A Lie-Poisson structure (and integrator) for the reduced n-body problem

Holger Dullin, University of Sydney
Toronto 2012

preprint at http://arxiv.org/find/math/1/au:+Dullin_H/0/1/0/all/0/1

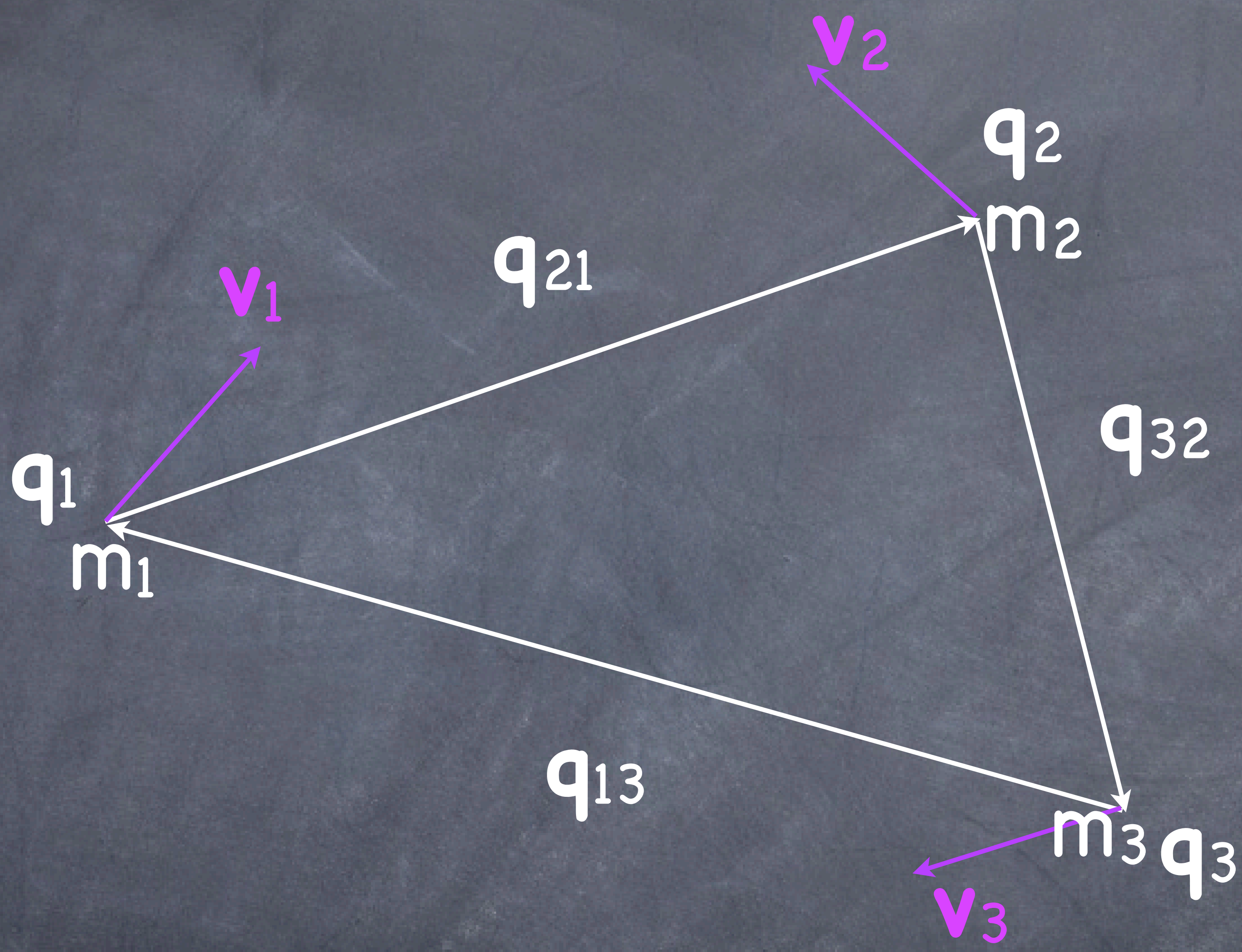
Plan of the Talk

- Introduction: Hamiltonian & Symmetries
- Reduction using Invariants
- The reduced Poisson structure
- Poisson Integrator
- Figure 8 in 18 steps: An example

n-body problem

- Celestial Mechanics (Solar system, n=9 or 10?)
- Classical Atoms (e.g. He, n=3)
- in particular n=3 bodies in d=3 dimensions

n=3, d=2,3



$$\dot{\mathbf{q}}_i = \mathbf{v}_i$$

$\mathbf{q}_{ij} = \mathbf{q}_i - \mathbf{q}_j$

$\mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j$

n-body Hamilton

$\mathbf{q}_i \in \mathbb{R}^d, \mathbf{p}_i \in \mathbb{R}^d, i = 1, \ldots, n$ canonically conjugate

$$H = \sum_{i=1}^{n} \frac{||\mathbf{p}_i||^2}{2m_i} + \sum_{1 \le i < j \le n} U_{ij}(||\mathbf{q}_i - \mathbf{q}_j||^2)$$

$$\begin{aligned} \dot{\mathbf{q}}_i &= \frac{\partial H}{\partial \mathbf{p}_i} = \mathbf{p}_i / m_i \\ \dot{\mathbf{p}}_i &= -\frac{\partial H}{\partial \mathbf{q}_i} = -\sum_{j \ne i} (\mathbf{q}_i - \mathbf{q}_j) U'_{ij}(||\mathbf{q}_i - \mathbf{q}_j||^2) \end{aligned}$$

