Martin's Axiom and initially ω_1-compact spaces

Alan Dow

Department of Mathematics and Statistics
University of North Carolina Charlotte

October 22, 2012

initially ω_1-compact

Definition

A space X is initially ω_1-compact if every open cover of size at most ω_1 has a finite subcover. Equivalently, each $A \in [X]^{\leq \omega_1}$ has a CAP (complete accumulation point).

initially ω_1-compact

Definition

A space X is initially ω_1-compact if every open cover of size at most ω_1 has a finite subcover. Equivalently, each $A \in [X]^{\leq\omega_1}$ has a CAP (complete accumulation point).

Definition

A space X has countable tightness, $t = \omega$, if for each $A \subset X$, $\overline{A} = \bigcup\{\overline{B} : B \in [A]^\omega\}$. For compact X, this is equivalent to having no (converging) uncountable free sequence (initial segments and final segments have disjoint closures).

C-embedding

Question [1978]

What property of first-countable X is sufficient to guarantee that for any first-countable $Y \supset X$, X is C-embedded in Y?

C-embedding

Question [1978]

What property of first-countable X is sufficient to guarantee that for any first-countable $Y \supset X$, X is C-embedded in Y?

Answer

X should be almost initially ω_1-compact – meaning, given two disjoint zero sets, Z_1, Z_2 of X, at least one should be initially ω_1-compact. But, is this just almost compact?

C-embedding

Question [1978]

What property of first-countable X is sufficient to guarantee that for any first-countable $Y \supset X$, X is C-embedded in Y?

Answer

X should be almost initially ω_1-compact – meaning, given two disjoint zero sets, Z_1, Z_2 of X, at least one should be initially ω_1-compact. But, is this just almost compact?

1. the properties $t = \omega$, $\aleph_0$-bounded, plus initially ω_1-compact implies compact,

C-embedding

Question [1978]

What property of first-countable X is sufficient to guarantee that for any first-countable $Y \supset X$, X is C-embedded in Y?

Answer

X should be almost initially ω_1-compact – meaning, given two disjoint zero sets, Z_1, Z_2 of X, at least one should be initially ω_1-compact. But, is this just almost compact?

1. the properties $t = \omega$, $\aleph_0$-bounded, plus initially ω_1-compact implies compact,
2. hence CH implies that $t = \omega$ plus initially ω_1-compact will be compact

C-embedding

Question [1978]

What property of first-countable X is sufficient to guarantee that for any first-countable $Y \supset X$, X is C-embedded in Y?

Answer

X should be almost initially ω_1-compact – meaning, given two disjoint zero sets, Z_1, Z_2 of X, at least one should be initially ω_1-compact. But, is this just almost compact?

1. the properties $t = \omega$, $\aleph_0$-bounded, plus initially ω_1-compact implies compact,
2. hence CH implies that $t = \omega$ plus initially ω_1-compact will be compact
3. and if there is a counterexample, there is a separable one.

Question

Is it consistent to have a first-countable initially ω_1-compact non-compact space? [let's call one "a counterexample"].

Question

Is it consistent to have a first-countable initially ω_1-compact non-compact space? [let's call one "a counterexample"].

Since first-countable compact spaces have cardinality $\mathfrak{c}$, we have a follow-up question by Arhangelskii

Question

Is it consistent to have a first-countable initially ω_1-compact non-compact space? [let's call one "a counterexample"].

Since first-countable compact spaces have cardinality $\mathfrak{c}$, we have a follow-up question by Arhangelskii

question

Does each first-countable initially ω_1-compact space have cardinality at most $\mathfrak{c}$?

Question

Is it consistent to have a first-countable initially ω_1-compact non-compact space? [let's call one "a counterexample"].

Since first-countable compact spaces have cardinality $\mathfrak{c}$, we have a follow-up question by Arhangelskii

question

Does each first-countable initially ω_1-compact space have cardinality at most $\mathfrak{c}$?

A remaining open problem in this area

Question

Is it consistent to have a first-countable initially ω_1-compact non-compact space? [let's call one "a counterexample"].

Since first-countable compact spaces have cardinality $\mathfrak{c}$, we have a follow-up question by Arhangelskii

question

Does each first-countable initially ω_1-compact space have cardinality at most $\mathfrak{c}$?

A remaining open problem in this area

Question [Juhasz]

Does each compact space with $t = \omega$ have a point with character at most ω_1?

Moore-Mrowka spaces

Moore-Mrowka spaces

Theorem

If X is a initially ω_1-compact, $t = \omega$ non-compact space, then βX is a Moore-Mrowka space (compact, $t = \omega$ and not sequential).

Moore-Mrowka spaces

Theorem

If X is a initially ω_1-compact, $t = \omega$ non-compact space, then βX is a Moore-Mrowka space (compact, $t = \omega$ and not sequential).

Theorem (Balogh)

PFA implies there is no Moore-Mrowka space (and so there is no "counterexample").

Moore-Mrowka spaces

Theorem

If X is a initially ω_1-compact, $t = \omega$ non-compact space, then βX is a Moore-Mrowka space (compact, $t = \omega$ and not sequential).

Theorem (Balogh)

PFA implies there is no Moore-Mrowka space (and so there is no "counterexample").

Proof.

Assume that K is compact, $t = \omega$ and $X \subset K$ is sequentially compact but not closed.

□

Moore-Mrowka spaces

Theorem

If X is a initially ω_1-compact, $t = \omega$ non-compact space, then βX is a Moore-Mrowka space (compact, $t = \omega$ and not sequential).

Theorem (Balogh)

PFA implies there is no Moore-Mrowka space (and so there is no "counterexample").

Proof.

Assume that K is compact, $t = \omega$ and $X \subset K$ is sequentially compact but not closed. MA(ω_1) implies that X can be assumed to be countably compact and cardinality at most $\mathfrak{c}$.

□

Moore-Mrowka spaces

Theorem

If X is a initially ω_1-compact, $t = \omega$ non-compact space, then βX is a Moore-Mrowka space (compact, $t = \omega$ and not sequential).

Theorem (Balogh)

PFA implies there is no Moore-Mrowka space (and so there is no "counterexample").

Proof.

Assume that K is compact, $t = \omega$ and $X \subset K$ is sequentially compact but not closed. MA(ω_1) implies that X can be assumed to be countably compact and cardinality at most $\mathfrak{c}$. Fix any $\mathcal{F}$ – a free countably complete filter of closed subsets of X.

□

Moore-Mrowka spaces

Theorem

If X is a initially ω_1-compact, $t = \omega$ non-compact space, then βX is a Moore-Mrowka space (compact, $t = \omega$ and not sequential).

Theorem (Balogh)

PFA implies there is no Moore-Mrowka space (and so there is no "counterexample").

Proof.

Assume that K is compact, $t = \omega$ and $X \subset K$ is sequentially compact but not closed. MA(ω_1) implies that X can be assumed to be countably compact and cardinality at most $\mathfrak{c}$. Fix any $\mathcal{F}$ – a free countably complete filter of closed subsets of X. For each $x \in X$, specify $x \in W_x \subset U_x$ – open in K – with $\overline{W_x} \subset U_x$ and $X \setminus U_x \in \mathcal{F}$. □

Proof continued

Proof continued

1. The poset $\mathbb{P}$ is proper, where $p \in \mathbb{P}$ has the form $(\langle x_\alpha : \alpha \in L_p \rangle, \mathcal{M}_p = \langle M_\alpha : \alpha \in L_p \rangle)$ providing $L_p \in [\omega_1]^{<\omega}$, $\mathcal{M}_p$ is an ϵ-chain of $M \prec H(\lambda)$, $M_\alpha \cap \omega_1 = \alpha$, and $x_\alpha \in \bigcap\{\overline{F \cap M_\alpha} : F \in \mathcal{F} \cap M_\alpha\}$.

