Hamel's Formalism and Nonholonomic Mechanics

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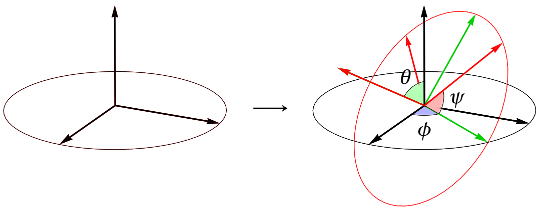
Motivation

- $L\colon TQ \to \mathbb{R}$ is the Lagrangian (kinetic minus potential energy), where Q is the configuration space
- The dynamics is given by the Euler–Lagrange equations $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$
- These equations appeared in Lagrange [1788]. Lagrange obtained them in an attempt to derive the equations of motion for mechanical systems in a covariant (i.e., coordinate-independent) form
- Substitutions are a major tool for studying differential equations. However, it may be impossible to find coordinates on the configuration space in which the Euler–Lagrange equations are sufficiently simple

Motivation

- For instance, consider the rotational dynamics of a rigid body. The configuration space is the **rotation group**, i.e., the group of orientation-preserving orthogonal matrices
- The **Euler angles** are often used as configuration coordinates



Motivation

- Using the Euler angles as local coordinates on the rotation group SO(3), the Euler–Lagrange equations for the Euler top are

$$
\begin{aligned}
&(J_1\cos^2\psi + J_2\sin^2\psi)\,\ddot\theta + (J_1 - J_2)\sin\theta\sin\psi\cos\psi\,\ddot\phi + (J_2 - J_1)\sin 2\psi\,\dot\theta\dot\psi \\
&-\left(J_1\sin^2\psi + J_2\cos^2\psi - J_3\right)\sin 2\theta\,\dot\phi^2 + \left(J_1\cos 2\psi - J_2\cos 2\psi + J_3\right)\sin\theta\,\dot\phi\dot\psi = 0, \\
&\left(J_1 - J_2\right)\sin\theta\sin\psi\cos\psi\,\ddot\theta + \left(J_1\sin^2\theta\sin^2\psi + J_2\sin^2\theta\cos^2\psi + J_3\cos^2\theta\right)\ddot\phi \\
&\quad + J_3\cos\theta\,\ddot\psi + \left(J_1 - J_2\right)\cos\theta\sin 2\psi\,\dot\theta^2 + \left(J_1\sin^2\psi + J_2\cos^2\psi - J_3\right)\sin 2\theta\,\dot\theta\dot\phi \\
&\qquad + \left(J_1\cos 2\psi - J_2\cos 2\psi - J_3\right)\sin\theta\,\dot\theta\dot\psi + \left(J_1 - J_2\right)\sin^2\theta\sin 2\psi\,\dot\phi\dot\psi = 0, \\
&J_3\left(\cos\theta\,\ddot\phi + \ddot\psi - \sin\theta\,\dot\theta\dot\phi\right) = 0,
\end{aligned}
$$

where J_1, J_2, and J_3 are the principal moments of inertia of the top

The Euler Top

- Using the body angular velocity components (ξ^1, ξ^2, ξ^3), the dynamics of the Euler top reads (Euler [1752])

$$
\begin{aligned}
J_1\dot{\xi}^1 &= (J_2 - J_3)\xi^2\xi^3, & \dot{\theta} &= \xi^1\cos\psi - \xi^2\sin\psi, \\
J_2\dot{\xi}^2 &= (J_3 - J_1)\xi^3\xi^1, & \dot{\phi} &= \left(\xi^1\sin\psi + \xi^2\cos\psi\right)\csc\theta, \\
J_3\dot{\xi}^3 &= (J_1 - J_2)\xi^1\xi^2, & \dot{\psi} &= \xi^3 - \left(\xi^1\sin\psi + \xi^2\cos\psi\right)\cot\theta
\end{aligned}
$$