n-body Poisson bracket

$$X = (\mathbf{q}_1, \ldots, \mathbf{q}_n, \mathbf{p}_1, \ldots, \mathbf{p}_n) \in \mathbb{R}^{2nd}$$

$$\{f, g\} = (\nabla f, J \nabla g), \quad J = \begin{pmatrix} 0 & I_{nd} \\ -I_{nd} & 0 \end{pmatrix}$$

$$\dot{f} = \{f, H\} \quad \Rightarrow \quad \dot{X} = J \nabla_X H$$

OR

$$Y = (\mathbf{q}_1, \ldots, \mathbf{q}_n, \mathbf{v}_1, \ldots, \mathbf{v}_n) \in \mathbb{R}^{2nd}$$

$$\{f, g\} = (\nabla f, J_M \nabla g), \quad J_M = \begin{pmatrix} 0 & M_{nd} \\ -M_{nd} & 0 \end{pmatrix}$$

M_{nd} diagonal such that $\{(\mathbf{q}_i)_k, (\mathbf{v}_j)_l\} = \delta_{ij}\delta_{kl}/m_j$

$$\dot{f} = \{f, H\} \quad \Rightarrow \quad \dot{Y} = J_M \nabla_Y H$$

general Poisson

$$\{f, g\} = (\nabla f, B\nabla g) \qquad \dot{X} = B\nabla H$$

- B=B(X) is the Poisson structure matrix such that:
- {f,g} = -{g,f} (antisymmetric)
- {f,gh} = g{f,h} + {f,g}h (Leibniz identity)
- {f,{g,h}}+{g,{h,f}}+{h,{f,g}}=0 (Jacobi identity)
- rank B may not be full: Casimirs {., C} = 0

Symmetries

$$H = \sum_{i=1}^{n} \frac{||\mathbf{p}_i||^2}{2m_i} + \sum_{1 \le i < j \le n} U_{ij}(||\mathbf{q}_i - \mathbf{q}_j||^2)$$

- translations of q
- rotations of p and q
- boosts (symmetry of the ODE, not of above H)
- time translation (irrelevant for us)

n-body Galilean G(d) symmetries

- translations $\mathcal{T} : \mathbf{q}_i \mapsto \mathbf{q}_i + \xi, \quad \xi \in \mathbb{R}^d$

  momentum: $\mathbf{P} = \sum \mathbf{p}_i$

- rotations $\mathcal{R} : (\mathbf{q}_i, \mathbf{p}_i) \mapsto (R\mathbf{q}_i, R\mathbf{p}_i), \quad R \in SO(d)$

  angular momentum: $\mathbf{L} = \sum \mathbf{q}_i \wedge \mathbf{p}_i$

- boosts $\mathcal{B} : (\mathbf{q}_i, \mathbf{p}_i) \mapsto (\mathbf{q}_i + \mathbf{v}t, \mathbf{p}_i + m_i\mathbf{v}), \quad \mathbf{v} \in \mathbb{R}^d$

  COM: $\mathbf{C} = \sum m_i \mathbf{q}_i / M, \quad M = \sum m_i, \quad \dot{\mathbf{C}} = \mathbf{P}/M$

  B generated by MC-Pt

counting to 8 (n=3,d=3)

- nd=9 degrees of freedom (dof)
- 10 integrals (P, L, C, H), 2*9 - 10 = 8 (strange...)
- 9 dof - 3 dof (P) - 2 dof (L) = 4 dof, 4*2 = 8
- in general: n d - d - (d-1) = (n-2)d+1 dof

Symplectic reduction

of the n-body problem

- Marsden Weinstein 1972, for free actions; n-body problem see e.g. Littlejohn & Reinsch 1997
- Jacobi coordinates reduce translations
- rotations are more difficult to reduce, action of SO(d) is not free (collinear configurations!), hence singular reduction
- in general destroys simple form H = K(p) + V(q)

Invariants

(of the symmetry group, not of the dynamics)

- Hilbert-Weyl (1946) Theorem: A compact group acting linearly on a vector space has a ring of invariant functions with a finite Hilbert basis
- G. Schwarz (1975): A smooth invariant function is a smooth function of basic invariants
- Idea: use this basis of invariants as new coordinates

Poisson reduction

- Poisson manifold M, bracket {,}
- group action φ^g of G on M
- let $f,k : M \to R$ be invariant: $f \circ \varphi^g = f$, $k \circ \varphi^g = k$
- if φ^g is a Poisson map, then
  $$\{f,k\} = \{f \circ \varphi^g, k \circ \varphi^g\} = \{f,k\} \circ \varphi^g$$
  so that the Poisson bracket of invariants is invariant
- hence a complete set of invariants induces a reduced Poisson bracket

Reduction using Invariants

- no need to use canonical variables, so can use e.g. the Hilbert-Weyl invariants as new variables
- may preserve ability to use splitting method
- gives global description even for singular reduction, see e.g. Lerman, Montgomery & Sjamaar 1991.
- fixing Casimirs gives (singular) symplectic leaves
- n-body: Lagrange 1772, ..., Albouy & Chenciner 1998

Constructing the G(d) invariants

Invariants

of translations T and boosts B

$$\mathcal{T},\mathcal{B} \quad : \quad \mathbf{q}_{ij} = \mathbf{q}_i - \mathbf{q}_j, \quad i \neq j$$

$$\mathcal{T} \quad : \quad \mathbf{p}_i - \mathbf{p}_j, \quad \text{no good}$$

$$\mathcal{T},\mathcal{B} \quad : \quad \mathbf{v}_{ij} = \dot{\mathbf{q}}_i - \dot{\mathbf{q}}_j, \quad \text{better}$$

