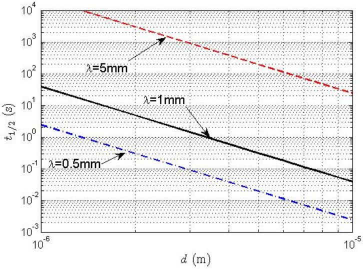
$$t_{1/2} = \frac{3\mu}{d^3\sigma} \left(\frac{\lambda}{2\pi} \right)^4 \ln 2$$

- Consequences?

Parametric dependence of decay rates in linear theory



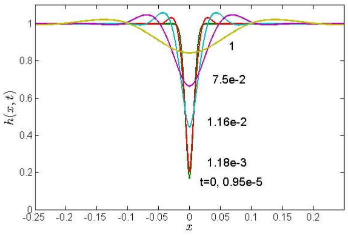
$t_{1/2}$ *vs.* λ

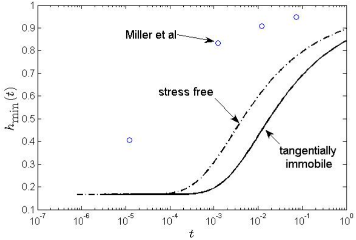


$t_{1/2}$ *vs.* d

- Note that one order of magnitude change in λ changes decay by 4 orders
- Example: 0.5mm is half of meibomium orifice spacing; 0.5cm is half of palpebral fissure
- Also proportional to d^{-3}
- Evaporating film can slow down decay

Nonlinear leveling of Gaussian valley





- Solving: $\partial_t h + \partial_x \left[\frac{h^3}{3} S \partial_x^3 h \right] = 0.$
- $d = 6.6\mu$m film after Miller et al (02); min film thickness 1.1μm
  $S = 1.38 \times 10^{-5}$, 1 std dev 0.025
- 1 second to recover to 0.8; 500 times slower than Miller et al (02)
- What if evaporation is happening too? Better thickness value?

Wetting film with evaporation: lubrication theory

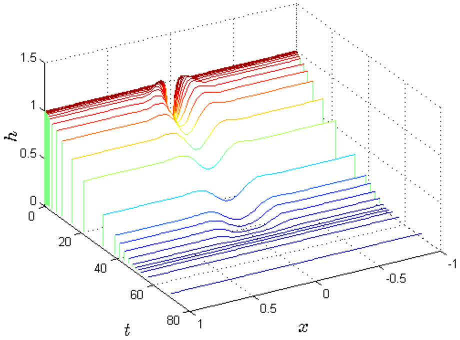
Still looking for h:

$$h_t + (\bar{u}h)_x = -EJ, \quad \bar{u} = -\frac{h^2}{3} p_x,$$

$$J = \frac{1}{\bar{K} + h}\left[1 + \delta p\right], \quad p = -Sh_{xx} - Ah^{-3}.$$

- Based on Ajaev & Homsy (01,05), Winter et al (2010)
- For $d = 3.5\mu$m, $S \sim 3 \times 10^{-6}$ is surface tension
- $E \sim 241$, $\bar{K} \sim 1.8 \times 10^4$ spec evap rate
- $A \sim 6.1 \times 10^{-6}$ is nondim'l Hamaker constant (conjoining)
- $\delta \sim 38$ is pressure contribution

Evaporation and tear film break up



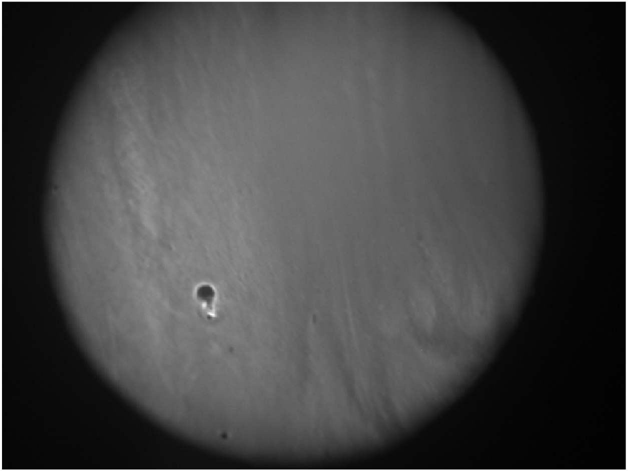
- $d = 3\mu$m film, std deviation 0.026 (width is one meibomium orifice)
- Deeper valley first rises, but still remains as entire film thins
- Disturbance mostly healed, but still was first place to break up
- Does this happen when we try to incorporate lipid layer, etc?

Part II: Evaporation, surfactant and osmolarity

(Braun, Siddiqui (PSU York), King-Smith (OSU))

Lipid layer dynamics: low mag

Interferometry (narrow band) for lipid layer thickness



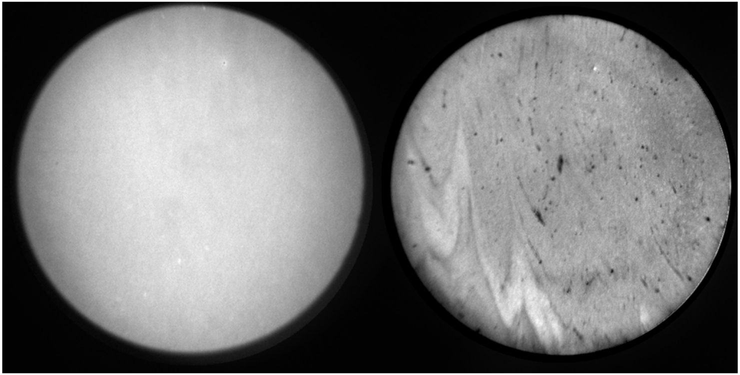
Tear film break up and lipid layer

No surfactant visible, and second fluid layer, but...