- The first three equations decouple from the full system
- The last three equations suffer from artificial singularities

Hamel's Equations

- Let $u_1(q), \ldots, u_n(q)$, $n = \dim Q$, be independent (local) vector fields on Q. These fields in general do not commute. Introduce the **structure functions** $c_{ij}^k(q)$ by the relation

  $$[u_i(q), u_j(q)](q) = c_{ij}^k(q) u_k(q)$$

- Use these fields to measure the velocity components,

  $$\dot{q} = \dot{q}^i \partial_{q^i} = \xi^i u_i(q), \quad \text{where} \quad \xi = (\xi^1, \ldots, \xi^n) \in \mathbb{R}^n,$$

  and write the Lagrangian as a function of (q, ξ):

  $$l(q, \xi) := L(q, \xi^i u_i(q))$$

Hamel's Equations

- The dynamics is (Euler, Lagrange, Poincaré, Boltzmann, Hamel...)

$$\frac{d}{dt}\frac{\partial l}{\partial \xi^j} = c_{ij}^a(q)\frac{\partial l}{\partial \xi^a}\xi^i + u_j[l]$$

- These equations generalize both the Euler–Lagrange and Euler–Poincaré equations
- Hamel's formalism is useful in
  - Nonholonomic mechanics (momentum conservation, integrability)
  - Control of mechanical systems
  - Discrete mechanics

Outline

1. Hamel's formalism in nonholonomic mechanics
2. Variational structures for Hamel's equations
3. Applications to nonholonomic integrators

Motivation: The nonholonomic integrator of Cortés and Martínez may fail to preserve the relative equilibria of nonholonomic systems and their stability

Nonholonomic System

- Nonholonomic systems are mechanical systems with **velocity constraints** that cannot be rewritten as position constraints

- Typically, the constraints are linear, i.e., they are given by a distribution $\mathscr{D} \subset TQ$ that defines a subspace $\mathscr{D}_q$ of admissible velocities at each configuration $q \in Q$

Nonholonomic System

- The constraints are **ideal:** They can be replaced with reaction forces R, and $\langle R, v\rangle = 0$ for any $v \in \mathscr{D}_q$
- For a suitable frame, the constraints read $\xi^{m+1} = \cdots = \xi^n$, and, according to the Lagrange–d'Alembert principle, the dynamics is given by the **constrained Hamel equations**

$$\frac{d}{dt}\frac{\partial l}{\partial \xi^j} = c_{ij}^a(q)\frac{\partial l}{\partial \xi^a}\xi^i + u_j[l], \quad i, j = 1, \ldots, m, \quad a = 1, \ldots, n,$$

coupled with the equation

$$\dot{q} = \xi^i u_i(q)$$

- Hamel's formalism allows to eliminate the Lagrange multipliers

Bundles and Frame Selection

- Nonholonomic constraints and/or symmetry define subbundles of the velocity phase space of the system. The base space of both subbundles is the configuration manifold Q
- Let $\mathscr{D}$ and $\mathscr{S}$ be the constraint and symmetry subbundles, respectively
- The frame u_i, $i = 1, \ldots, n$, is usually selected in such a way that there are subframes that span the fibers of $\mathscr{D}$, $\mathscr{S}$, and $\mathscr{D} \cap \mathscr{S}$
- For underactuated systems, the controlled directions are characterized by the fibers of a subbundle $\mathscr{T}^*$ of the momentum phase space T^*Q. One then may wish to select a frame that contains a subframe whose dual spans the fibers of $\mathscr{T}^*$

Variational Principles

- The Euler–Lagrange equations are equivalent to **Hamilton's principle**: $q(t)$ is a trajectory of the Euler–Lagrange equations if and only if $q(t)$ extremizes the action:

$$\delta \int_a^b L(q(t), \dot{q}(t))\, dt, \quad \text{where} \quad \delta q(a) = \delta q(b) = 0$$