Proof continued

1. The poset $\mathbb{P}$ is proper, where $p \in \mathbb{P}$ has the form $(\langle x_\alpha : \alpha \in L_p\rangle, \mathcal{M}_p = \langle M_\alpha : \alpha \in L_p\rangle)$ providing $L_p \in [\omega_1]^{<\omega}$, $\mathcal{M}_p$ is an ϵ-chain of $M \prec H(\lambda)$, $M_\alpha \cap \omega_1 = \alpha$, and $x_\alpha \in \bigcap\{\overline{F \cap M_\alpha} : F \in \mathcal{F} \cap M_\alpha\}$.
2. For $\alpha \in L_p$, define $W^p_\alpha = \bigcap\{W_{x_\delta} : x_\alpha \in W_{x_\delta}, \delta \in L_p\}$

Proof continued

1. The poset $\mathbb{P}$ is proper, where $p \in \mathbb{P}$ has the form $(\langle x_\alpha : \alpha \in L_p \rangle, \mathcal{M}_p = \langle M_\alpha : \alpha \in L_p \rangle)$ providing $L_p \in [\omega_1]^{<\omega}$, $\mathcal{M}_p$ is an ϵ-chain of $M \prec H(\lambda)$, $M_\alpha \cap \omega_1 = \alpha$, and $x_\alpha \in \bigcap \{ \overline{F \cap M_\alpha} : F \in \mathcal{F} \cap M_\alpha \}$.
2. For $\alpha \in L_p$, define $W^p_\alpha = \bigcap \{ W_{x_\delta} : x_\alpha \in W_{x_\delta}, \delta \in L_p \}$
3. define $p < q$ to require that for $\alpha \in L_p \setminus L_q$ and $\delta = \min(L_q \setminus \alpha)$ $x_\alpha \in W^q_\delta$

Proof continued

1. The poset $\mathbb{P}$ is proper, where $p \in \mathbb{P}$ has the form $(\langle x_\alpha : \alpha \in L_p \rangle, \mathcal{M}_p = \langle M_\alpha : \alpha \in L_p \rangle)$ providing $L_p \in [\omega_1]^{<\omega}$, $\mathcal{M}_p$ is an ϵ-chain of $M \prec H(\lambda)$, $M_\alpha \cap \omega_1 = \alpha$, and $x_\alpha \in \bigcap\{\overline{F \cap M_\alpha} : F \in \mathcal{F} \cap M_\alpha\}$.
2. For $\alpha \in L_p$, define $W^p_\alpha = \bigcap\{W_{x_\delta} : x_\alpha \in W_{x_\delta}, \delta \in L_p\}$
3. define $p < q$ to require that for $\alpha \in L_p \setminus L_q$ and $\delta = \min(L_q \setminus \alpha)$ $x_\alpha \in W^q_\delta$
4. Meeting ω_1-sets produces an ω_1-sequence, free in K,

Proof continued

1. The poset $\mathbb{P}$ is proper, where $p \in \mathbb{P}$ has the form $(\langle x_\alpha : \alpha \in L_p\rangle, \mathcal{M}_p = \langle M_\alpha : \alpha \in L_p\rangle)$ providing $L_p \in [\omega_1]^{<\omega}$, $\mathcal{M}_p$ is an ϵ-chain of $M \prec H(\lambda)$, $M_\alpha \cap \omega_1 = \alpha$, and $x_\alpha \in \bigcap\{\overline{F \cap M_\alpha} : F \in \mathcal{F} \cap M_\alpha\}$.
2. For $\alpha \in L_p$, define $W^p_\alpha = \bigcap\{W_{x_\delta} : x_\alpha \in W_{x_\delta}, \delta \in L_p\}$
3. define $p < q$ to require that for $\alpha \in L_p \setminus L_q$ and $\delta = \min(L_q \setminus \alpha)$ $x_\alpha \in W^q_\delta$
4. Meeting ω_1-sets produces an ω_1-sequence, free in K,

Corollary

Therefore PFA implies there is no first-countable initially ω_1-compact non-compact space.

Proof continued

1. The poset $\mathbb{P}$ is proper, where $p \in \mathbb{P}$ has the form $(\langle x_\alpha : \alpha \in L_p \rangle, \mathcal{M}_p = \langle M_\alpha : \alpha \in L_p \rangle)$ providing $L_p \in [\omega_1]^{<\omega}$, $\mathcal{M}_p$ is an ϵ-chain of $M \prec H(\lambda)$, $M_\alpha \cap \omega_1 = \alpha$, and $x_\alpha \in \bigcap \{ \overline{F \cap M_\alpha} : F \in \mathcal{F} \cap M_\alpha \}$.
2. For $\alpha \in L_p$, define $W^p_\alpha = \bigcap \{ W_{x_\delta} : x_\alpha \in W_{x_\delta}, \delta \in L_p \}$
3. define $p < q$ to require that for $\alpha \in L_p \setminus L_q$ and $\delta = \min(L_q \setminus \alpha)$ $x_\alpha \in W^q_\delta$
4. Meeting ω_1-sets produces an ω_1-sequence, free in K,

Corollary

Therefore PFA implies there is no first-countable initially ω_1-compact non-compact space.

What about Martin's Axiom? But we still don't know about ZFC

How to get a large 1st-ctble initially ω_1-compact space

Proposition

If there is a first-countable initially ω_1-compact space X such that $|\beta X| > \mathfrak{c}$, and each $A \in [\beta X \setminus X]^{\leq\omega_1}$ has a complete accumulation point in X, then there is a first-countable initially ω_1-compact space of cardinality greater than $\mathfrak{c}$

How to get a large 1st-ctble initially ω_1-compact space

Proposition

If there is a first-countable initially ω_1-compact space X such that $|\beta X| > \mathfrak{c}$, and each $A \in [\beta X \setminus X]^{\leq \omega_1}$ has a complete accumulation point in X, then there is a first-countable initially ω_1-compact space of cardinality greater than $\mathfrak{c}$

Proof.

Take the space to have base set βX and simply declare the points of $\beta X \setminus X$ to be isolated. The points of X retain their original neighborhood bases.

This space is first-countable and large.

This space is initially ω_1-compact simply by the hypotheses. □

Rabus forces a minimal Boolean algebra – solving $t = \omega$

Definition

A minimally generating sequence for a Boolean algebra is a sequence $\{a_\alpha : \alpha \in L\}$ (for $L \subset \kappa$) satisfying that

Rabus forces a minimal Boolean algebra – solving $t = \omega$

Definition

A minimally generating sequence for a Boolean algebra is a sequence $\{a_\alpha : \alpha \in L\}$ (for $L \subset \kappa$) satisfying that

1. $\alpha \in a_\alpha \subset L \cap \alpha{+}1$,

Rabus forces a minimal Boolean algebra – solving $t = \omega$

Definition

A minimally generating sequence for a Boolean algebra is a sequence $\{a_\alpha : \alpha \in L\}$ (for $L \subset \kappa$) satisfying that

1. $\alpha \in a_\alpha \subset L \cap \alpha{+}1$,
2. for $\alpha < \beta$ $a_\alpha \cap a_\beta \in B_{\alpha+1} = \langle a_\gamma : \gamma \leq \alpha \rangle_{BA}$

Rabus forces a minimal Boolean algebra – solving $t = \omega$

Definition

A minimally generating sequence for a Boolean algebra is a sequence $\{a_\alpha : \alpha \in L\}$ (for $L \subset \kappa$) satisfying that

1. $\alpha \in a_\alpha \subset L \cap \alpha{+}1$,
2. for $\alpha < \beta$ $a_\alpha \cap a_\beta \in B_{\alpha+1} = \langle a_\gamma : \gamma \leq \alpha \rangle_{BA}$
3. **strongly** will mean that $\langle a_\alpha : \alpha \in L \rangle$ generates a proper ideal.

Rabus forces a minimal Boolean algebra – solving $t = \omega$

Definition

A minimally generating sequence for a Boolean algebra is a sequence $\{a_\alpha : \alpha \in L\}$ (for $L \subset \kappa$) satisfying that

1. $\alpha \in a_\alpha \subset L \cap \alpha{+}1$,
2. for $\alpha < \beta$ $a_\alpha \cap a_\beta \in B_{\alpha+1} = \langle a_\gamma : \gamma \leq \alpha \rangle_{BA}$
3. **strongly** will mean that $\langle a_\alpha : \alpha \in L \rangle$ generates a proper ideal.