- Upward motion following a blink seen as Marangoni effect
  Berger & Corrsin (74), Owens & Philips (01), Jones et al (06), King-Smith et al (08)
- Burst of bubble causes spreading
  overall outline, Williams and Jensen (93), Zubkov et al (12)
- Complex pattern at upper edge
  Matar and Troian (90s), Matar et al
- Repeatability of pattern: why?

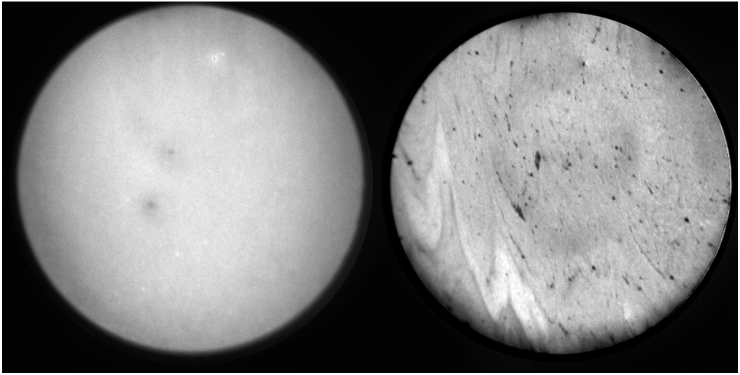
On to first try for lipid layer and evaporation...

Tear film break up and lipid layer



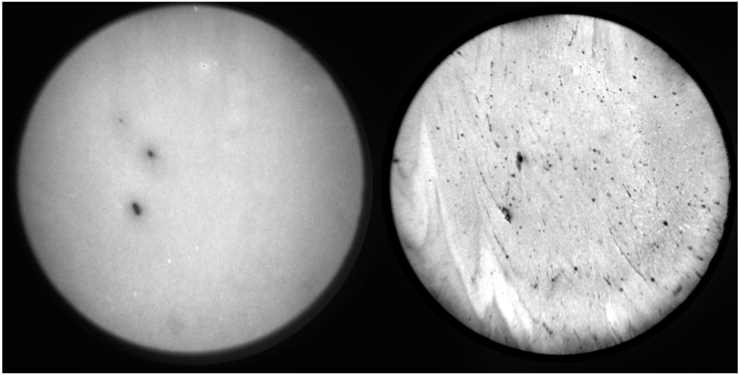
- Left: fluorescein; right: lipid interferogram
- 9 seconds after a blink
- Note two larger dark holes just left of center in lipid image

Tear film break up and lipid layer



- Left: fluorescein; right: lipid interferogram
- 15 seconds after a blink
- Dimming in fluorescein images under dark holes in lipid image

Tear film break up and lipid layer



- Left: fluorescein; right: lipid interferogram
- 20 seconds after a blink
- Dark patches in left image are break up under dark holes in lipid image
- First attempt at lipids effect on evaporation rate follows

Lubrication theory: leading order system of equations

Variables are free surface h, osmolarity c, surfactant concentration Γ:

$$h_t + (\bar{u}h)_x = -EJ + P_c(c-1), \quad \Gamma_t + (u_s\Gamma)_x = (Pe_s)^{-1}\Gamma_{xx}$$
$$h(c_t + \bar{u}c_x) = \mathrm{Pe}{-1}\,(hc_x)_x + EJc - P_c(c-1)c,$$
$$J = \frac{1}{\bar{K}+h}\left[1 + \delta p - \beta\Gamma\right], \quad \bar{u} = -\frac{h^2}{3}p_x - \frac{h}{2}M\Gamma_x,$$
$$u_s = -\frac{h^2}{2}p_x - M\Gamma_x h, \qquad p = -Sh_{xx} - Ah^{-3}.$$

- For $d = 3.5\mu$m, $S \sim 3 \times 10^{-6}$ is surface tension
- $E \sim 241$, $\bar{K} \sim 1.8 \times 10^4$ spec evap rate
- $A \sim 6.1 \times 10^{-6}$ is nondim'l Hamaker constant (conjoining)
- $\delta \sim 38$ is pressure contrib, β is surfactant effect
- Pe$\sim 10^4$ is Péclet in film, Pe$\sim 10^4$ is surface Péclet
- $P_c \sim 0.02$ is the nondim'l permeability of cornea (more below)

Constitutive eqn for evaporation

Evaporation from the film surface is hypothesized to be (nondimensional):

$$\bar{K}J = \delta p + T - \beta\Gamma$$
$$p = -Sh_{xx} - Ah^{-3}$$

- Surfactant concentration Γ lowers evaporation rate
- Linearized
- Equilibrium for flat, uniform solution, Γ a parameter,