- The equations for the angular velocity $\xi = (\xi^1, \xi^2, \xi^3)$ of the Euler top are equivalent to the principle

$$\delta \int_a^b l(\xi)\, dt = 0,$$

where

$$l(\xi) = \tfrac{1}{2}\left(J_1(\xi^1)^2 + J_2(\xi^2)^2 + J_3(\xi^3)^2\right), \quad \delta\xi = \dot{\zeta} + \xi \times \zeta, \quad \text{and} \quad \zeta(a) = \zeta(b) = 0;$$

observe that the variations are defined in a nontrivial way

Hamel's Equations

Theorem

The curve $\left(q(t), \xi(t)\right)$ *satisfies the Hamel equations*

$$\frac{d}{dt}\frac{\partial l}{\partial \xi^j} = c^a_{ij}(q)\frac{\partial l}{\partial \xi^a}\xi^i + u_j[l]$$

if and only if

$$\delta \int_a^b l(q,\xi)\,dt = 0,$$

where

$$\delta\xi^a(t) = \dot{\zeta}^a(t) + c^a_{ij}(q(t))\xi^i(t)\zeta^j(t), \quad \delta q(t) = \zeta^j(t)u_j(q(t)), \quad \textit{and} \quad \zeta(a) = \zeta(b) = 0$$

Hamel's Equations

The formula for variations

$$\delta\xi^a = \dot{\zeta}^a + c^a_{ij}(q)\xi^i\zeta^j,$$

follows form the formula

$$\frac{d}{dt}\delta q = \delta\dot{q},$$

which, when written relative to the frame $u_1, \ldots, u_n$, becomes

$$\frac{d}{dt}\left(\zeta^i u_i\right) = \delta\left(\xi^i u_i\right)$$

The Hamilton–Pontryagin Principle

- Q is a manifold, TQ and T^*Q are its tangent and cotangent bundles. Let q, (q,v), and (q,p) be local coordinates on Q, TQ, and T^*Q, respectively
- $t \mapsto (q(t), v(t), p(t))$, $t \in [a,b]$, is a curve in the **Pontryagin bundle** $TQ \oplus T^*Q$
- The action on $TQ \oplus T^*Q$ is defined by

$$S = \int_a^b \left[L(q(t), v(t)) + \left\langle p(t), \dot{q}(t) - v(t) \right\rangle \right] dt$$

- Setting the variation of this action equal to zero produces the Euler–Lagrange equations, the Legendre transform, and the **second order condition** $\dot{q} = v$
- See the papers of Yoshimura and Marsden for details

The Hamilton–Pontryagin Principle

Theorem

The following statements are equivalent:

(i) *The curve* $(q(t), \eta(t), \mu(t))$, $a \le t \le b$, *is a critical point of the action functional*

$$\int_a^b \left[l(q(t), \eta(t)) + \langle \mu_i u^i(q(t)), \dot{q}(t) - \eta^j(t) u_j(q(t)) \rangle \right] dt$$

on the space of curves in $TQ \oplus T^*Q$ *connecting* q_a *and* q_b *on the interval* $[a, b]$, *where we choose variations of the curve* $(q(t), \eta(t), \mu(t))$ *that satisfy* $\delta q(a) = \delta q(b) = 0$

(ii) *The* ***implicit Hamel equations****, the Legendre transform, the second order condition, and the 'reconstruction equation',*

$$u_j[l] - \dot{\mu}_j - \frac{\partial \phi_b^a}{\partial q^r} \psi_i^r \psi_j^b \mu_a \xi^i - \frac{\partial \psi_i^b}{\partial q^r} \phi_b^a \psi_j^r \mu_a \eta^i = 0, \ \mu = \frac{\partial l}{\partial \eta}, \ \xi = \eta, \ \dot{q} = \xi^i u_i(q)$$