Proposition

Given such $\langle a_\alpha : \alpha \in L \rangle$ then, for each $\alpha \in L$,

Rabus forces a minimal Boolean algebra – solving $t = \omega$

Definition

A minimally generating sequence for a Boolean algebra is a sequence $\{a_\alpha : \alpha \in L\}$ (for $L \subset \kappa$) satisfying that

1. $\alpha \in a_\alpha \subset L \cap \alpha{+}1$,
2. for $\alpha < \beta$ $a_\alpha \cap a_\beta \in B_{\alpha+1} = \langle a_\gamma : \gamma \leq \alpha \rangle_{BA}$
3. **strongly** will mean that $\langle a_\alpha : \alpha \in L \rangle$ generates a proper ideal.

Proposition

Given such $\langle a_\alpha : \alpha \in L \rangle$ then, for each $\alpha \in L$,

1. the set $\{a_\alpha \setminus a_\gamma : \gamma \in \alpha \cap L\}$ generates an ultrafilter on B

Rabus forces a minimal Boolean algebra – solving $t = \omega$

Definition

A minimally generating sequence for a Boolean algebra is a sequence $\{a_\alpha : \alpha \in L\}$ (for $L \subset \kappa$) satisfying that

1. $\alpha \in a_\alpha \subset L \cap \alpha{+}1$,
2. for $\alpha < \beta$ $a_\alpha \cap a_\beta \in B_{\alpha+1} = \langle a_\gamma : \gamma \leq \alpha \rangle_{BA}$
3. **strongly** will mean that $\langle a_\alpha : \alpha \in L \rangle$ generates a proper ideal.

Proposition

Given such $\langle a_\alpha : \alpha \in L \rangle$ then, for each $\alpha \in L$,

1. the set $\{a_\alpha \setminus a_\gamma : \gamma \in \alpha \cap L\}$ generates an ultrafilter on B
2. the only other ultrafilter is generated by the complements.

Rabus forces a minimal Boolean algebra – solving $t = \omega$

Definition

A minimally generating sequence for a Boolean algebra is a sequence $\{a_\alpha : \alpha \in L\}$ (for $L \subset \kappa$) satisfying that

1. $\alpha \in a_\alpha \subset L \cap \alpha{+}1$,
2. for $\alpha < \beta$ $a_\alpha \cap a_\beta \in B_{\alpha+1} = \langle a_\gamma : \gamma \leq \alpha \rangle_{BA}$
3. **strongly** will mean that $\langle a_\alpha : \alpha \in L \rangle$ generates a proper ideal.

Proposition

Given such $\langle a_\alpha : \alpha \in L \rangle$ then, for each $\alpha \in L$,

1. the set $\{a_\alpha \setminus a_\gamma : \gamma \in \alpha \cap L\}$ generates an ultrafilter on B
2. the only other ultrafilter is generated by the complements.

Proof.

the key is that $\alpha \in a_\beta$ implies that $\alpha \notin a_\alpha \setminus (a_\alpha \cap a_\beta)$ □

Baumgartner-Shelah Δ-function to get ccc poset

Theorem (Rabus)

Adding, with σ-closed forcing, $f : [\omega_2]^2 \mapsto [\omega_2]^\omega$, and force with

Baumgartner-Shelah Δ-function to get ccc poset

Theorem (Rabus)

Adding, with σ-closed forcing, $f : [\omega_2]^2 \mapsto [\omega_2]^\omega$, and force with

1. *$q \in Q$ providing $q = \langle a_\alpha^q : \alpha \in L_q \in [\omega_2]^{<\omega} \rangle$ is strongly minimal but also*

Baumgartner-Shelah Δ-function to get ccc poset

Theorem (Rabus)

Adding, with σ-closed forcing, $f : [\omega_2]^2 \mapsto [\omega_2]^\omega$, and force with

1. $q \in Q$ *providing* $q = \langle a^q_\alpha : \alpha \in L_q \in [\omega_2]^{<\omega} \rangle$ *is strongly minimal but also*
2. $\alpha < \beta \in L_q$ *implies* $a^q_\alpha \cap a^q_\beta \in B_{L_q \cap f(\alpha,\beta)}$

Baumgartner-Shelah Δ-function to get ccc poset

Theorem (Rabus)

Adding, with σ-closed forcing, $f : [\omega_2]^2 \mapsto [\omega_2]^\omega$, and force with

1. $q \in Q$ *providing* $q = \langle a^q_\alpha : \alpha \in L_q \in [\omega_2]^{<\omega} \rangle$ *is strongly minimal but also*
2. $\alpha < \beta \in L_q$ *implies* $a^q_\alpha \cap a^q_\beta \in B_{L_q \cap f(\alpha,\beta)}$
3. $p < q$ *requires* $a^p_\alpha \mapsto a^p_\alpha \cap L_q = a^q_\alpha$ *for* $\alpha \in L_q$ *is an isomorphism.*

Baumgartner-Shelah Δ-function to get ccc poset

Theorem (Rabus)

Adding, with σ-closed forcing, $f : [\omega_2]^2 \mapsto [\omega_2]^\omega$, and force with

1. *$q \in Q$ providing $q = \langle a^q_\alpha : \alpha \in L_q \in [\omega_2]^{<\omega} \rangle$ is strongly minimal but also*
2. *$\alpha < \beta \in L_q$ implies $a^q_\alpha \cap a^q_\beta \in B_{L_q \cap f(\alpha,\beta)}$*
3. *$p < q$ requires $a^p_\alpha \mapsto a^p_\alpha \cap L_q = a^q_\alpha$ for $\alpha \in L_q$ is an isomorphism.*

Properties in extension by Q

If G is Q-generic, then $a_\alpha = \bigcup \{ a^q_\alpha : q \in G \} \subset \alpha+1$

Baumgartner-Shelah Δ-function to get ccc poset

Theorem (Rabus)

Adding, with σ-closed forcing, $f : [\omega_2]^2 \mapsto [\omega_2]^\omega$, and force with

1. *$q \in Q$ providing $q = \langle a^q_\alpha : \alpha \in L_q \in [\omega_2]^{<\omega} \rangle$ is strongly minimal but also*
2. *$\alpha < \beta \in L_q$ implies $a^q_\alpha \cap a^q_\beta \in B_{L_q \cap f(\alpha,\beta)}$*
3. *$p < q$ requires $a^p_\alpha \mapsto a^p_\alpha \cap L_q = a^q_\alpha$ for $\alpha \in L_q$ is an isomorphism.*

Properties in extension by Q

If G is Q-generic, then $a_\alpha = \bigcup\{a^q_\alpha : q \in G\} \subset \alpha{+}1$

1. yields a locally compact, scattered thin very-tall space on ω_2 which is $t = \omega$ and initially ω_1-compact

Baumgartner-Shelah Δ-function to get ccc poset

Theorem (Rabus)

Adding, with σ-closed forcing, $f : [\omega_2]^2 \mapsto [\omega_2]^\omega$, and force with

1. *$q \in Q$ providing $q = \langle a^q_\alpha : \alpha \in L_q \in [\omega_2]^{<\omega}\rangle$ is strongly minimal but also*
2. *$\alpha < \beta \in L_q$ implies $a^q_\alpha \cap a^q_\beta \in B_{L_q \cap f(\alpha,\beta)}$*
3. *$p < q$ requires $a^p_\alpha \mapsto a^p_\alpha \cap L_q = a^q_\alpha$ for $\alpha \in L_q$ is an isomorphism.*

Properties in extension by Q

If G is Q-generic, then $a_\alpha = \bigcup\{a^q_\alpha : q \in G\} \subset \alpha{+}1$

1. yields a locally compact, scattered thin very-tall space on ω_2 which is $t = \omega$ and initially ω_1-compact
2. for the α-th limit λ_α, $[\lambda_\alpha, \lambda_\alpha + \omega)$ is the α scattering level,