- 2(n-1) independent difference vectors
- take any fixed index j, e.g. j=1:
  gives basis for T & B invariant difference vectors
- now the SO(d) (compact!) action remains

Invariants

of T, B and rotations R

Take scalar products of difference vectors!

$$\mathcal{T},\mathcal{B},\mathcal{R} \quad : \quad \mathbf{q}_{kj}\cdot\mathbf{q}_{ij} \quad i\le k, i,k\neq j$$

$$\mathcal{T},\mathcal{B},\mathcal{R} \quad : \quad \mathbf{q}_{kj}\cdot\mathbf{v}_{ij} \quad i,k\neq j$$

$$\mathcal{T},\mathcal{B},\mathcal{R} \quad : \quad \mathbf{v}_{kj}\cdot\mathbf{v}_{ij} \quad i\le k, i,k\neq j$$

- total of (2n-1)(n-1) invariants, n=3: 3+4+3=10
- quadratic invariants form a vector space
- find “nice” basis; write H,L in invariants:

Hamiltonian in Invariants

$$\rho_{ij} = ||\mathbf{q}_{ij}||^2, \quad i < j \leq n$$
$$\nu_{ij} = ||\mathbf{v}_{ij}||^2, \quad i < j \leq n$$

$$U = U(\rho_{12}, \ldots, \rho_{n-1,n}) = \sum_{1 \leq i < j \leq n} U_{ij}(\rho_{ij})$$

$$K_c = \frac{1}{2}\sum_{i=1}^{n} m_i ||\dot{\mathbf{q}}_i - \dot{\mathbf{C}}||^2 = \frac{1}{2}\sum_{i=1}^{n} m_i ||\dot{\mathbf{q}}_i||^2 - \frac{1}{2} M ||\dot{\mathbf{C}}||^2$$
$$= \frac{1}{2M} \sum_{1 \leq i < j \leq n} m_i m_j \nu_{ij}$$

(kinetic energy relative to centre of mass)

Angular momentum in invariants

$$\mathbf{L}_c = \sum_{i=1}^{n} m_i(\mathbf{q}_i - \mathbf{C}) \wedge (\dot{\mathbf{q}}_i - \dot{\mathbf{C}})$$
$$= \frac{1}{M^2} \sum_{i,j,k} m_i m_j m_k \mathbf{q}_{ij} \wedge \mathbf{v}_{ik}$$

(angular momentum relative to centre of mass)

- Now compute scalar invariant $||\mathbf{L}_c||^2$
- expand (a∧b).(c∧d) = a.c b.d - a.d b.c
- thus quadratic in invariants

"nice" invariants:

$$
\begin{aligned}
\rho_{ij} &= ||\mathbf{q}_{ij}||^2, \quad i < j \leq n \\
\nu_{ij} &= ||\mathbf{v}_{ij}||^2, \quad i < j \leq n \\
\sigma_{ij} &= \mathbf{q}_{ij} \cdot \mathbf{v}_{ij}, \quad i < j \leq n
\end{aligned}
$$

- not quite enough: n=3: $3+3+3 < 10$
- additional invariant $\delta = \mathbf{q}_{23}\mathbf{v}_{13} - \mathbf{q}_{13}\mathbf{v}_{23}$ ("why" and general case later)
- ready for reduction: use these invariants as coordinates, compute Poisson brackets!

Exercise: 2 bodies

$$\rho_{12} = ||\mathbf{q}_{12}||^2, \quad \nu_{12} = ||\mathbf{v}_{12}||^2, \quad \sigma_{12} = \mathbf{q}_{12} \cdot \mathbf{v}_{12}$$

new variables: $Z = (\rho_{12}, \nu_{12}, \sigma_{12})$

Poisson bracket of Z is closed: (μ = reduced mass)

$$\{\rho_{12}, \nu_{12}\} = \frac{4}{\mu}\sigma_{12}, \quad \{\rho_{12}, \sigma_{12}\} = \frac{2}{\mu}\rho_{12}, \quad \{\nu_{12}, \sigma_{12}\} = -\frac{2}{\mu}\nu_{12}$$

hence define Poisson structure matrix for Z variables:

$$B = \frac{2}{\mu}\begin{pmatrix} 0 & 2\sigma_{12} & \rho_{12} \\ -2\sigma_{12} & 0 & -\nu_{12} \\ -\rho_{12} & \nu_{12} & 0 \end{pmatrix}$$

computing a Poisson bracket, e.g.

$$
\begin{aligned}
\{\rho_{12}, \nu_{12}\} &= \{||\mathbf{q}_1 - \mathbf{q}_2||^2, ||\mathbf{v}_1 - \mathbf{v}_2||^2\} \\
&= \sum 4((\mathbf{q}_1)_i - (\mathbf{q}_2)_i)\{(\mathbf{q}_1)_i - (\mathbf{q}_2)_i, (\mathbf{v}_1)_i - (\mathbf{v}_2)_i\}((\mathbf{v}_1)_i - (\mathbf{v}_2)_i) \\
&= \sum 4((\mathbf{q}_1)_i - (\mathbf{q}_2)_i)\left(\frac{1}{m_1} + \frac{1}{m_2}\right)((\mathbf{v}_1)_i - (\mathbf{v}_2)_i) \\
&= \frac{4}{\mu}(\mathbf{q}_1 - \mathbf{q}_2) \cdot (\mathbf{v}_1 - \mathbf{v}_2) = \frac{4}{\mu}\sigma_{12}
\end{aligned}
$$

independent of dimension d!