hold, where ϕ_b^a *are the elements of the inverse of* ψ_i^j

Discrete Hamilton's Principle and Variational Integrators

- Assuming Q is a vector space, $L : TQ \to \mathbb{R}$ is a Lagrangian. A **discrete Lagrangian** is a map $L^d : Q \times Q \to \mathbb{R}$ that approximates the action integral along an exact solution of the Euler–Lagrange equations joining the configurations $q_k, q_{k+1} \in Q$:

$$L^d(q_k, q_{k+1}) = hL\big(\tfrac{1}{2}(q_k + q_{k+1}), \tfrac{1}{h}(q_{k+1} - q_k)\big) \approx \int_0^h L(q, \dot{q})\, dt$$

- Continuous-time trajectories $q(t)$ are replaced with finite sequences $\{q_k\}_{k=0}^N$ in the configuration space Q
- The states of the system are the pairs $(q_k, q_{k+1}) \in Q \times Q$

Discrete Hamilton's Principle and Variational Integrators

- The action integral is replaced with the **action sum**

$$S^d(q_0, q_1, \dots, q_N) = \sum_{k=0}^{N-1} L^d(q_k, q_{k+1}),$$

where $q_k \in Q$, $k = 0, 1, \dots, N$, is a finite sequence of points in the configuration space with given fixed endpoints q_0 and q_N

- The dynamics is determined by the **discrete Hamilton principle**

$$\delta S^d(q_0, q_1, \dots, q_N) = 0, \quad \delta q_0 = \delta q_N = 0$$

- The **discrete Euler–Lagrange equations** are

$$D_2 L^d(q_{k-1}, q_k) + D_1 L^d(q_k, q_{k+1}) = 0,$$

where D_i denotes differentiation with respect to the i^{th} input

Discrete Hamilton's Principle and Variational Integrators

- The **update map** $(q_{k-1}, q_k) \mapsto (q_k, q_{k+1})$ defined by the discrete Euler–Lagrange equations is the analogue of the phase flow of the continuous-time mechanics. The update map preserves the discrete Lagrangian symplectic form, and thus is volume-preserving. If symmetry is present, the update map preserves the discrete momentum map
- Numerical experiments suggest that discretizations like that adequately model continuous-time dynamics and accurately track system's energy over long time intervals, i.e., they are free from the so-called **numerical dissipation** (an artificial dissipation introduced by some numerical methods)

Discrete Hamel's Equations

- Assuming Q is a vector space and adopting the midpoint rule, let $q_{k+1/2} = \frac{1}{2}\left(q_k + q_{k+1}\right)$, and let $\xi_{k,k+1}$ be the discrete analogue of ξ. Define the discrete Lagrangian by the formula

$$l^d(q_{k+1/2}, \xi_{k,k+1}) := h l(q_{k+1/2}, \xi_{k,k+1})$$

- Set $\mu_{k,k+1} := D_2 l^d(q_{k+1/2}, \xi_{k,k+1})$, which is a discrete analogue of

$$\mu = D_2 l(q, \xi) \equiv \partial l / \partial \xi$$

Discrete Hamel's Equations

Theorem

The sequence $(q_{k+1/2}, \xi_{k,k+1})$ *satisfies the* ***discrete Hamel equations***

$$\frac{1}{h}\big(\mu_{k-1,k;j} - \mu_{k,k+1;j}\big) + \frac{1}{2}\big(u_j[l^d](q_{k+1/2}, \xi_{k,k+1}) + u_j[l^d](q_{k-1/2}, \xi_{k-1,k})\big)$$
$$+ \frac{1}{2}\big(c^a_{ij}(q_{k+1/2})\xi^i_{k,k+1}\mu_{k,k+1;a} + c^a_{ij}(q_{k-1/2})\xi^i_{k-1,k}\mu_{k-1,k;a}\big) = 0$$