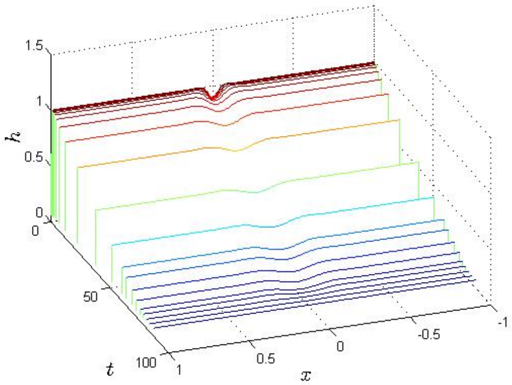
$$h_{eq}c_{eq} = 1 \text{ for our ICs}$$
$$\frac{E}{\bar{K} + h_{eq}}\left[1 - \delta Ah^{-3} - \beta\Gamma\right] = P_c\left(c_{eq} - 1\right)$$

- If $P_c = 0$,

$$h_{eq} = \left(\frac{\delta A}{1 - \beta\Gamma}\right)^{1/3}$$

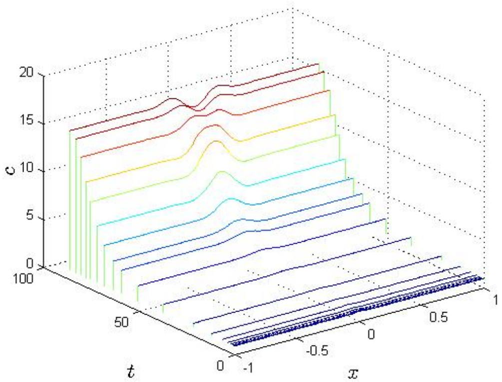
- $\beta = 0$ is Ajaev and Homsy result

Results: Thickness, $P_c = 0$, $\beta = 0.1$, $M = 0.01$



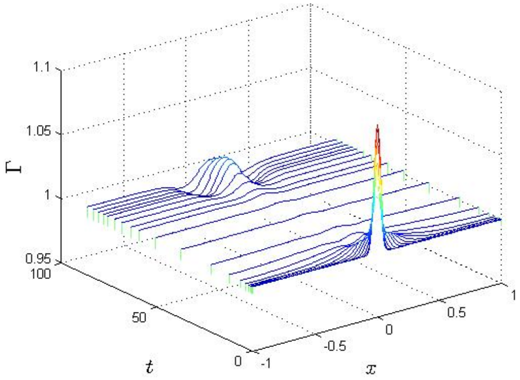
- $h(x, t)$, time increasing toward viewer
- IC has $h(x, 0)c(x, 0) = 1$, $h(x, 0) = 1 - \epsilon_1 e^{x^2/2/0.026^2}$, $\epsilon_1 = 0.25$, $\Gamma(0, 0) = 1.1$
- Dent in film decays slowly, first location for breakup

Osmolarity variation



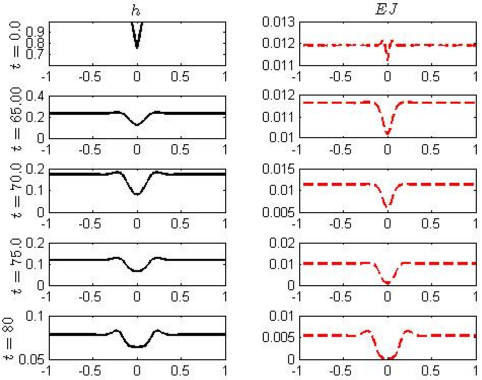
- $c(x,t)$ for $P_c = 0$, $\beta = 0.1$, $M = 0.01$, time increasing away from viewer
- IC has $h(x,0)c(x,0) = 1$, $h(x,0) = 1 - \epsilon_1 e^{x^2/2/0.026^2}$, $\epsilon_1 = 0.25$, $\Gamma(0,0) = 1.1$
- c increases due to conservation, decays slowly to constant

$\Gamma(x,t)$ for $P_c = 0$, $\beta = 0.1$ and $M = 0.01$



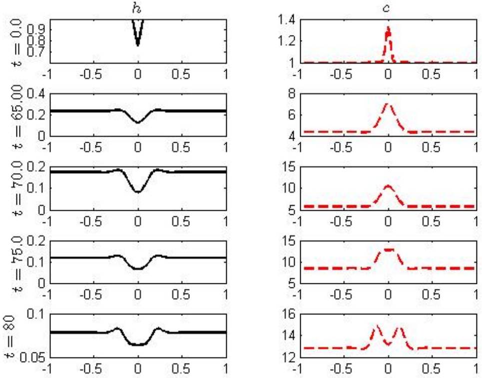
- IC has $\Gamma(x,0) = 1 + 0.1e^{x^2/2/0.026^2}$, $\Gamma(0,0) = 1.1$
- $\Gamma(x,t)$ has rapid decay compared to other variables
- Some perturbation when $h \to h_{eq}$ if $M \ll 1$

h and EJ for $P_c = 0$, $\beta = 0.1$, $M = 0.1$



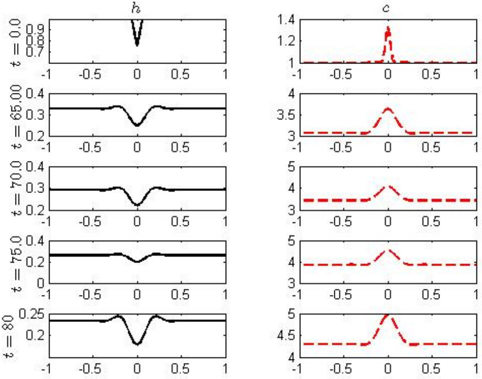
- $h \to h_{eq}$ and evaporations shuts off
- Break up region spreads as in Winter et al (10)

