**Remark 20.** *For every set of $\aleph_1$ many reals, there is a $\lambda < \omega_2$ so that this set of reals is in the model obtained by forcing with $Fn(\lambda, 2) \times \mathcal{M}_\lambda$. Thus it will have measure 0. However, from the point of view of the model $V[H]$, there is a set of $\aleph_1$-many reals added by $Fn(\omega_1, 2)$ which will not be meager.*

## 1. Iterated forcing

Iterated forcing is, in some sense, really just iterated construction of posets combined with reflection and factoring of the poset to prove the properties. But the initial stages of the forcing allows us to pick the future posets.

**Definition 21.** If $P$ is a poset, and $\dot{Q}$ is forced by $1_P$ to be a poset with ordering $\dot{<}_Q$, then $P * \dot{Q}$ is defined so that forcing with $P * \dot{Q}$ is the same as forcing with $P$ and then forcing with $\dot{Q}$.

But we have to be precise about definition of the elements of $P * \dot{Q}$ because there are too many possibilities. On the other hand, for many technical reasons we do want a lot of choices.

**Definition 22.** Let $P$ be a poset and assume that $1_P$ forces that $\dot{Q}$ – i.e. $(\dot{Q}, \dot{<}_Q, 1_Q)$ – is a poset in the extension. Let $\theta$ be a regular cardinal such that $P$ and $\dot{Q}$ are in $H(\theta)$. Let $P * \dot{Q}$ denote the set of pairs $(p, \dot{q})$ such that $p \in P$, $\dot{q} \in H(\theta)$, and $p \Vdash \dot{q} \in \dot{Q}$. The ordering $<$ on $P * \dot{Q}$ is clear: $(p_1, \dot{q}_1) < (p_0, \dot{q}_0)$ providing $p_1 <_P p_0$ and $p_1 \Vdash \dot{q}_1 <_Q \dot{q}_0$.

**Remark 23.** *We have seen, for example, that if $P = Fn(\omega_2, 2)$, then $P * P$ is the same as $P \times P$, and that $P * \mathcal{M}_{\omega_2}$ is not the same as $P * \check{\mathcal{M}}_{\omega_2}$. Similarly, $P * \check{\mathcal{H}}$ is the same as $P \times \mathcal{H}$, but is different from $P * \mathcal{H}$.*

*The jargon here is that things like $\mathcal{M}_{\omega_2}$ and $\mathcal{H}$ are "forcing notions" which are interpreted in the model under discussion, and the actual set of elements (but not usually the ordering) changes when we move to a forcing extension.*

We mostly prove things about the extension by $P * \dot{Q}$ by first forcing with $P$ and then forcing with $val_G(\dot{Q})$ – so why have it?

The answer is that there is no end to how many times you can extend your poset $P_\alpha$ by attaching a $\dot{Q}_\alpha$ because you are just building confusing posets, but it is very unclear what it means to force infinitely many times – a point which is handled at limit stages in the construction of $P_\alpha$'s.

**Definition 24** (V.3.11 Kunen - Set Theory)**.** An $\alpha$-stage iterated forcing construction is a pair

$$( \ \{(P_\xi, <_\xi, 1_\xi) : \xi \leq \alpha\} \ \ , \ \ \{(\dot{Q}_\xi, <_{\dot{Q}_\xi}, 1_{\dot{Q}_\xi}) : \xi < \alpha\}$$

such that

(1) Each $P_\xi$ is a forcing poset,
(2) each $(\dot{Q}_\xi, <_{\dot{Q}_\xi}, 1_{\dot{Q}_\xi})$ is a $P_\xi$-name for a forcing poset,
(3) each $p \in P_\xi$ is a function of the form $\langle \dot{q}_\mu : \mu < \xi \rangle$ where each $\dot{q}_\mu$ is an element of $dom(\dot{Q}_\xi)$, and $p(\mu)$ would denote this $\dot{q}_\mu$
(4) if $\xi < \eta$ and $p \in P_\eta$, then $p \restriction \xi \in P_\xi$
(5) if $\xi < \eta$ and $p \in P_\xi$, then there is $p' \in P_\eta$ such that $p' \restriction \xi = p$ and $p'(\mu) = 1_{\dot{Q}_\mu}$ for all $\xi \leq \mu < \eta$
(6) $1_\xi$ is the sequence $\langle 1_{\dot{Q}_\mu} : \mu < \xi \rangle$