if and only if

$$\delta \sum_{k=0}^{N-1} l^d(q_{k+1/2}, \xi_{k,k+1}) = 0,$$

where

$$\delta q^i_{k+1/2} = \frac{1}{2}\psi^i_b(q_{k+1/2})(\zeta^b_{k+1} + \zeta^b_k),$$
$$\delta \xi^b_{k,k+1} = \frac{1}{h}\big(\zeta^b_{k+1} - \zeta^b_k\big) + \frac{1}{2}c^b_{ij}(q_{k+1/2})\,\xi^i_{k,k+1}\big(\zeta^j_{k+1} + \zeta^j_k\big),$$

where $\zeta_0 = \zeta_N = 0$, *and where* $u_i(q) = \psi^j_i(q)\partial_{q^j}$

One then writes a discrete analogue of the kinematic equation $\dot{q} = \xi^i u_i(q)$, there is a certain freedom in doing that

Discrete Hamel's Equations

- The principal step is to obtain the formulae

$$\delta q^i_{k+1/2} = \tfrac{1}{2}\psi^i_b(q_{k+1/2})\big(\zeta^b_{k+1} + \zeta^b_k\big),$$
$$\delta \xi^b_{k,k+1} = \tfrac{1}{h}\big(\zeta^b_{k+1} - \zeta^b_k\big) + \tfrac{1}{2} c^b_{ij}(q_{k+1/2})\, \xi^i_{k,k+1}\big(\zeta^j_{k+1} + \zeta^j_k\big)$$

- In the continuous-time case the formulae for variations

$$\delta q^i = \psi^i_b(q)\zeta^b, \quad \delta\xi^b = \dot{\zeta}^b + c^b_{ij}(q)\xi^i\zeta^j$$

  are straightforward to obtain and follow form the formula

$$\tfrac{d}{dt}\big(\zeta^i u_i\big) = \delta\big(\xi^i u_i\big)$$

- In the discrete case it becomes less straightforward because of the absence of time-differentiation
- Alternatively, one can utilize the discrete version of the Hamilton–Pontryagin principle to derive the implicit discrete Hamel equations

Nonholonomic Integrators

According to Cortés and Martínez, the discrete Lagrange–d'Alembert principle is

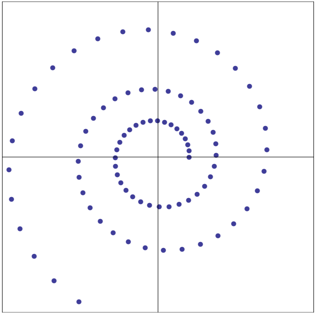
$$\delta \sum_{k=0}^{N-1} L^d(q_k, q_{k+1}) = 0, \quad \delta q_k \in \mathscr{D}_{q_k}, \quad (q_k, q_{k+1}) \in \mathscr{D}^d,$$

where $\mathscr{D} \subset TQ$ is a continuous-time constraint distribution and $\mathscr{D}^d \subset Q \times Q$ is discrete constraint space

Nonholonomic Integrators

Nonholonomic integrators are sensitive to how the constraints are discretized. For instance a trajectory of the contact point of a discrete balanced Chaplygin sleigh (a platform supported by a skate) may become a spiral, which is not what the continuous-time model predicts

- For a proper discrete constraint one obtains the anticipated trajectory



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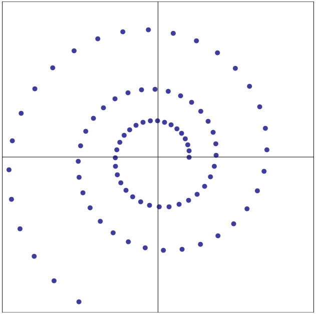
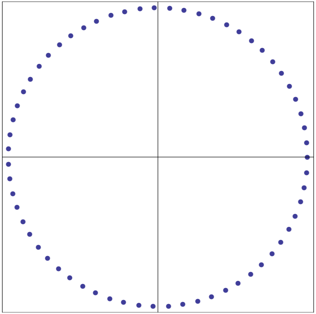