Baumgartner-Shelah Δ-function to get ccc poset

Theorem (Rabus)

Adding, with σ-closed forcing, $f : [\omega_2]^2 \mapsto [\omega_2]^\omega$, and force with

1. *$q \in Q$ providing $q = \langle a^q_\alpha : \alpha \in L_q \in [\omega_2]^{<\omega}\rangle$ is strongly minimal but also*
2. *$\alpha < \beta \in L_q$ implies $a^q_\alpha \cap a^q_\beta \in B_{L_q \cap f(\alpha,\beta)}$*
3. *$p < q$ requires $a^p_\alpha \mapsto a^p_\alpha \cap L_q = a^q_\alpha$ for $\alpha \in L_q$ is an isomorphism.*

Properties in extension by Q

If G is Q-generic, then $a_\alpha = \bigcup\{a^q_\alpha : q \in G\} \subset \alpha+1$

1. yields a locally compact, scattered thin very-tall space on ω_2 which is $t = \omega$ and initially ω_1-compact
2. for the α-th limit λ_α, $[\lambda_\alpha, \lambda_\alpha + \omega)$ is the α scattering level,
3. more generally, each ground model infinite set has coinitial closure

A T-algebra reformulation

the hardest step!

if $A \subset \omega_2$ does not have compact closure, then $\overline{A}$ is coinitial (even a single limit for countable A was the innovative step).

A T-algebra reformulation

the hardest step!

if $A \subset \omega_2$ does not have compact closure, then $\overline{A}$ is coinitial (even a single limit for countable A was the innovative step).

Definition (Koszmider)

Fix a tree $T \subset 2^{<\kappa}$ such that for all $t \in T$, $T \cap \{t0, t1\}$ is not a singleton (for $t \in 2^{\alpha+1}$, let $t^\dagger$ denote its twin). A T-algebra generating sequence $\mathfrak{a}_T = \{a_t : t \in Succ(T)\}$ satisfies

A T-algebra reformulation

the hardest step!

if $A \subset \omega_2$ does not have compact closure, then $\overline{A}$ is coinitial (even a single limit for countable A was the innovative step).

Definition (Koszmider)

Fix a tree $T \subset 2^{<\kappa}$ such that for all $t \in T$, $T \cap \{t0, t1\}$ is not a singleton (for $t \in 2^{\alpha+1}$, let $t^\dagger$ denote its twin). A T-algebra generating sequence $\mathfrak{a}_T = \{a_t : t \in Succ(T)\}$ satisfies

1. $t \in a_t \subset C_t = \{s \in Succ(T) : s \leq t\}$

A T-algebra reformulation

the hardest step!

if $A \subset \omega_2$ does not have compact closure, then $\overline{A}$ is coinitial (even a single limit for countable A was the innovative step).

Definition (Koszmider)

Fix a tree $T \subset 2^{<\kappa}$ such that for all $t \in T$, $T \cap \{t0, t1\}$ is not a singleton (for $t \in 2^{\alpha+1}$, let $t^\dagger$ denote its twin). A T-algebra generating sequence $\mathfrak{a}_T = \{a_t : t \in Succ(T)\}$ satisfies

1. $t \in a_t \subset C_t = \{s \in Succ(T) : s \leq t\}$
2. $a_{t^\dagger} = C_{t^\dagger} \setminus a_t$

A T-algebra reformulation

the hardest step!

if $A \subset \omega_2$ does not have compact closure, then $\overline{A}$ is coinitial (even a single limit for countable A was the innovative step).

Definition (Koszmider)

Fix a tree $T \subset 2^{<\kappa}$ such that for all $t \in T$, $T \cap \{t0, t1\}$ is not a singleton (for $t \in 2^{\alpha+1}$, let $t^\dagger$ denote its twin). A T-algebra generating sequence $\mathfrak{a}_T = \{a_t : t \in Succ(T)\}$ satisfies

1. $t \in a_t \subset C_t = \{s \in Succ(T) : s \leq t\}$
2. $a_{t^\dagger} = C_{t^\dagger} \setminus a_t$
3. $\{a_s : s \in C_t\}$ is strongly minimal (as above)

a topology $\mathcal{A}_T$ on the maximal branches bT

Definition

For each $t \in Succ(T)$, we define $A_t \in \mathcal{A}_T$

$$A_t = \{x \in bT : (\exists \rho \in a_t) \;\; \rho^\dagger \subset x\}$$

a topology $\mathcal{A}_T$ on the maximal branches bT

Definition

For each $t \in Succ(T)$, we define $A_t \in \mathcal{A}_T$

$$A_t = \{x \in bT : (\exists \rho \in a_t) \ \ \rho^\dagger \subset x\}$$

A_t is also equal to $\{x \in bT : t \notin x \text{ and } \min(C_t \setminus x) \in a_t\}$; A_t and $A_{t^\dagger}$ are complements; $t \in A_{t^\dagger}$.

each point $x \in bT$ has neighborhood base generated by $\{A_{t^\dagger} : t \in C_x\}$.
A set Y of branches accumulates to a branch x if its $\dagger$-projection does.

a topology $\mathcal{A}_T$ on the maximal branches bT

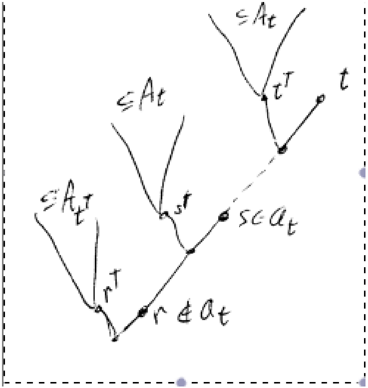
Definition

For each $t \in Succ(T)$, we define $A_t \in \mathcal{A}_T$

$$A_t = \{x \in bT : (\exists \rho \in a_t)\ \ \rho^\dagger \subset x\}$$

A_t is also equal to $\{x \in bT : t \notin x \text{ and } \min(C_t \setminus x) \in a_t\}$; A_t and $A_{t^\dagger}$ are complements; $t \in A_{t^\dagger}$.

each point $x \in bT$ has neighborhood base generated by $\{A_{t^\dagger} : t \in C_x\}$.
A set Y of branches accumulates to a branch x if its $\dagger$-projection does.



Rabus example as a T-algebra

Definition

For Rabus' example, we let
$T_R = \{t_\alpha \restriction \alpha, t_\alpha \in 2^{<\omega_2} : t_\alpha(\alpha) = 0 \text{ and } t_\alpha \restriction \alpha \equiv 1\}$.

Rabus example as a T-algebra

Definition

For Rabus' example, we let
$T_R = \{t_\alpha \restriction \alpha, t_\alpha \in 2^{<\omega_2} : t_\alpha(\alpha) = 0 \text{ and } t_\alpha \restriction \alpha \equiv 1\}$.

The point $t_\alpha \in bT_R$ because it is maximal, and strangely, $a_{t_\alpha^\dagger}$ codes the canonical neighborhood $A_{t_\alpha^\dagger}$ of t_α. $a_{t_\alpha^\dagger}$ is describing a set of t_β which have bounded closure.

Rabus example as a T-algebra

Definition

For Rabus' example, we let
$T_R = \{t_\alpha \restriction \alpha, t_\alpha \in 2^{<\omega_2} : t_\alpha(\alpha) = 0 \text{ and } t_\alpha \restriction \alpha \equiv 1\}$.

The point $t_\alpha \in bT_R$ because it is maximal, and strangely, $a_{t_\alpha^\dagger}$ codes the canonical neighborhood $A_{t_\alpha^\dagger}$ of t_α. $a_{t_\alpha^\dagger}$ is describing a set of t_β which have bounded closure.

$X = bT_R \setminus \{\vec{1}\}$ is initially ω_1-compact and $t = \omega$. a_{t_α} codes a set which is dense in a tail.