(7) $p \leq_\xi p'$ if, for each $\mu < \xi$, $p \restriction \mu \Vdash p(\mu) <_{\dot{Q}_\mu} p'(\mu)$

(8) for each $\xi < \alpha$, $P_{\xi+1}$ is (basically) $P_\xi * \dot{Q}_\xi$ – more precisely, for each $p \in P_\xi$ and each $\dot{q} \in dom(\dot{Q}_\xi)$ such that $p \Vdash \dot{q} \in \dot{Q}_\xi$, we have $p \cup \{(\xi, \dot{q})\} \in P_{\xi+1}$

The above is a prescription, but it still needs more precision at limits. We will mainly study FS-iteration sequences and CS-iteration sequences. For each $p \in P_\alpha$, the set $supp(p) = \{\mu : p(\mu) \neq 1_{\dot{Q}_\mu}\}$ is finite or countable respectively. I find it useful to replace $p$ by $p \restriction supp(p)$ and modify the definitions accordingly.

**Lemma 25.** *Suppose that $p, q$ are in $P_\alpha$ which is constructed from an interation sequence as above. Assume that $supp(p) \cap supp(q) \subset \xi$. Then $p \not\perp q$ if and only if $p \restriction \xi \not\perp q \restriction \xi$*

**Theorem 26.** *A finite support iteration of ccc posets is ccc.*

*Proof.* We leave as an informal exercise that the iteration of two ccc posets is ccc. Here we are more interested in the handling of the supports. Suppose that $P_\alpha$ is built from a FS-iteration of ccc posets. Naturally this theorem is proven by induction on $\alpha$ and, by the exercise we assume that $\alpha$ is a limit. Let $\{p_\gamma : \gamma \in \omega_1\} \subset P_\alpha$. For each $\gamma$, let $F_\gamma = supp(p_\gamma)$. Clearly by the induction hypothesis, we must have that supports are cofinal in $\alpha$ and that $\alpha$ has cofinality $\omega_1$. Apply the $\Delta$-system and obtain some $\delta < \alpha$ so that the root $F$ of the $\Delta$-system is below $\delta$. By the induction hypothesis, there are $\gamma, \zeta$ (indices from the $\Delta$-system) so that $p_\gamma \restriction \delta$ and $p_\zeta \restriction \delta$ are compatible. By Lemma 25, $p_\gamma$ and $p_\zeta$ are compatible. $\square$

**Question 5.** *Suppose that $P_\kappa$ is the result of the FS-support iteration $\{P_\xi, \dot{Q}_\xi : \xi < \alpha\}$. It is pretty clear that for each $\mu < \kappa$, the poset $P_\mu$ is completely embedded (is a subposet by my convention) in $P_\kappa$. Thus, if $G_\mu$ is $P_\mu$-generic, what does $G_\mu^+$ look like?*

**Remark 27.** *The answer is that it also looks like a FS-support iteration, and if the selection of $\dot{Q}_\xi$'s were made by some natural formula, this "tail" of the forcing, perhaps denoted $P_\kappa^\xi$ is, for all intents and purposes the same as $P_\kappa$ (as a forcing notion). In particular, if $\dot{Q}_\xi$ is always chosen to be $\mathcal{H}$, then this is true in the sense that $P_\kappa$ and $P_\kappa^\xi$ are both "the FS-iteration of Hechler posets".*

**Remark 28.** *The details are messy but straightforward.*

**Question 6.** *What are some of the theorems of $V[G_\kappa]$ when CH holds in $V$ and $P_\kappa$ is the FS-iteration of $\mathcal{H}$ for some regular cardinal $\kappa$?*

**Exercise 8.** *The value of $\mathfrak{b}, \mathfrak{d}, \mathfrak{a}, \mathfrak{u}$ are all equal to $\mathfrak{c} = \kappa$. (Hint: factor the forcing)*