h and c for $P_c = 0$, $\beta = 0.1$, $M = 0.1$



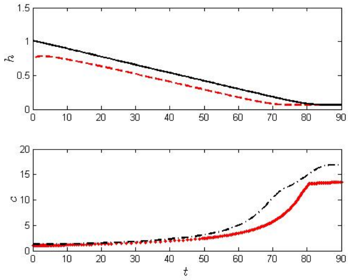
- c tending to constant but slowly due to small diffusion
- Have not confirmed by computing to very long times

h and c for $P_c = 0.00206$, $\beta = 0.1$, $M = 0.1$

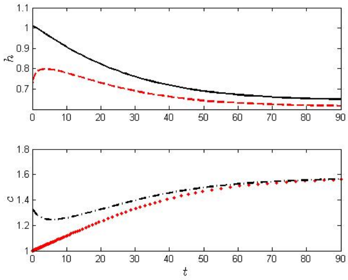


- Larger final tear film thickness
- M-shape at late times in $c(x,t)$ is lost

Max and Min of h, c for $\beta = 0.1$, $M = 0.1$



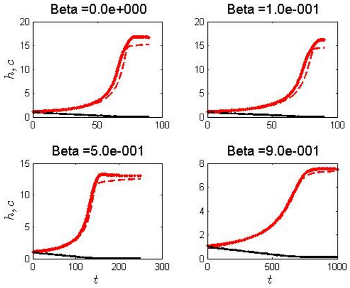
$P_c = 0$



$P_c = 0.0206$

- If $P_c = 0$, then the c difference persists
- If $P_c \neq 0$, then the h difference persists

Min and Max of h, c for $M = 0.1$ and vary β



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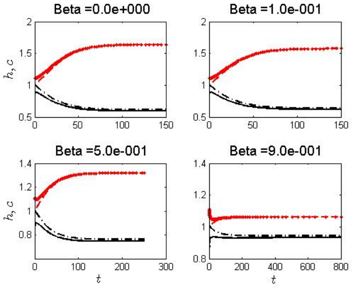
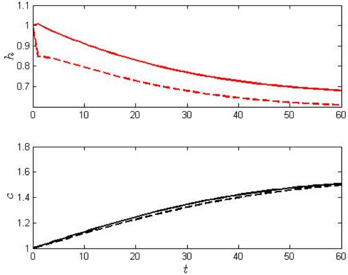
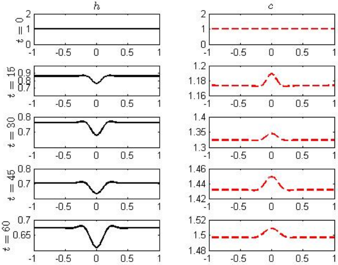


$P_c = 0.0206$

- If $\beta \leq 0.1$, then small effect
- β can't be too large or J switches sign





$P_c = 0.0206$ $\qquad$ $P_c = 0.0206$

- Uniform initial h and c develop valley, peak respectively
- Thinning freezes in valley again

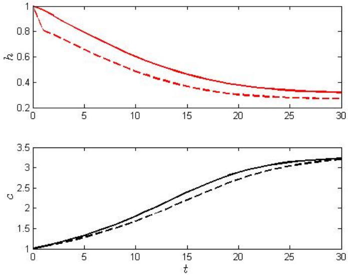
3.5μm tear film with 10μm/min thinning rate

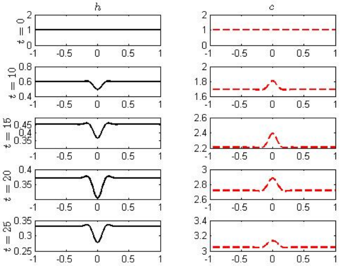
High rate (avg) seen in Nichols et al (05)

$$
\begin{aligned}
&h_t + (\bar{u}h)_x = -EJ + P_c(c-1), \quad \Gamma_t + (u_s\Gamma)_x = (Pe_s)^{-1}\Gamma_{xx} \\
&h\left(c_t + \bar{u}c_x\right) = \mathrm{Pe}{-1}\left(hc_x\right)_x + EJc - P_c(c-1)c, \\
&J = \frac{1}{\bar{K}+h}\left[1 + \delta p - \beta\Gamma\right], \quad \bar{u} = -\frac{h^2}{3}p_x - \frac{h}{2}M\Gamma_x, \\
&u_s = -\frac{h^2}{2}p_x - M\Gamma_x h, \qquad p = -Sh_{xx} - Ah^{-3}.
\end{aligned}
$$

- $S \sim 3 \times 10^{-6}$ is surface tension
- $E \sim 241$, $\bar{K} \sim 4.6 \times 10^3$ spec evap rate
- $A \sim 6.1 \times 10^{-6}$ is nondim'l Hamaker constant (conjoining)
- $\delta \sim 38$ is pressure contrib, $\beta = 0.1$ here
- Pe$\sim 10^4$ is Péclet number in film, Pe$_s \sim 10^4$ is surface Péclet
- $P_c \sim 0.02$ is the nondim'l permeability of cornea (more below)