Forcing minimal extensions of Boolean algebras

Theorem (P. Koszmider, TAMS 351, 1999)

Using a FS-support iteration of (Souslin-free) ccc posets, and $T_0 = 2^{<\omega_1}$, $T = 2^{<\omega_1+\omega}$, *there is* $\mathfrak{a}_T$ *extending* $\mathfrak{a}_{T_0}$ *s.t.*

Forcing minimal extensions of Boolean algebras

Theorem (P. Koszmider, TAMS 351, 1999)

Using a FS-support iteration of (Souslin-free) ccc posets, and $T_0 = 2^{<\omega_1}$, $T = 2^{<\omega_1+\omega}$, *there is* $\mathfrak{a}_T$ *extending* $\mathfrak{a}_{T_0}$ *s.t.*

1. *the space* bT_0 *has no points of countable character*

Forcing minimal extensions of Boolean algebras

Theorem (P. Koszmider, TAMS 351, 1999)

Using a FS-support iteration of (Souslin-free) ccc posets, and $T_0 = 2^{<\omega_1}$, $T = 2^{<\omega_1+\omega}$, *there is* $\mathfrak{a}_T$ *extending* $\mathfrak{a}_{T_0}$ *s.t.*

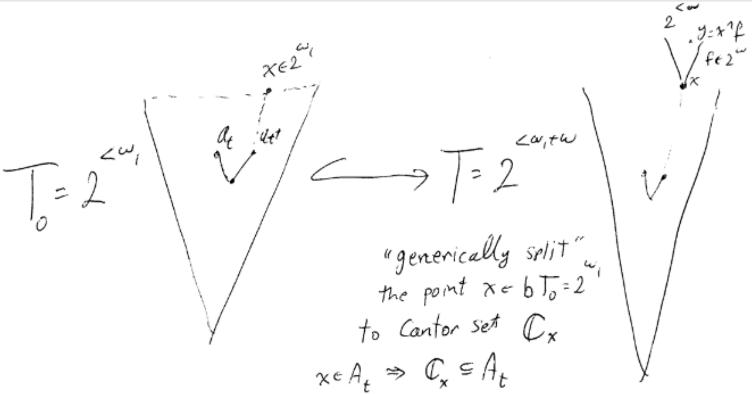
1. *the space* bT_0 *has no points of countable character*
2. *the space* bT *is first-countable*

Forcing minimal extensions of Boolean algebras

Theorem (P. Koszmider, TAMS 351, 1999)

Using a FS-support iteration of (Souslin-free) ccc posets, and $T_0 = 2^{<\omega_1}$, $T = 2^{<\omega_1+\omega}$, *there is* $\mathfrak{a}_T$ *extending* $\mathfrak{a}_{T_0}$ *s.t.*

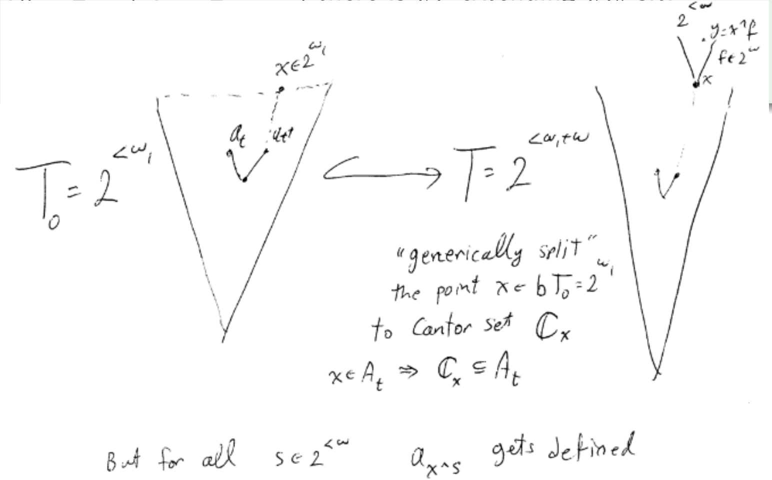
1. *the space* bT_0 *has no points of countable character*
2. *the space* bT *is first-countable*



Forcing minimal extensions of Boolean algebras

Theorem (P. Koszmider, TAMS 351, 1999)

Using a FS-support iteration of (Souslin-free) ccc posets, and $T_0 = 2^{<\omega_1}$, $T = 2^{<\omega_1+\omega}$, *there is* $\mathfrak{a}_T$ *extending* $\mathfrak{a}_{T_0}$ *s.t.*



Apply this general idea to Rabus space

Borrowing from the paper

Juhasz, Koszmider, and Soukup, **A first countable, initially ω_1-compact but non-compact space**, TopAppl 156, 2009

Apply this general idea to Rabus space

Borrowing from the paper

Juhasz, Koszmider, and Soukup, **A first countable, initially ω_1-compact but non-compact space**, TopAppl 156, 2009

Definition

Set
$T_\omega = \{t \in 2^{<\omega_2} : t(\alpha) = 0 \text{ implies } (\exists\beta < \alpha < \beta + \omega) t \restriction \beta \equiv 1\}$
each Rabus t_α has all possible finite extensions

Apply this general idea to Rabus space

Borrowing from the paper

Juhasz, Koszmider, and Soukup, **A first countable, initially ω_1-compact but non-compact space**, TopAppl 156, 2009

Definition

Set
$T_\omega = \{t \in 2^{<\omega_2} : t(\alpha) = 0 \text{ implies } (\exists \beta < \alpha < \beta + \omega) t \restriction \beta \equiv 1\}$
each Rabus t_α has all possible finite extensions

Theorem

Again, forcing (using Δ-function) an $\mathfrak{a}_{T_\omega}$ makes $X = bT_\omega \setminus \{\vec{1}\}$ initially ω_1-compact and (somewhat remarkably) first-countable.

can we use simply $T = 2^{<\omega_2}$? and make it large

References

can we use simply $T = 2^{<\omega_2}$? and make it large

References

1. Itay Neeman, **Forcing with sequences of models of two types**, 2011 www.math.ucla.edu/∼ineeman/ttms.pdf.

can we use simply $T = 2^{<\omega_2}$? and make it large

References

1. Itay Neeman, **Forcing with sequences of models of two types**, 2011 www.math.ucla.edu/∼ineeman/ttms.pdf.
2. Boban Velickovic and Giorgio Venturi, **Proper forcing remastered** , 2011, www.math.cmu.edu/∼eschimme/Appalachian/Index.html,

References

1. Itay Neeman, **Forcing with sequences of models of two types**, 2011 www.math.ucla.edu/∼ineeman/ttms.pdf.
2. Boban Velickovic and Giorgio Venturi, **Proper forcing remastered** , 2011, www.math.cmu.edu/∼eschimme/Appalachian/Index.html,

Definition

Assume GCH and let $\trianglelefteq$ be a well-ordering of $H(\omega_2)$ in type ω_2. Let $\mathcal{E}^2 = \mathcal{E}^2_0 \cup \mathcal{E}^2_1$

can we use simply $T = 2^{<\omega_2}$? and make it large

References

1. Itay Neeman, **Forcing with sequences of models of two types**, 2011 www.math.ucla.edu/∼ineeman/ttms.pdf.
2. Boban Velickovic and Giorgio Venturi, **Proper forcing remastered** , 2011, www.math.cmu.edu/∼eschimme/Appalachian/Index.html,

Definition

Assume GCH and let $\trianglelefteq$ be a well-ordering of $H(\omega_2)$ in type ω_2. Let $\mathcal{E}^2 = \mathcal{E}^2_0 \cup \mathcal{E}^2_1$

1. $E \in \mathcal{E}^2_0$ providing $E \prec (H(\omega_2), \trianglelefteq)$ is countable

References

1. Itay Neeman, **Forcing with sequences of models of two types**, 2011 www.math.ucla.edu/∼ineeman/ttms.pdf.
2. Boban Velickovic and Giorgio Venturi, **Proper forcing remastered** , 2011, www.math.cmu.edu/∼eschimme/Appalachian/Index.html,

Definition

Assume GCH and let $\trianglelefteq$ be a well-ordering of $H(\omega_2)$ in type ω_2. Let $\mathcal{E}^2 = \mathcal{E}^2_0 \cup \mathcal{E}^2_1$