general computation done using quadratic forms

Poisson equations n=2

$$H_c = \frac{1}{2}\mu\nu_{12} - \mu\frac{M}{\sqrt{\rho_{12}}}$$

$$\dot{Z} = B\nabla_Z H_c = \begin{pmatrix} 0 & 2\sigma_{12} & \rho_{12} \\ -2\sigma_{12} & 0 & -\nu_{12} \\ -\rho_{12} & \nu_{12} & 0 \end{pmatrix} \begin{pmatrix} M/\rho_{12}^{3/2} \\ 1 \\ 0 \end{pmatrix}$$

$$||L_c||^2 = \mu^2(\rho_{12}\nu_{12} - \sigma_{12}^2)$$

$||L_c||^2$ is Casimir of B: $\{\ .\ ,||L_c||^2\} = 0$

so $\nabla||L_c||^2$ spans kernel of B (rank B = 2)

Geometry of Solutions (n=2)

- { Casimir = l^2 } one sheet of two-sheeted hyperboloid (for $l^2 > 0$)
- { $H_c = h$ } hyperbolic cylinder
- Intersection gives reduced trajectory
- Tangency corresponds to circular orbit

Poisson Structure for the reduced n-body problem

Theorem (Albouy & Chenciner 1998, HRD 2012):
The vector space of quadratic invariants has a basis $Z = (\rho_{ij}, \nu_{ij}, \sigma_{ij}, \delta_{ij}), \quad i < j \leq n, \quad i < j < n \text{ for } \delta$.
Z is closed under the original Poisson bracket.
The induced structure matrix B is linear in Z.
The induced bracket is isomorphic to the Lie-Poisson bracket of sp(2n-2).

Steps of the proof:

- Z_i are quadratic forms → symmetric matrices
- Algebra of symmetric matrices, independent of d,
  $A*B = (AJ_MB - BJ_MA) = 2[AJ_MB]_{sym}$
- Shift invariance of quadratic forms gives Laplacian block matrices (zero row sum of blocks)
- Composition * preserves Laplacian block structure
- Sub-algebra gives reduced Poisson Bracket
- It is Lie-Poisson (sp(2n-2)), so Jacobi identity holds

Poisson structure matrix

n=2

$$B_2 = \frac{2}{\mu}\begin{pmatrix} 0 & 2\sigma_{12} & \rho_{12} \\ -2\sigma_{12} & 0 & -\nu_{12} \\ -\rho_{12} & \nu_{12} & 0 \end{pmatrix}$$

$$1/\mu = 1/m_1 + 1/m_2$$

Casimir: Angular momentum

$$||L_c||^2 = \mu^2(\rho_{12}\nu_{12} - \sigma_{12}^2)$$

n=3

$$B_3 = \begin{pmatrix} 0 & 2L(\sigma) + 2\Delta(\delta) & L(\rho) & G(\rho) \\ -2L(\sigma) + 2\Delta(\delta) & 0 & -L(\nu) & G(\nu) \\ -L(\rho) & L(\nu) & \Delta(\delta) & G(\sigma) \\ -G(\rho)^t & -G(\nu)^t & -G(\sigma)^t & 0 \end{pmatrix}$$

where L, Δ are 3x3, L^t=L, Δ^t = -Δ, G is 3x1

L, G, Δ are linear in their argument

degree -1 in masses; 2 Casimirs

2 Casimirs (n=3):

1) angular momentum $|L_c|^2$ (a quadric in Z)

$$\frac{m_1 m_2 m_3}{2} \sum_{i<j} m_k((\rho_s - 2\rho_{ij})(\nu_s - 2\nu_{ij}) - (\sigma_s - 2\sigma_{ij})^2) +$$

$$+ \frac{m_1 m_2 m_3}{2} M \delta_{12}^2 + \sum_{i<j} m_i^2 m_j^2 (\rho_{ij}\nu_{ij} - \sigma_{ij}^2)$$

$$\rho_s = \sum_{i<j} \rho_{ij}, \quad \nu_s = \sum_{i<j} \nu_{ij}, \quad \sigma_s = \sum_{i<j} \sigma_{ij}$$

2) Gram determinant of difference vectors (a determinantal variety)

$$\det \begin{pmatrix} 2\rho_{12} & \rho_{23} - \rho_{12} - \rho_{13} & 2\sigma_{12} & \delta_{12} - \sigma_{12} - \sigma_{13} + \sigma_{23} \\ \cdot & 2\rho_{13} & -\delta_{12} - \sigma_{12} - \sigma_{13} + \sigma_{23} & 2\sigma_{13} \\ \cdot & \cdot & 2\nu_{12} & -\nu_{12} - \nu_{13} + \nu_{23} \\ \cdot & \cdot & \cdot & 2\nu_{13} \end{pmatrix}$$

d-Independence (n=3)

- The Poisson structure is independent of d
- the only way the reduced system "knows" about d is through the value (!) of the Gram Casimir
- rank of Gram matrix remembers d=1,2,3
- for d > 3 Gram determinant may be non-zero

the details...

$$L(a,b,c) = \begin{pmatrix} 2a/\mu_{12} & (a+b-c)/m_1 & (a-b+c)/m_2 \\ (a+b-c)/m_1 & 2b/\mu_{13} & (-a+b+c)/m_3 \\ (a-b+c)/m_2 & (-a+b+c)/m_3 & 2c/\mu_{23} \end{pmatrix}$$

$$G(a,b,c) = \left(-\frac{b-c}{\mu_{12}} - \frac{m_1-m_2}{m_1 m_2}a, \frac{a-c}{\mu_{13}} + \frac{m_1-m_3}{m_1 m_3}b, -\frac{a-b}{\mu_{23}} - \frac{m_2-m_3}{m_2 m_3}c\right)$$

$$\Delta(\delta) = \delta \begin{pmatrix} 0 & -1/m_1 & 1/m_2 \\ 1/m_1 & 0 & -1/m_3 \\ -1/m_2 & 1/m_3 & 0 \end{pmatrix}$$

$$1/\mu_{ij} = 1/m_i + 1/m_j$$

$$(a,b,c) = (\rho_{12}, \rho_{13}, \rho_{23}), (\nu_{12}, \nu_{13}, \nu_{23}), \text{ or } (\sigma_{12}, \sigma_{13}, \sigma_{23})$$