$h(x,0) = c(x,0) = 1$, $M = 0.1$, $\beta = 0.1$, $\Gamma(0,0) = 1.5$





$P_c = 0.0206$ $P_c = 0.0206$

- Uniform initial h and c develop much faster
- Thinning freezes in features quicker
- If uniform initial h and c with dip in Γ, no dip in h
- Dip in initial h and Γ, can end with dip in h there

Summary for this part

- Simple model with evaporation plus wetting including surfactant
  - Evaporation slowed if Γ increased
  - Marangoni effect more important in this model for getting breakup at specific location for flat initial h and c
  - Dip in initial h can overcome dip in Γ
- Could use more physics and 2D computations: second layer, etc

Part III: A start at two-layer dynamics

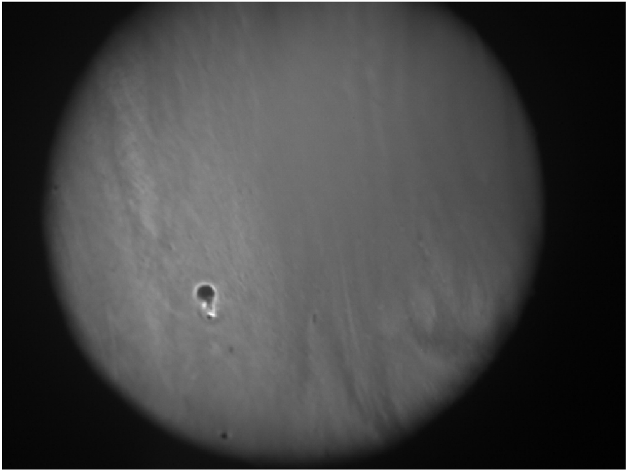
(Braun, Gewecke (UD), Breward (Oxford), King-Smith (OSU))

Lipid microscope images

- King-Smith developed lipid microscope (The Ocular Surface, 2011)
- about 1μm depth of focus; thickness computed from reflectance
- Each run is 2000 images, only a few usable images from each
- almost no time sequence info available
- At low magnification, streaks, different directions, frequency, persistence
- Now looking with high mag; what is seen?
- Grayscale images collected from more than 400 subjects
  (with KK and JJ Nichols)

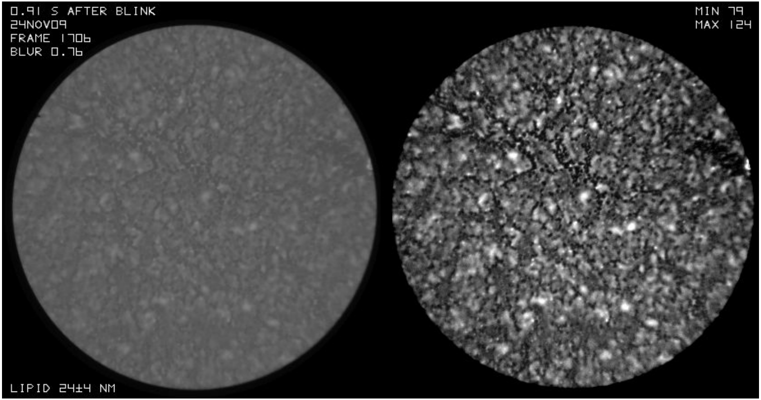
Lipid layer dynamics: low mag

Interferometry (narrow band) for lipid layer thickness



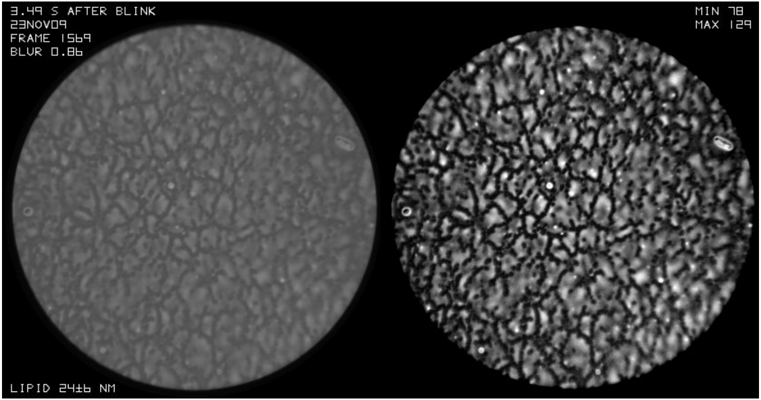
Lipid microscope images

Flux vector field at 1 seconds:



Not long after blink.
Small thin dark spots; lighter is thicker.

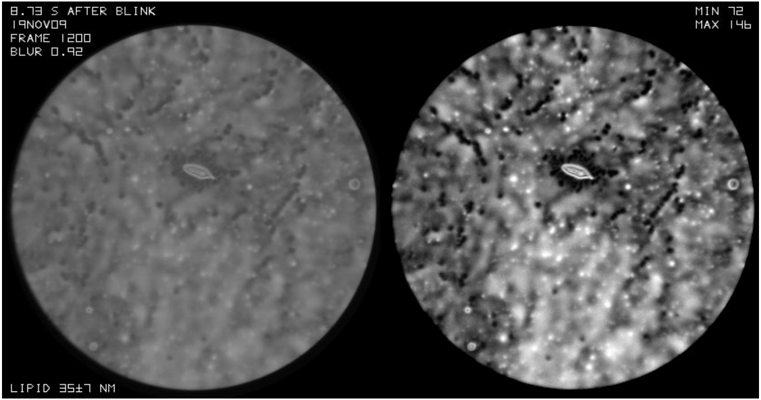
Flux vector field at 1 seconds:



Longer after blink.
More dark spots; they join together.