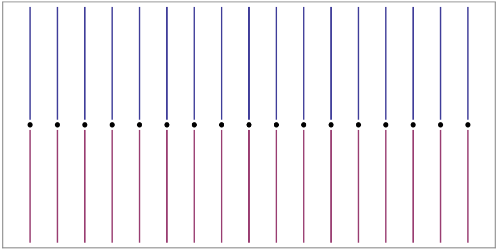
Nonholonomic Integrators

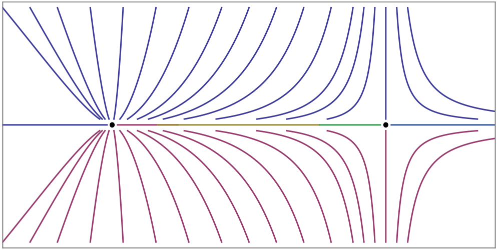
- Lynch and Zenkov [2009] observed that the stability of relative equilibria of Chaplygin systems may be corrupted by the discretization of Cortés and Martínez
- $L = \frac{1}{2}\left(\dot{r}^2 + \dot{g}^2 + (e^{-g}\dot{s})^2\right) - U(r), \quad \dot{s} + e^{g}\left(a(r)\dot{r} + b(r)\dot{g}\right) = 0$

Equilibria and Structural Stability

- Relative equilibria of nonholonomic systems with symmetry are never isolated and often are partially asymptotically stable
- Relative equilibria are not robust under perturbations

Equilibria and Structural Stability





Dynamics before and after a perturbation

Equilibria and Structural Stability

- Let $g^t : M \to M$ be the phase flow of a continuous-time system of interest. An **ideal integrator** is a discrete dynamical system $F : M \to M$, where F is a diffeomorphism such that $F^k = g^{kh}$ and where h is the time step
- Constructing an ideal integrator is equivalent to ability to solve the system analytically
- Real-life integrators may be interpreted as perturbations of the ideal integrator
- We ask for the preservation of the manifold of relative equilibria and their stability type. This is important for long-term numerical integration. If these are not preserved, the α- and ω-limit sets of the original continuous-time system and of its discretization are certain to be not the same. Therefore, the asymptotic dynamics of the discrete system is different from that of the original system, resulting in a structurally-unstable discretization

Discrete Nonholonomic Systems

One of the challenges in the nonholonomic setting is discretizing the constraints. Discrete Hamel's formalism suggests a natural procedure:

1. Select the fields $u_i(q)$ such that the continuous-time constraints read $\xi^{m+1} = \xi^{m+2} = \cdots = \xi^n = 0$
2. Define the discrete constraints to be $\xi^{m+1}_{k,k+1} = \xi^{m+2}_{k,k+1} = \cdots = \xi^n_{k,k+1} = 0$
3. The discrete nonholonomic dynamics becomes

$$\frac{1}{h}\big(\mu_{k-1,k;j} - \mu_{k,k+1;j}\big) + \frac{1}{2}\big(u_j\big[l^d\big](q_{k+1/2}, \xi_{k,k+1}) + u_j\big[l^d\big](q_{k-1/2}, \xi_{k-1,k})\big) \\ + \frac{1}{2}\big(c^c_{ij}(q_{k+1/2})\xi^i_{k,k+1}\mu_{k,k+1;a} + c^a_{ij}(q_{k-1/2})\xi^i_{k-1,k}\mu_{k-1,k;a}\big) = 0,$$

$$i, j = 1, \ldots, m, \quad a = 1, \ldots, n$$

Discrete Nonholonomic Systems

- Discrete Hamel's formalism modifies the discrete Lagrange–d'Alembert principle of Cortés and Martínez and brings back the preservation of manifolds of relative equilibria and their stability
- The proof is based on the center manifold stability analysis, which produces the same stability conditions in both the continuous-time and discrete settings