1. $E \in \mathcal{E}^2_0$ providing $E \prec (H(\omega_2), \trianglelefteq)$ is countable
2. $E \in \mathcal{E}^2_1$ providing $E = \bigcup E_\alpha$ for some continuous increasing $\in$-chain $\{E_\alpha : \alpha \in \omega_1\} \subset \mathcal{E}^2_0$

we can use $T = T_\omega$ or $T = 2^{<\omega_2}$ – cofinality of branches

Following Velickovic's lead on the Baumgartner-Shelah example:

we can use $T = T_\omega$ or $T = 2^{<\omega_2}$ – cofinality of branches

Following Velickovic's lead on the Baumgartner-Shelah example:

Definition

A condition $p \in \mathbb{P}_T$ if $p = (\mathfrak{a}_p = \{a_t : t \in H_p \cap Succ(T)\}, \mathcal{M}_p)$ where

we can use $T = T_\omega$ or $T = 2^{<\omega_2}$ – cofinality of branches

Following Velickovic's lead on the Baumgartner-Shelah example:

Definition

A condition $p \in \mathbb{P}_T$ if $p = (\mathfrak{a}_p = \{a_t : t \in H_p \cap Succ(T)\}, \mathcal{M}_p)$ where

1. $H_p \in [T]^{<\omega}$ is closed under $\cap$, $\dagger$ and Y's (i.e. $t < s_0, s_1 \in H_p$ with $t = s_0 \cap s_1$, then $\{t0, t1\} \subset H_p$)

we can use $T = T_\omega$ or $T = 2^{<\omega_2}$ – cofinality of branches

Following Velickovic's lead on the Baumgartner-Shelah example:

Definition

A condition $p \in \mathbb{P}_T$ if $p = (\mathfrak{a}_p = \{a_t : t \in H_p \cap Succ(T)\}, \mathcal{M}_p)$ where

1. $H_p \in [T]^{<\omega}$ is closed under $\cap$, $\dagger$ and Y's (i.e. $t < s_0, s_1 \in H_p$ with $t = s_0 \cap s_1$, then $\{t0, t1\} \subset H_p$)
2. $E \in \mathcal{M}_p$, $t \subset s$ both in $H_p \cap Succ(T) \cap E$ implies $a_t \cap a_s \in \langle a_r : r \in C_t \cap H_p \cap E \rangle_{BA}$ (same as Rabus)

we can use $T = T_\omega$ or $T = 2^{<\omega_2}$ – cofinality of branches

Following Velickovic's lead on the Baumgartner-Shelah example:

Definition

A condition $p \in \mathbb{P}_T$ if $p = (\mathfrak{a}_p = \{a_t : t \in H_p \cap Succ(T)\}, \mathcal{M}_p)$ where

1. $H_p \in [T]^{<\omega}$ is closed under $\cap$, $\dagger$ and Y's (i.e. $t < s_0, s_1 \in H_p$ with $t = s_0 \cap s_1$, then $\{t0, t1\} \subset H_p$)
2. $E \in \mathcal{M}_p$, $t \subset s$ both in $H_p \cap Succ(T) \cap E$ implies $a_t \cap a_s \in \langle a_r : r \in C_t \cap H_p \cap E \rangle_{BA}$ (same as Rabus)
3. $a_{t^\dagger} = C_{t^\dagger} \setminus a_t$ for $\{t, t^\dagger\} \subset H_p$

we can use $T = T_\omega$ or $T = 2^{<\omega_2}$ – cofinality of branches

Following Velickovic's lead on the Baumgartner-Shelah example:

Definition

A condition $p \in \mathbb{P}_T$ if $p = (\mathfrak{a}_p = \{a_t : t \in H_p \cap Succ(T)\}, \mathcal{M}_p)$ where

1. $H_p \in [T]^{<\omega}$ is closed under $\cap$, $\dagger$ and Y's (i.e. $t < s_0, s_1 \in H_p$ with $t = s_0 \cap s_1$, then $\{t0, t1\} \subset H_p$)
2. $E \in \mathcal{M}_p$, $t \subset s$ both in $H_p \cap Succ(T) \cap E$ implies $a_t \cap a_s \in \langle a_r : r \in C_t \cap H_p \cap E\rangle_{BA}$ (same as Rabus)
3. $a_{t^\dagger} = C_{t^\dagger} \setminus a_t$ for $\{t, t^\dagger\} \subset H_p$
4. and $p < q$ if $\mathcal{M}_p \supset \mathcal{M}_q$, $H_p \supset H_q$, and $\mathfrak{a}_q$ canonically embeds in $\mathfrak{a}_p$.

sketch of proper and ω_1-compact

Lemma

Suppose that $\mathbb{P}_T, p \in M^* \prec H(\theta)$, $M = M^* \cap H(\omega_2) \in \mathcal{M}_r$, $r < p$ *and* $r \in D$ *for some dense* $D \in M^*$; *prepare* H_r, *then*

sketch of proper and ω_1-compact

Lemma

Suppose that $\mathbb{P}_T, p \in M^* \prec H(\theta)$, $M = M^* \cap H(\omega_2) \in \mathcal{M}_r$, $r < p$ *and* $r \in D$ *for some dense* $D \in M^*$; *prepare* H_r, *then*

1. *Then there is* $q \in D \cap M$ *such that* $H_q \setminus H_r$ *and* $\mathcal{M}_q \setminus \mathcal{M}_r$ *are each disjoint from* E *for all* $E \in \mathcal{M}_r \cap M^* \cap \mathcal{E}_0^2$;

sketch of proper and ω_1-compact

Lemma

Suppose that $\mathbb{P}_T, p \in M^* \prec H(\theta)$*,* $M = M^* \cap H(\omega_2) \in \mathcal{M}_r$*,* $r < p$ *and* $r \in D$ *for some dense* $D \in M^*$*; prepare* H_r*, then*

1. *Then there is* $q \in D \cap M$ *such that* $H_q \setminus H_r$ *and* $\mathcal{M}_q \setminus \mathcal{M}_r$ *are each disjoint from* E *for all* $E \in \mathcal{M}_r \cap M^* \cap \mathcal{E}_0^2$*;*
2. *to define* $\bar{q} < q, r$*,* $\mathcal{M}_{\bar{q}}$ *is handled by Neeman/Velickovic as the smallest set containing* $\mathcal{M}_q \cup \mathcal{M}_r$ *closed under finite intersections, and* $H_{\bar{q}} = H_q \cup H_r$ *(this takes some work),*

Lemma

Suppose that $\mathbb{P}_T, p \in M^* \prec H(\theta)$, $M = M^* \cap H(\omega_2) \in \mathcal{M}_r$, $r < p$ and $r \in D$ for some dense $D \in M^*$; prepare H_r, then

1. *Then there is $q \in D \cap M$ such that $H_q \setminus H_r$ and $\mathcal{M}_q \setminus \mathcal{M}_r$ are each disjoint from E for all $E \in \mathcal{M}_r \cap M^* \cap \mathcal{E}_0^2$;*
2. *to define $\bar{q} < q, r$, $\mathcal{M}_{\bar{q}}$ is handled by Neeman/Velickovic as the smallest set containing $\mathcal{M}_q \cup \mathcal{M}_r$ closed under finite intersections, and $H_{\bar{q}} = H_q \cup H_r$ (this takes some work),*
3. *for each $t \in H_q \setminus H_r$, we have $t^\dagger \in H_q \setminus H_r$ and (we designate t* as primary *if) $t^\dagger$ has no extension in H_r (one of them has this property)*

sketch of proper and ω_1-compact

Lemma

Suppose that $\mathbb{P}_T, p \in M^* \prec H(\theta)$, $M = M^* \cap H(\omega_2) \in \mathcal{M}_r$, $r < p$ *and* $r \in D$ *for some dense* $D \in M^*$*; prepare* H_r*, then*