Lipid microscope images

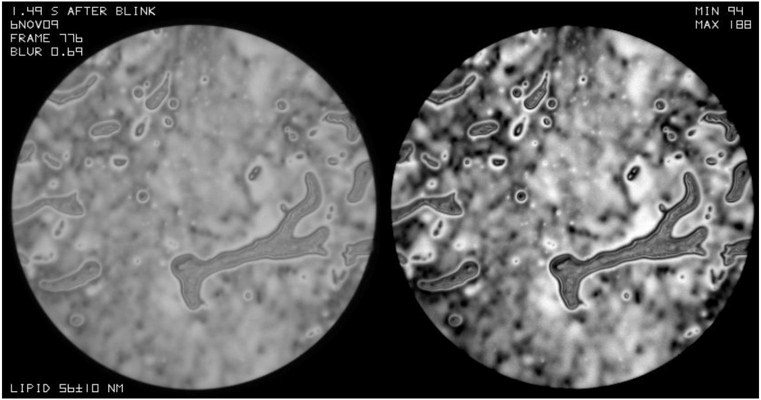
Flux vector field at 1 seconds:



Longer after blink.
More dark spots; they cluster here.

Lipid microscope images

Flux vector field at 1 seconds:



Big bright areas: nonpolar lipid drops?
Not Newtonian fluid: ignore for now

Lipid layer dynamics

- Polar lipids nominally at aqueous-lipid later interface
- Non-polar lipids floating on top: spreading, then dewetting?
- Surface active proteins like in lungs (SP-A,B,C,D) may be important: neglected
- Salts/osmolarity: not in first attempt
- Local rate of evaporation: not in starting results, but really want
- But: what are dominant ingredients for a model?

Lipid layer dynamics

- Start with liquid bilayer dewetting
  Matar et al (02), Pototsky et al (04,05), Fisher & Golovin (05), ...
- Let $h^{(1)}$ be aqueous thickness, $h^{(2)}$ be lipid thickness
- Total thickness is $h = h^{(1)} + h^{(2)}$
- Use van der Waals terms to get dewetting

$$
\begin{aligned}
\Pi^{(1)} &= A_1 \left(h^{(1)}\right)^{-3} + A_2 h^{-3} - A_4 \left(h^{(2)}\right)^{-3} \\
\Pi^{(2)} &= A_3 \left(h^{(1)}\right)^{-3} + A_4 \left(h^{(2)}\right)^{-3}
\end{aligned}
$$

- Dewetting to small nonzero thickness separating bumps seen with only these terms.
- But, the lipid layer is much thinner than the aqueous layer... necessitating different terms

Lipid layer dynamics

- Average lipid thickness 50 to 100 times less than aqueous thickness: $h^{(2)} \to \delta\epsilon h^{(2)}$, $\delta \sim 10^{-2}$
- Viscosity of upper layer is much larger; $\eta = \eta_2/eta_1 = \tilde{\eta}\epsilon^{-2}$
- Use van der Waals terms to get dewetting (Israelachvili 11)

$$\begin{aligned} \Pi^{(1)} &= A_1 \left(h^{(1)}\right)^{-3} - \left[A_4 \left(h^{(2)}\right)^{-3} + A_5\delta \left(h^{(2)}\right)^{-4}\right] \\ \Pi^{(2)} &= \frac{1}{\tilde{\eta}} \left[A_4 \left(h^{(2)}\right)^{-3} + A_5\delta \left(h^{(2)}\right)^{-4}\right] \end{aligned}$$

- Smaller thickness means terms with $h^{(2)}$ in denominator are dominant.
- Added short range terms to stabilize the lipid layer
- $A_1 \neq 0$ only if evaporation present (Ajaev & Homsy; Winter et al)
- Put together these contributions with extensional lipid layer Matar et al (02), Bruna-Estrach, Breward and Gaffney (09, 12)

Lubrication theory: leading order system of equations

Variables free surface h, osmolarity c:

$$h_t^{(1)} + \left(\bar{u}^{(1)} h^{(1)}\right)_x = 0, \quad h = h^{(1)} + \hat{\delta} h^{(2)},$$

$$h_t^{(2)} + \left(u^{(2)} h^{(2)}\right)_x = 0,$$

$$\bar{u}^{(1)} = -\left(p_x^{(1)} - \mathrm{St}\right) \frac{\left(h^{(1)}\right)^2}{12} + u^{(2)}/2,$$

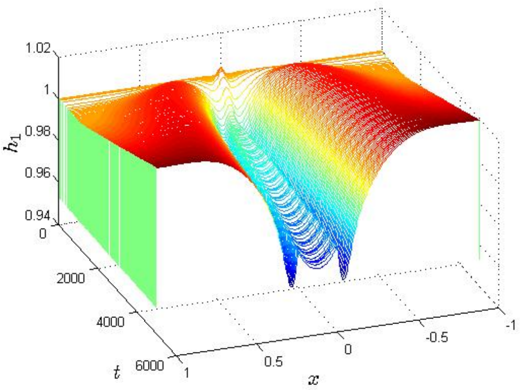
$$p^{(1)} = -Sh_{xx}^{(1)} - \Pi^{(1)} - \gamma S h_{xx} + \tilde{\eta} p^{(2)} + \tilde{\eta} 2 u_x^{(2)},$$

$$p^{(2)} = -2u_x^{(2)} - (\gamma S/\tilde{\eta}) h_{xx} - \Pi^{(2)},$$

$$\hat{\delta}\tilde{\eta}\left(4u_x^{(2)} h^{(2)}\right)_x = -\hat{\delta}\left[\tilde{\eta}\Pi_x^{(2)} + \gamma S h_{xxx} + \rho \mathrm{St}\right] h^{(2)} + \frac{4u^{(2)} - 6\bar{u}^{(1)}}{h^{(1)}}$$