**Question 7.** *What are the values of $\mathfrak{t}$ and $\mathfrak{s}$?*

**Lemma 29.** *If any finite support iteration $P_\kappa$ over a model of CH fails to preserve a witness to $\mathfrak{t} \leq \omega_1$ or (most witnesses to) $\mathfrak{s} \leq \omega_1$, then it does so at some successor step.*

This next result is an excellent one: even though $\mathcal{H}$ fills the filter on $\omega \times \omega$ generated by all sets of the form $\{(m, f(m)) : m > n, f \in \omega^\omega\}$, it will not fill a linear tower.

but first, let's go back to Truss

**Remark 30.** *If $D \subset \mathcal{H}$ is dense, define $T_D$ to be the minimal elements of $\{t \in \omega^{<\omega\uparrow} : (\exists h)(t,h) \in D\}$. Then define $\tilde{T}_D = \{s \in \omega^{<\omega\uparrow} : (\forall t \in T_D)|s| = |t| \Rightarrow s \not\leq t\}$ (nothing in $T_D$ above it with the same domain). Check that $\tilde{T}_D$ has no cofinal branches and that this is absolute. Truss theorem follows from this observation. The basic notion is the well-foundedness of $\tilde{T}_D$.*

**Lemma 31** (Baumgartner)**.** *Say that a family $\mathcal{A}$ is $\omega$-hitting if for each countable $\mathcal{S} \subset [\omega]^\omega$, there is an $a \in \mathcal{A}$ such that $a \cap S$ is infinite for all $S \in \mathcal{S}$. Forcing with $\mathcal{H}$ preserves that $\mathcal{A}$ is $\omega$-hitting.*

*Proof.* Let $\mathcal{A}$ be a member of any countable elementary submodel $M$. Choose $a \in \mathcal{A}$ so that $a \cap b$ is infinite for all infinite $b \subset \omega$ in $M$. We prove that for any $\tau \in M$ and any $(s,f) \in \mathcal{H}$ such that $(s,f) \Vdash \tau \in [\omega]^\omega$, we have that $(s,f) \Vdash \check{a} \cap \tau \in [\omega]^\omega$.

The trick is the rank function. Let $D \subset \mathcal{H}$ be any dense set, and let $T_D$ be as above. For each $t \in T_D$ we define $rk_D(t) = 0$. Actually every $s$ extending any $t \in T_D$ will also have $rk_D(s) = 0$. For other $s \in \omega^{<\omega\uparrow}$, we want to measure how long is the walk to $T_D$ from $(s,g)$ regardless of which $g \in \omega^{\omega\uparrow}$ we are given. The right definition for $\mathcal{H}$ (this is an idea that works for some other posets) is that $rk_D(s) \leq \alpha$ if there is an integer $\ell$ so that for all $n$, there is an $s_n$ extending $s$ and in $\omega^\ell$ so that $s_n(|s|) > n$ and $rk(s_n) < \alpha$. Okay, it's a mouthful. But notice that for any $g$, there is an $n$ so that $(s_n, g) < (s,g)$.

**Claim 1.** For all $s \in \omega^{<\omega\uparrow}$, $rk_D(s) < \omega_1$ (i.e. "it exists").

Recursively define $g \in \omega^{\omega\uparrow}$ so that for each $\ell$ and each $s'$ such that $(s',g) < (s,g)$ we have that $rk_D(s')$ also does not exist. Suppose we have defined $g \restriction \ell$. We suppose we fail to define $g(\ell)$, hence for each $n$ we have $t_n \in \omega^\ell$ such that $(t_n, g) < (s,g)$, $t_n(\ell) > n$, and $rk_D(t_n) < \omega_1$. Find minimal $j$ so that the family $\{t_n(j) : n \in \omega\}$ is unbounded. Find $\bar{t} \in \omega^j$ so that there are infinitely many $n$ such that $\bar{t} = t_n \restriction j$ and this subfamily is also unbounded at $j$. Notice then that $rk_D(\bar{t})$ will then be less than $\omega_1$ and $(\bar{t}, g) < (s,g)$.