1. *Then there is* $q \in D \cap M$ *such that* $H_q \setminus H_r$ *and* $\mathcal{M}_q \setminus \mathcal{M}_r$ *are each disjoint from* E *for all* $E \in \mathcal{M}_r \cap M^* \cap \mathcal{E}_0^2$*;*
2. *to define* $\bar{q} < q, r$, $\mathcal{M}_{\bar{q}}$ *is handled by Neeman/Velickovic as the smallest set containing* $\mathcal{M}_q \cup \mathcal{M}_r$ *closed under finite intersections, and* $H_{\bar{q}} = H_q \cup H_r$ *(this takes some work),*
3. *for each* $t \in H_q \setminus H_r$*, we have* $t^\dagger \in H_q \setminus H_r$ *and (we designate* t as primary *if)* $t^\dagger$ *has no extension in* H_r *(one of them has this property)*
4. *for* $t \in H_r \setminus H_q$ *(which may have extensions in* H_q*!) we designate it primary if* $t^\dagger$ *has no extension in* H_q

sketch of proper and ω_1-compact

Lemma

Suppose that $\mathbb{P}_T, p \in M^* \prec H(\theta)$, $M = M^* \cap H(\omega_2) \in \mathcal{M}_r$, $r < p$ and $r \in D$ for some dense $D \in M^*$; prepare H_r, then

1. *Then there is $q \in D \cap M$ such that $H_q \setminus H_r$ and $\mathcal{M}_q \setminus \mathcal{M}_r$ are each disjoint from E for all $E \in \mathcal{M}_r \cap M^* \cap \mathcal{E}_0^2$;*
2. *to define $\bar{q} < q, r$, $\mathcal{M}_{\bar{q}}$ is handled by Neeman/Velickovic as the smallest set containing $\mathcal{M}_q \cup \mathcal{M}_r$ closed under finite intersections, and $H_{\bar{q}} = H_q \cup H_r$ (this takes some work),*
3. *for each $t \in H_q \setminus H_r$, we have $t^\dagger \in H_q \setminus H_r$ and (we designate t* as primary *if) $t^\dagger$ has no extension in H_r (one of them has this property)*
4. *for $t \in H_r \setminus H_q$ (which may have extensions in H_q!) we designate it primary if $t^\dagger$ has no extension in H_q*
5. *for $t \in H_q \cup H_r$, we define $a_t^{\bar{q}}$ from $\mathfrak{a}_q$ and $\mathfrak{a}_r$ and we let $a_{t^\dagger}^{\bar{q}}$ be equal $(H_q \cup H_r) \cap C_{t^\dagger} \setminus a_t^{\bar{q}}$.*

Definition (of amalgamation)

We define the values for $a_t^{\bar{q}}$ by recursion on primary t – choose minimal $P_t \in \mathcal{M}_{\bar{q}}$ with $t \in P_t$ (for $t \in H_q$, same as $P_t \in \mathcal{M}_q$)

Definition (of amalgamation)

We define the values for $a_t^{\bar{q}}$ by recursion on primary t – choose minimal $P_t \in \mathcal{M}_{\bar{q}}$ with $t \in P_t$ (for $t \in H_q$, same as $P_t \in \mathcal{M}_q$)

1. if $t \in H_q \cap H_r$, then $a_t^{\bar{q}} = a_t^q \cup a_t^r$

more proof and another lemma

Definition (of amalgamation)

We define the values for $a_t^{\bar{q}}$ by recursion on primary t – choose minimal $P_t \in \mathcal{M}_{\bar{q}}$ with $t \in P_t$ (for $t \in H_q$, same as $P_t \in \mathcal{M}_q$)

1. if $t \in H_q \cap H_r$, then $a_t^{\bar{q}} = a_t^q \cup a_t^r$
2. for $t \in H_q \setminus H_r$ $a_t^{\bar{q}}$ equals a_t^q union of all sets of the form $\bigcap_{s\in L_0} a_s^{\bar{q}} \setminus \bigcup_{u\in L_1} a_u^{\bar{q}}$ where $L_0 \cup L_1 \subset P_t \cap C_t \cap H_q$ satisfies $\bigcap_{s\in L_0} a_s^q \setminus \bigcup_{u\in L_1} a_u^q \subset a_t^q$

Definition (of amalgamation)

We define the values for $a_t^{\bar{q}}$ by recursion on primary t – choose minimal $P_t \in \mathcal{M}_{\bar{q}}$ with $t \in P_t$ (for $t \in H_q$, same as $P_t \in \mathcal{M}_q$)

1. if $t \in H_q \cap H_r$, then $a_t^{\bar{q}} = a_t^q \cup a_t^r$
2. for $t \in H_q \setminus H_r$ $a_t^{\bar{q}}$ equals a_t^q union of all sets of the form $\bigcap_{s \in L_0} a_s^{\bar{q}} \setminus \bigcup_{u \in L_1} a_u^{\bar{q}}$ where $L_0 \cup L_1 \subset P_t \cap C_t \cap H_q$ satisfies $\bigcap_{s \in L_0} a_s^q \setminus \bigcup_{u \in L_1} a_u^q \subset a_t^q$
3. symmetric definition for $t \in H_r \setminus H_q$

more proof and another lemma

Definition (of amalgamation)

We define the values for $a^{\bar{q}}_t$ by recursion on primary t – choose minimal $P_t \in \mathcal{M}_{\bar{q}}$ with $t \in P_t$ (for $t \in H_q$, same as $P_t \in \mathcal{M}_q$)

1. if $t \in H_q \cap H_r$, then $a^{\bar{q}}_t = a^q_t \cup a^r_t$
2. for $t \in H_q \setminus H_r$ $a^{\bar{q}}_t$ equals a^q_t union of all sets of the form $\bigcap_{s \in L_0} a^{\bar{q}}_s \setminus \bigcup_{u \in L_1} a^{\bar{q}}_u$ where $L_0 \cup L_1 \subset P_t \cap C_t \cap H_q$ satisfies $\bigcap_{s \in L_0} a^q_s \setminus \bigcup_{u \in L_1} a^q_u \subset a^q_t$
3. symmetric definition for $t \in H_r \setminus H_q$
4. we note that for $t \in H_r \setminus H_q$ primary, $a^{\bar{q}}_t$ is minimum possible – hence $a^{\bar{q}}_{t^\dagger}$ is pretty big

more proof and another lemma

Definition (of amalgamation)

We define the values for $a_t^{\bar{q}}$ by recursion on primary t – choose minimal $P_t \in \mathcal{M}_{\bar{q}}$ with $t \in P_t$ (for $t \in H_q$, same as $P_t \in \mathcal{M}_q$)

1. if $t \in H_q \cap H_r$, then $a_t^{\bar{q}} = a_t^q \cup a_t^r$
2. for $t \in H_q \setminus H_r$ $a_t^{\bar{q}}$ equals a_t^q union of all sets of the form $\bigcap_{s\in L_0} a_s^{\bar{q}} \setminus \bigcup_{u\in L_1} a_u^{\bar{q}}$ where $L_0 \cup L_1 \subset P_t \cap C_t \cap H_q$ satisfies $\bigcap_{s\in L_0} a_s^q \setminus \bigcup_{u\in L_1} a_u^q \subset a_t^q$
3. symmetric definition for $t \in H_r \setminus H_q$
4. we note that for $t \in H_r \setminus H_q$ primary, $a_t^{\bar{q}}$ is minimum possible – hence $a_{t^\dagger}^{\bar{q}}$ is pretty big

Lemma

Forcing with $\mathbb{P}_T$ adds many new branches to bT, but none with uncountable cofinality. Thus $bT \setminus (2^{\omega_2})^V$ consists of countable cofinality branches.