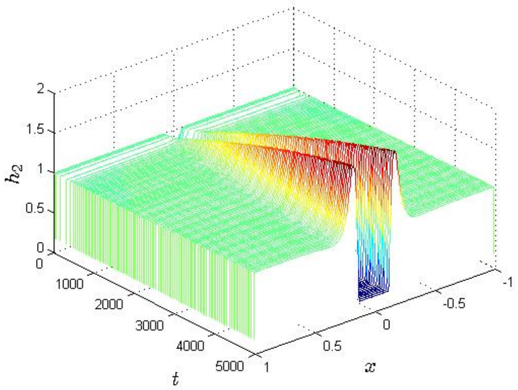
- No evaporation, surfactant or osmolarity
- Typical $d = 3.5\mu$m, $\bar{h}_2 = 140$nm, $\delta = 0.04$
- $\sigma = \sigma_2/\sigma_1 = 18/27$; $\tilde{\eta} = 0.01$
- $A_5 = -r_{eq} A_4 \sim 10^{-4}$; $A_4 = 25\times$ Israelachvili formula (naive)
- $r_{eq} = 1/4$ or $1/5$

Results: 2 layer, no evap, surfactant or osmolarity



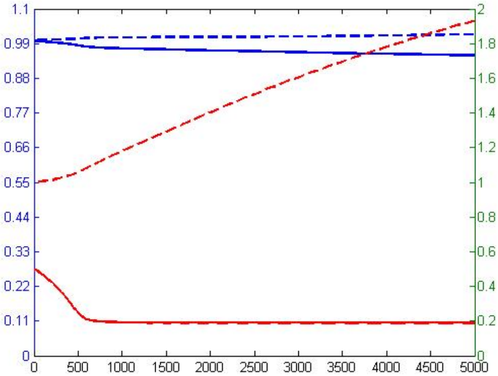
- $h_1(x, t)$, aqueous layer, time increasing toward viewer
- Note scale; perturbation to thickness
- No breakup

Results: 2 layer, no evap, surfactant or osmolarity



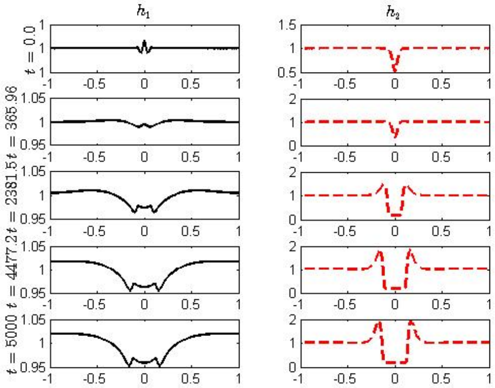
- $h_2(x, t)$, lipid layer, time increasing toward viewer
- Drops to 1/5 thickness, only a few molecules thick
- Dewets after minimum thickness found

Results: 2 layer, no evap, surfactant or osmolarity



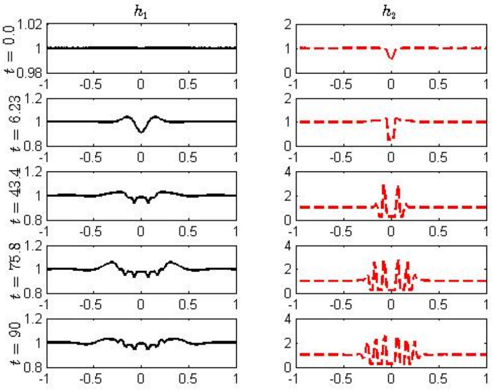
- Red (right axis): Lipid piles up and minimum becomes constant
- Blue (left axis): mild variation

Snapshots of h_i



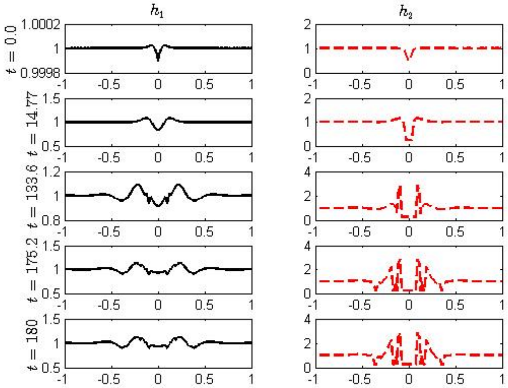
- Snapshots of thicknesses with time
-

Results: 2 layer, no evap, surfactant or osmolarity



- $d = 3.5\mu$m, $\bar{h}_2 = 140nm$, $\delta = 0.04$
- Pushing integration to 90s; note spreading of instability
- No breakup of aqueous: just perturbation
- Note extent of instability 1/3 of domain; spacing under 1mm