The Spherical Pendulum (with A. Bloch and M. Leok)

- Consider a point mass moving on a sphere in the presence of gravity. One can use the spherical coordinates to study the motion, but this is not the best idea
- Alternatively, one may view the pendulum as a *degenerate rigid body*. The inertia tensor $J = \operatorname{diag}\{mr^2, mr^2, 0\}$ is non-invertible, and the Lagrangian $\frac{1}{2}\langle J\xi, \xi\rangle - mgr\gamma^3$ is degenerate. Here m is the mass and r is the length of the pendulum
- The dynamics is captured by the equations

  $$\dot{\mu} = \tau, \quad \dot{\gamma} = \gamma \times \xi$$

  where ξ is the angular velocity, $\mu = J\xi$ is the angular momentum, γ is a unit vertical vector relative to the body frame, and τ is the torque due to gravity
- Observe that for a given $\mu = (\mu_1, \mu_2, 0)$ there may be many corresponding angular velocities, but the third component of angular velocity does not affect the motion of the pendulum.

The Spherical Pendulum (with A. Bloch and M. Leok)

- Knowing $\gamma = (\gamma^1, \gamma^2, \gamma^3)$ is equivalent to knowing the position of the pendulum. The components of γ are not independent as $\|\gamma\| = 1$, and thus are redundant coordinates for the pendulum. The redundancy is minimal and the components of γ give a global and singularity-free coordinate system
- Let

$$u_1 = \gamma^3 \frac{\partial}{\partial \gamma^2} - \gamma^2 \frac{\partial}{\partial \gamma^3}, \quad u_2 = \gamma^1 \frac{\partial}{\partial \gamma^3} - \gamma^3 \frac{\partial}{\partial \gamma^1},$$

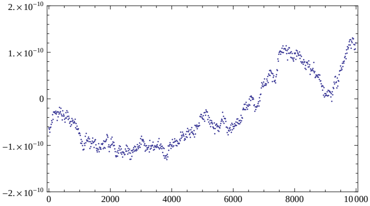
  these are tangent to the unit sphere. Then

$$\dot{\mu}_1 = mgr\gamma^2, \quad \dot{\mu}_2 = -mgr\gamma^1, \quad \dot{\gamma}^1 = -\xi^2\gamma^3, \quad \dot{\gamma}^2 = \xi^1\gamma^3, \quad \dot{\gamma}^3 = \xi^2\gamma^1 - \xi^1\gamma^2$$

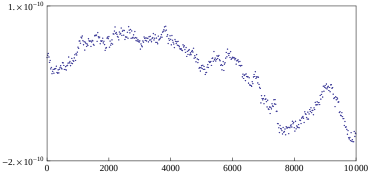
  are the Hamel equations for the pendulum written in the redundant configuration coordinates $(\gamma^1, \gamma^2, \gamma^3)$

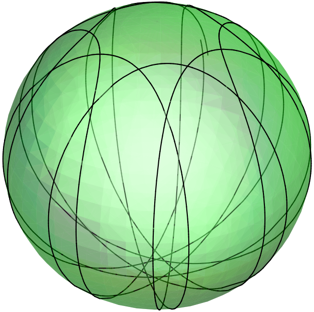
The Spherical Pendulum (with A. Bloch and M. Leok)

- Understanding the variational structure of Hamel's equations assisted in the derivation of discrete Hamel's equations and an energy- and momentum-preserving integrator for a spherical pendulum
- Preservation of the length of γ



- Conservation of energy





Concluding Remarks

- Discrete Hamel's formalism modifies the discrete Lagrange–d'Alembert principle of Cortés and Martínez and brings back the preservation of manifolds of relative equilibria and their stability
- A discretizations causing stability loss for systems with holonomic constraints has been observed in a recent work of Peng, Huynh, Zenkov, and Bloch. Hamel's formalism is likely to repair this situation

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