So what is δ?

row sum 0

invariants Q=(Y,AY), A is block Laplacian

$$A = \begin{pmatrix} R & S+D \\ S-D & P \end{pmatrix}$$

D is antisymmetric Laplacian, R, P, S symmetric Laplacian

δ_{ij} is a basis for this set of matrices!

For n=3 there is only one such matrix (up to scale):

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Corresponding invariant $Z_{10} = -q_{23} \cdot v_{13} + q_{13} \cdot v_{23}$

Splitting Integrator

- $H = K(\nu) + U(\varrho), \quad Z = (\varrho, \nu, \sigma, \delta)^t$
- take K as Hamiltonian: $Z' = B\,\nabla K$, solve: ϕ_K
- take V as Hamiltonian: $Z' = B\,\nabla U$, solve: ϕ_U
- construct approximation to (unknown) flow of H by composing the known flows ϕ_K and ϕ_U, which are Poisson maps
- composition of Poisson maps is a Poisson map: structure preserving integrator for the reduced n-body problem

ODE generated by K(ν)

$$\begin{pmatrix} \rho \\ \nu \\ \sigma \\ \delta \end{pmatrix}' = \begin{pmatrix} 0 & 2L(\sigma)+2\Delta(\delta) & L(\rho) & G(\rho) \\ -2L(\sigma)+2\Delta(\delta) & 0 & -L(\nu) & G(\nu) \\ -L(\rho) & L(\nu) & \Delta(\delta) & G(\sigma) \\ -G(\rho)^t & -G(\nu)^t & -G(\sigma)^t & 0 \end{pmatrix} \begin{pmatrix} 0 \\ K' \\ 0 \\ 0 \end{pmatrix}$$

flow of K

- K depends on ν only, linearly, ∇K = const
- linear equations $\dot\rho = 2\sigma, \dot\nu = 0, \dot\sigma = \nu, \dot\delta = 0$
- ν, δ constant, σ linear, ϱ quadratic in time
- flow $\phi^t_K(\rho,\nu,\sigma,\delta) = (\rho + 2t\sigma + t^2\nu, \nu, \sigma + t\nu, \delta)$
- Newton's first law in strange coordinates

ODE generated by U(ϱ)

$$\begin{pmatrix} \rho \\ \nu \\ \sigma \\ \delta \end{pmatrix}' = \begin{pmatrix} 0 & 2L(\sigma) + 2\Delta(\delta) & L(\rho) & G(\rho) \\ -2L(\sigma) + 2\Delta(\delta) & 0 & -L(\nu) & G(\nu) \\ -L(\rho) & L(\nu) & \Delta\ \ \delta) & G((\sigma) \\ -G(\rho)^t & -G(\nu)^t & -G(\sigma)^t & 0 \end{pmatrix} \begin{pmatrix} U'(\rho) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

flow of U

- U depends on ϱ only, but not linear
- non-linear equations (in ϱ), but ϱ is constant, so again easy to integrate
- ϱ constant, σ, δ linear, ν quadratic in time t

$$
\begin{aligned}
\phi_V^t(\rho, \nu, \sigma, \delta) &= (\rho, \nu + tc + t^2 d, \sigma + ta, \delta + tb) \\
a &= -L(\rho) U'(\rho) \\
b &= -G(\rho)^t U'(\rho) \\
c &= -2(L(\sigma) + \Delta(\delta)) U'(\rho) \\
d &= -(L(a) + \Delta(b)) U'(\rho)
\end{aligned}
$$