Lemma (sample)

Let $x \in 2^{\omega_2}$, $x \supset \{t_\alpha : \alpha \in Succ(\omega_2)\}$. *If* p *forces that* $\dot{A}$ *is an uncountable set of successor ordinals,* $p, \dot{A} \in M_1 \prec H(\theta)$, $E = M_1 \cap H(\omega_2) \in \mathcal{E}^2_1$, *and* p *forces uncountable* $\{\alpha \in \dot{A} : t_\alpha \notin \bigcup\{\dot{a}_{t_\beta} : \beta \in L\}\}$ *for any finite* $L \in [\lambda]^{<\omega}$,

Lemma (sample)

Let $x \in 2^{\omega_2}$, $x \supset \{t_\alpha : \alpha \in Succ(\omega_2)\}$. *If* p *forces that* $\dot{A}$ *is an uncountable set of successor ordinals,* $p, \dot{A} \in M_1 \prec H(\theta)$, $E = M_1 \cap H(\omega_2) \in \mathcal{E}_1^2$, *and* p *forces uncountable* $\{\alpha \in \dot{A} : t_\alpha \notin \bigcup\{\dot{a}_{t_\beta} : \beta \in L\}\}$ *for any finite* $L \in [\lambda]^{<\omega}$,

1. *then for each* n, p *forces that* $\dot{a}_{t_\lambda+n} \setminus \bigcup_{k<n} \dot{a}_{t_\lambda+k}$ *contains uncountably many members of* $\{t_\alpha : \alpha \in \dot{A}\}$.

Lemma (sample)

Let $x \in 2^{\omega_2}$, $x \supset \{t_\alpha : \alpha \in Succ(\omega_2)\}$. *If* p *forces that* $\dot{A}$ *is an uncountable set of successor ordinals,* $p, \dot{A} \in M_1 \prec H(\theta)$, $E = M_1 \cap H(\omega_2) \in \mathcal{E}^2_1$, *and* p *forces uncountable* $\{\alpha \in \dot{A} : t_\alpha \notin \bigcup\{\dot{a}_{t_\beta} : \beta \in L\}\}$ *for any finite* $L \in [\lambda]^{<\omega}$,

1. *then for each* n, p *forces that* $\dot{a}_{t_\lambda+n} \setminus \bigcup_{k<n} \dot{a}_{t_\lambda+k}$ *contains uncountably many members of* $\{t_\alpha : \alpha \in \dot{A}\}$.
2. *and so if* $Y \subset bT$ *is any set that projects onto* $\{t_\alpha : \alpha \in \dot{A}\}$ *then the closure of* Y *contains the entire set* $bT \cap \{x \supset t_\lambda\}$ *for some* $\lambda \in \omega_2$.

Lemma (sample)

Let $x \in 2^{\omega_2}$, $x \supset \{t_\alpha : \alpha \in Succ(\omega_2)\}$. If p forces that $\dot{A}$ is an uncountable set of successor ordinals, $p, \dot{A} \in M_1 \prec H(\theta)$, $E = M_1 \cap H(\omega_2) \in \mathcal{E}^2_1$, and p forces uncountable $\{\alpha \in \dot{A} : t_\alpha \notin \bigcup\{\dot{a}_{t_\beta} : \beta \in L\}\}$ for any finite $L \in [\lambda]^{<\omega}$,

1. *then for each n, p forces that $\dot{a}_{t_{\lambda+n}} \setminus \bigcup_{k<n} \dot{a}_{t_{\lambda+k}}$ contains uncountably many members of $\{t_\alpha : \alpha \in \dot{A}\}$.*
2. *and so if $Y \subset bT$ is any set that projects onto $\{t_\alpha : \alpha \in \dot{A}\}$ then the closure of Y contains the entire set $bT \cap \{x \supset t_\lambda\}$ for some $\lambda \in \omega_2$.*

Proof.

We can work inside of E to decide members of $\dot{A}$ and designate each $t_{\lambda+k}$ ($k < n$) as primary. □

Now we can force Martin's Axiom

Proposition

Let $G \subset \mathbb{P}_T$ be generic. Let Q be a Souslin-free ccc poset of cardinality at most ω_1. What are the properties of $X = bT \cap 2^{<\omega_2}$ in this extension?

Now we can force Martin's Axiom

Proposition

Let $G \subset \mathbb{P}_T$ be generic. Let Q be a Souslin-free ccc poset of cardinality at most ω_1. What are the properties of $X = bT \cap 2^{<\omega_2}$ in this extension?

1. bT acquires only new branches of countable cofinality, and all such points will have countable character

Now we can force Martin's Axiom

Proposition

Let $G \subset \mathbb{P}_T$ be generic. Let Q be a Souslin-free ccc poset of cardinality at most ω_1. What are the properties of $X = bT \cap 2^{<\omega_2}$ in this extension?

1. bT acquires only new branches of countable cofinality, and all such points will have countable character
2. each countable subset of X which does not have compact closure, still contains a set of the form $bT \cap \{x \supset t\}$ for some $t \in T$

Now we can force Martin's Axiom

Proposition

Let $G \subset \mathbb{P}_T$ be generic. Let Q be a Souslin-free ccc poset of cardinality at most ω_1. What are the properties of $X = bT \cap 2^{<\omega_2}$ in this extension?

1. bT acquires only new branches of countable cofinality, and all such points will have countable character
2. each countable subset of X which does not have compact closure, still contains a set of the form $bT \cap \{x \supset t\}$ for some $t \in T$
3. each ω_1-sized subset of X does the same

Now we can force Martin's Axiom

Proposition

Let $G \subset \mathbb{P}_T$ be generic. Let Q be a Souslin-free ccc poset of cardinality at most ω_1. What are the properties of $X = bT \cap 2^{<\omega_2}$ in this extension?

1. bT acquires only new branches of countable cofinality, and all such points will have countable character
2. each countable subset of X which does not have compact closure, still contains a set of the form $bT \cap \{x \supset t\}$ for some $t \in T$
3. each ω_1-sized subset of X does the same
4. therefore X is still first-countable and initially ω_1-compact in the forcing extension by Q.

Corollary

It is consistent with MA $+\mathfrak{c} = \omega_2$ that there is a first-countable initially ω_1-compact space which is not compact.

Corollary

It is consistent with MA $+\mathfrak{c} = \omega_2$ that there is a first-countable initially ω_1-compact space which is not compact.

Corollary

It is consistent with MA $+\mathfrak{c} = \omega_2$ that there is a Moore-Mrowka space.

Corollary

It is consistent with MA $+\mathfrak{c} = \omega_2$ that there is a first-countable initially ω_1-compact space which is not compact.

Corollary

It is consistent with MA $+\mathfrak{c} = \omega_2$ that there is a Moore-Mrowka space.

Corollary

Our bT is βX and it has the desired property that every $Y \in [bT \cap 2^{\omega_2}]^{\leq\omega_1}$ accumulates to points in X, thus there is a first-countable initially ω_1-compact space of cardinality greater than $\mathfrak{c}$.

Proof.

We skip the proof of part 1.
To prove parts 2 and 3, we note that the space we get from $\mathfrak{a}_{T_\omega}$ maps perfectly onto the space we get from $\mathfrak{a}_{T_R}$. I.e. our only points are the t_α with neighborhoods given by $A_{t^\dagger_\alpha}$. It suffices to show that this space remains initially ω_1-compact.
We just consider countably compact.
Assume that $\dot{A} = \{\dot{\xi}_n : n \in \omega\}$ is a Q-name of an infinite set of successor ordinals. Assume that for all $q \in Q$, and all finite $L \subset Succ(\omega_2)$, there is $q' < q$ and an n such that $q' \Vdash t_{\dot{\xi}_n} \in A_{t_\beta}$ for each $\beta \in L$ (a typical neighborhood of the point $\vec{1}$).
For each uncountable limit λ, choose finite $F_\lambda \subset \lambda$, n_λ, and $q_\lambda \in Q$, so that for $n \geq n_\lambda$, if $q_\lambda \Vdash \dot{x}_n \in A_{t^\dagger_{\lambda+1}}$ then $q_\lambda \Vdash \dot{x}_n \in A_{t^\dagger_\beta}$ for some $\beta \in F_\lambda$.
But now, there is a stationary S and fixed $\bar{q}, \bar{n}, F$ so that $(q_\lambda, n_\lambda, F_\lambda) = (\bar{q}, \bar{n}, F)$ for all $\lambda \in S$.
However, it now follows that $Y = \{t_\xi : (\exists q' < \bar{q})\xi \in \dot{A}\}$ must have compact closure, since its closure misses $\{t_{\lambda+1} : \lambda \in S\}$.