Results: 2 layer, no evap, surfactant or osmolarity



- $d = 2\mu$m, $\bar{h}_2 = 140nm$, $\delta = 0.07$
- Spreading of instability is slowed, smaller extent
- No breakup of aqueous: still perturbation
- Instability development is complicated

Summary for this part

- Simple dewetting model with tear film parameters
    - Increased naive estimate of vdW constant gives reasonable time scale
    - Reasonable thickness for small and thick regions
    - Instability of lipid layer can spread from one defect
    - Spacing a little smaller than 1mm holes spacing on lid margins
- Could use more physics and 2D computations

Lubrication theory: with surfactant now

Variables are free surface h, osmolarity c, surfactant concentration Γ:

$$h_t^{(1)} + \left(\bar{u}^{(1)} h^{(1)}\right)_x = 0, \quad h = h^{(1)} + \hat{\delta} h^{(2)},$$

$$\bar{u}^{(1)} = -p_x^{(1)} \frac{\left(h^{(1)}\right)^2}{12} + u^{(2)}/2,$$

$$h_t^{(2)} + \left(u^{(2)} h^{(2)}\right)_x = 0,$$

$$p^{(1)} = -S h_{xx}^{(1)} - \Pi^{(1)} - \gamma S h_{xx} + \tilde{\eta} p^{(2)} + \tilde{\eta} 2 u_x^{(2)},$$

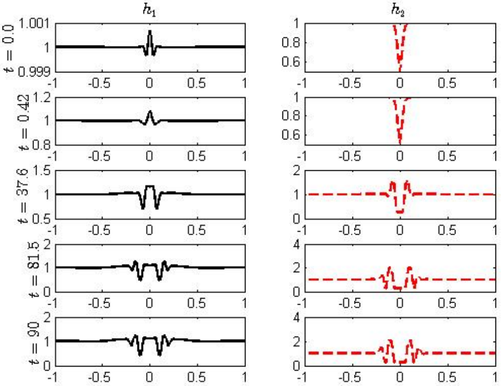
$$p^{(2)} = -2 u_x^{(2)} - (\gamma S/\tilde{\eta}) h_{xx} - \Pi^{(2)},$$

$$\hat{\delta} \tilde{\eta} \left(4 u_x^{(2)} h^{(2)}\right)_x = M \Gamma_x - \hat{\delta} \left[\tilde{\eta} \Pi_x^{(2)} + \gamma S h_{xxx} + \rho \mathrm{St}\right] h^{(2)} + \frac{4u^{(2)} - 6\bar{u}^{(1)}}{h^{(1)}}$$

$$\Gamma_t + \left(u^{(2)} \Gamma\right)_x = (Pe_s)^{-1} \Gamma_{xx}$$

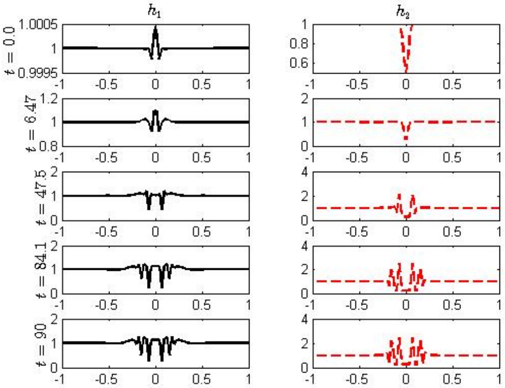
- No evaporation or osmolarity
- For $d = 3.5\mu$m, $S \sim 10^{-6}$ is surface tension
- $A_5 = -r_{eq} A_4 \sim 10^{-4}$ are nondim'l Hamaker constants
- $\delta = 0.05$, $r_{eq} = 1/5$

Snapshots of $h^{(i)}$



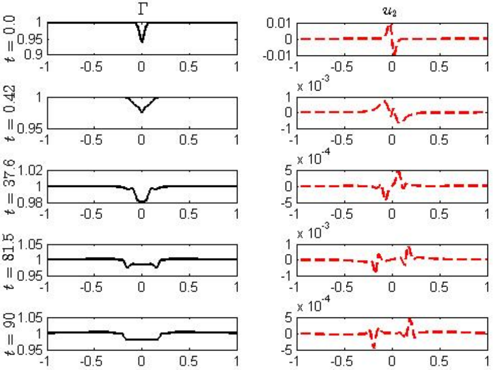
- $d = 3.5\mu$m, $\bar{h}_2 = 140$nm, $\hat{\delta} = 0.04$
- Spreading of instability on reasonable time scale
- Significant effect on underlying layer

Snapshots of $h^{(i)}$



- $d = 2\mu$m, $\bar{h}_2 = 100$nm, $\hat{\delta} = 0.05$
- Spreading of instability slowed significantly
- Complicated effect on underlying layer

Snapshots of Γ and $u^{(2)}$



- $d = 3.5\mu$m, $\bar{h}_2 = 140$nm, $\hat{\delta} = 0.05$
- Gradients in Γ line up well with change in $u^{(2)}$
- Significant effect on underlying layer

Summary and Future directions

- Today: no menisci
- Evaporation can freeze in features
- First break up locations?
- Surfactant models for lipid layer appear to have mixed results
- 2 layers models just started
- Future/other directions
  - More physics/chemistry in 2D on eye shape: with Li
  - Moving geometry for blinks
  - contintue two layer models
  - Wetting, osmosis, fluorescein: with Begley et al
- Recent review: Annual Review of Fluid Mechanics, 2012

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- Thank You!