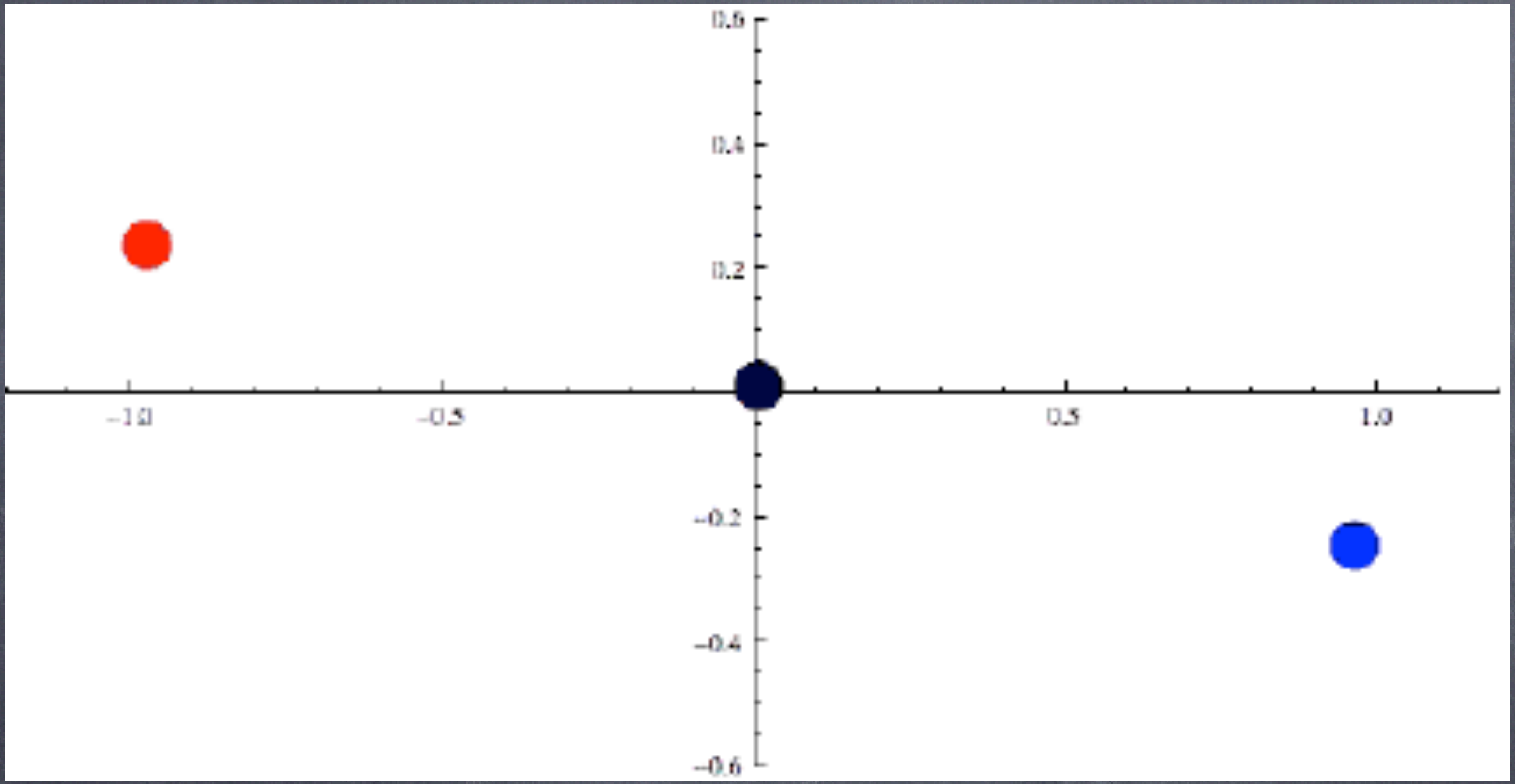
Poisson integrator

- composition $\phi_K \phi_U$ is first order integrator
- composition $\phi_U \phi_K$ is first order integrator
- composition of two 1/2 steps gives 2nd order reversible integrator, step size h:

$$\Phi^h = \phi_K^{h/2} \circ \phi_U^h \circ \phi_K^{h/2}$$

- higher order methods can be constructed (Yoshida '90)

Figure 8 choreography



Chenciner & Montgomery 2000, existence proof

animation from Chenciner's Scholarpedia article

figure 8 choreography, h=0.02

n=3, d=3, equal masses

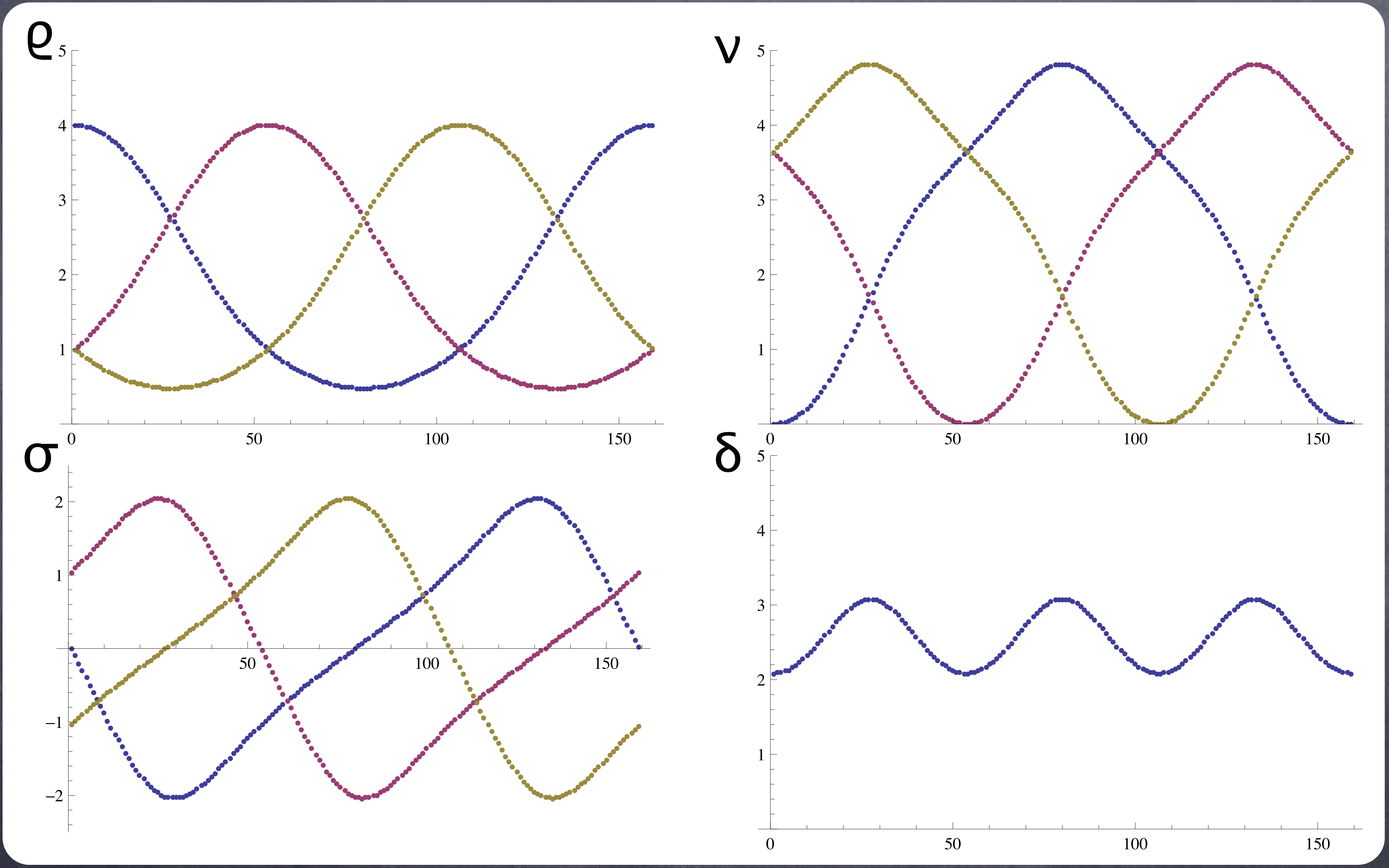


figure 8, h=0.02, one period

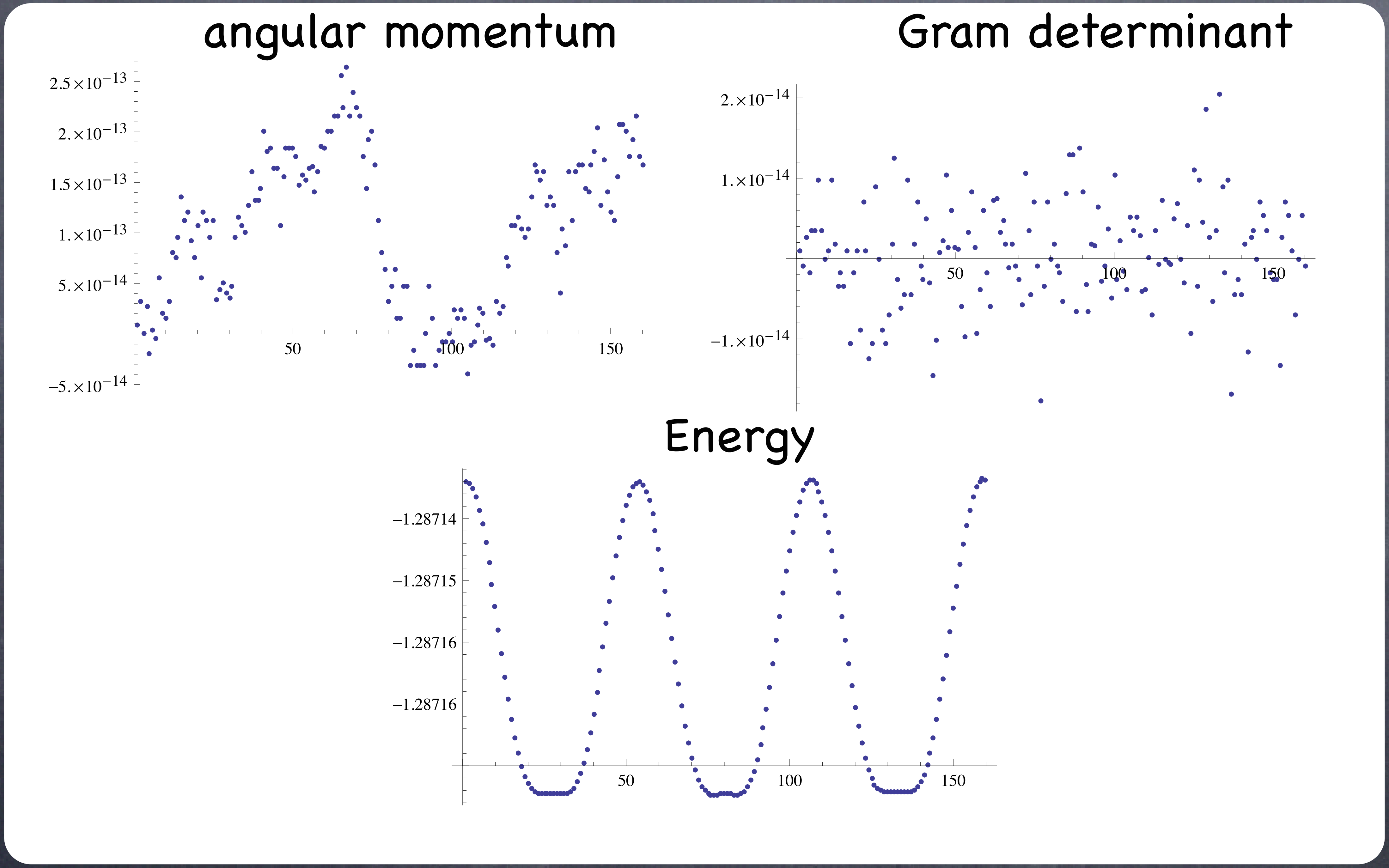


figure 8, h=0.02, 100 periods

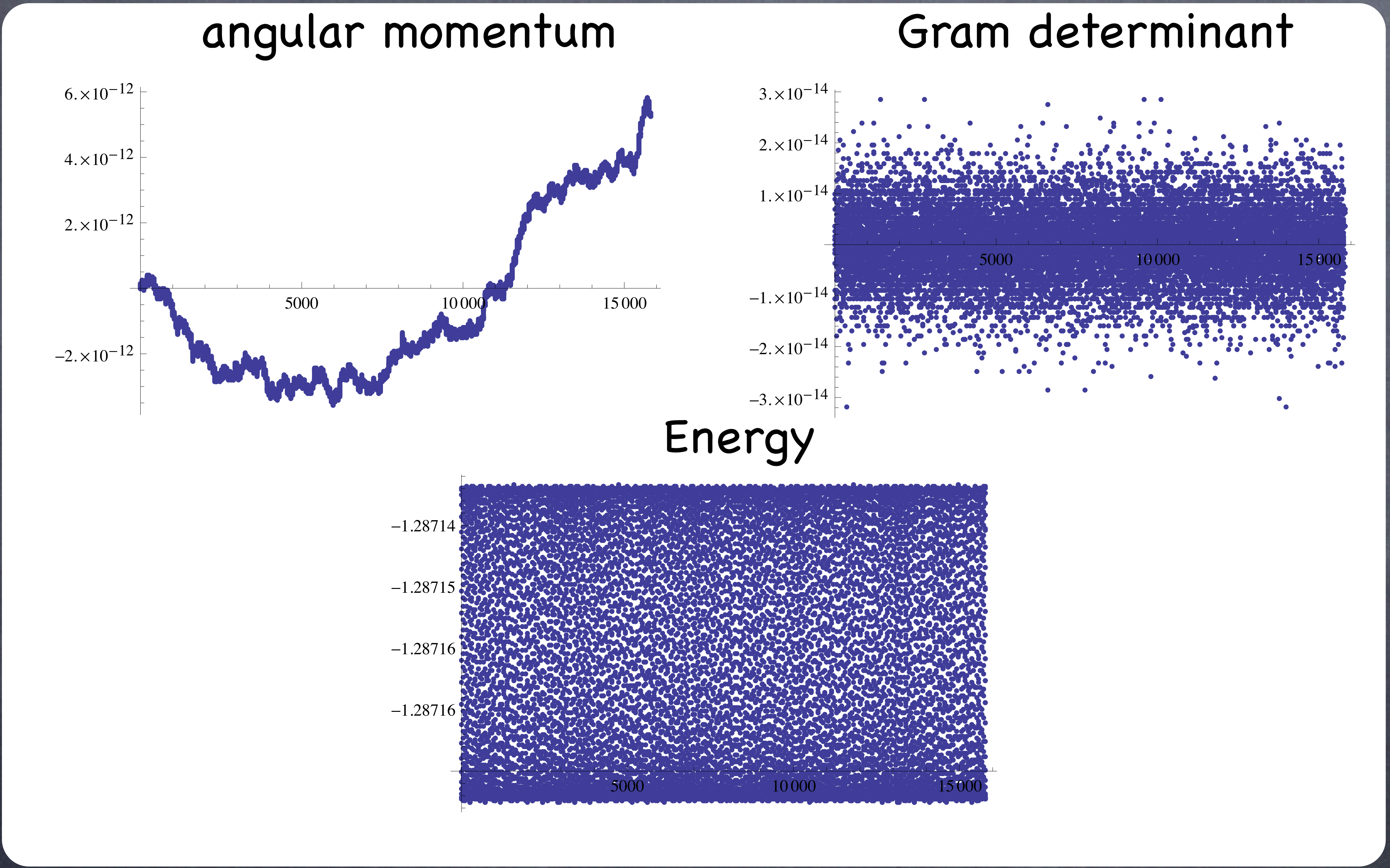
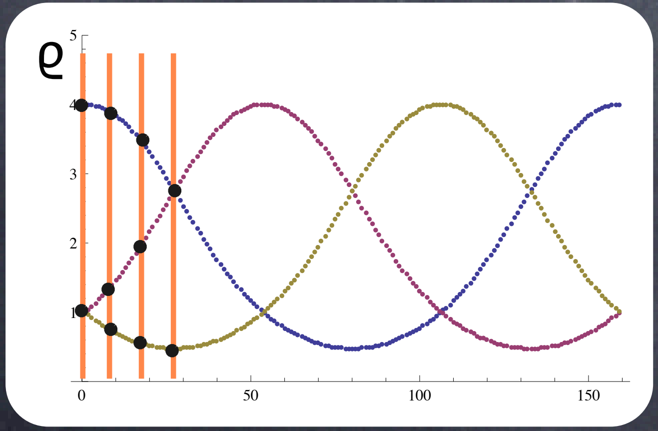


figure 8 in 18 steps, h=0.18

(huge!)

- period 18 orbit of ϕ^h with figure 8 symmetry
- multipliers are on unit circle (elliptic)
- 3 iterates, rest by symmetry (D_6/Z_2):

ρ_{12}	ρ_{13}	ρ_{23}	ν_{12}	ν_{13}	ν_{23}	σ_{12}	σ_{13}	σ_{23}	δ_{12}
4.10022	1.02505	1.02505	0	3.54541	3.54541	0	1.04418	−1.04418	2.08837
3.94247	1.48533	0.71439	0.191188	3.09007	4.03274	−0.867775	1.4924	−0.659803	2.31442
3.48824	2.10346	0.533511	0.779696	2.50767	4.54029	−1.62926	1.91548	−0.322236	2.8289
2.8214	2.8214	0.473492	1.66922	1.66922	4.78925	−2.03534	2.03534	0	3.12529



start: collinear
finish: equilateral

Conclusion

- Invariants are useful for reduction of the n-body problem
- They give a reduced Lie-Poisson structure that is independent of the spatial dimension d
- There is a Poisson structure preserving splitting integrator for the reduced n-body problem
- Efficiency gain for n=3, d>2?
- Project: Study singular reduction using invariants