Assume there is a $(\bar{s}, \bar{f})$ and $\bar{n}$ so that $(\bar{s}, \bar{f}) \Vdash \check{a} \cap \tau \subset \bar{n}$. Let $h_G$ denote the generic Hechler real added by $\mathcal{H}$. We may assume that $dom(\bar{s}) = \bar{n}$. Choose, in $M$, $\sigma$ a name so that, for some $f \in M$, $(\bar{s}, f) \Vdash \sigma = \min(\tau \setminus h_G(\bar{n}))$ (i.e. if $(t,f)$ decides $\sigma$, then it forces $\sigma > t(\bar{n})$). We just used the maximum principle. We have the dense set $D$ of conditions which force a value on $\sigma$. For each condition $(s,f)$ extending $(\bar{s}, f)$, define the set $L(s) = \{m \in \omega : (\forall g)(s,g) \not\Vdash \check{m} \neq \sigma\}$. In words, $m$ is in $L(s)$ if the second coordinate can not prevent $\sigma$ from having value $m$. By induction on $rk_D(s)$, if $(s, \bar{f}) < (\bar{s}, \bar{f})$, then $L(s)$ is a finite subset of $\omega \setminus \bar{n}$ and non-empty. In fact it should be a subset of $\omega \setminus s(\bar{n})$, and so, in fact $L(\bar{s})$ can not be finite and non-empty.

This clearly holds if $rk_D(s) = 0$. Assume $(s, \bar{f}) < (\bar{s}, \bar{f})$ and that it holds for all $s'$ with $rk_D(s') < rk_D(s)$, and choose the sequence $\{s_n : n \in \omega\}$ and $\ell$ witnessing the rank of $rk_D(s)$. Thus, for all but finitely many $n$, $L(s_n)$ is finite and non-empty (in fact for all $n > \bar{f}(\ell)$). Also, the set $\bigcup\{L(s_n) : n > \bar{f}(\ell)\}$ is necessarily disjoint from $a$ and so it is finite. And $L(s)$ contains the set of all $k$ which appear in infinitely many. $\square$

Perhaps more important is what we actually proved in the proof.

**Corollary 32.** *If $N \prec H(\theta)$ is countable, and $a \in [\omega]^\omega$ meets every member of $N \cap [\omega]^\omega$, then that $a$ meets every member of $N[G] \cap [\omega]^\omega$. And, this holds if $G$ is a generic for the FS-iteration of Hechler.*

Another important forcing is the forcing that does diagonalize a filter.

**Definition 33.** For a filter $\mathcal{F}$ on $\omega$ define $Pr(\mathcal{F})$ to be the set of conditions $[\omega]^{<\omega} \times \mathcal{F}$ with the ordering $(a, A) < (b, B)$ providing $a \supset b$, $A \subset B$, and $a \setminus b \subset B$.

If $\mathcal{A}$ is an almost disjoint family, then let $\mathcal{F}_\mathcal{A}$ denote the dual filter to the ideal generated by $\mathcal{A}$. Then $Pr(\mathcal{F}_\mathcal{A})$ adds a set almost disjoint from every member of $\mathcal{A}$.

If $\mathcal{F}$ is a filter that already has such diagonalizing sets, let $\mathcal{J}$ be the family of all such sets, which diagonalize it, then by genericity, forcing $Pr(\mathcal{F})$ adds a new set which diagonalizes $\mathcal{F}$ and meets every set in $\mathcal{J}$ in an infinite set.

Choosing a suitable regular cardinal $\kappa$ and standard bookkeeping:

**Proposition 34.** *There is a finite support iteration of $\sigma$-centered posets of the form $Pr(\mathcal{F})$ so that in $V[G_\kappa]$ we have that $\mathfrak{p} = \mathfrak{c} = \kappa$. Also, there is a further forcing of length $\kappa + \omega_1$ in which we have $\mathfrak{d} = \mathfrak{u} = \omega_1$. And a forcing $\mathbb{P}_{\beth_{\omega_1}}$ in which the $\pi$-character of $\beta\omega \setminus \omega$ is $\omega_1$.*

By the way, recall that there does exist Souslin trees in such models – similar to the Exercise about Cohen reals not adding uncountable antichains, it is also true that FS-iteration of $\sigma$-centered does not add uncountable antichains.

**Remark 35.** *It is a theorem of Bell and Kunen (using independent matrices) that there is an ultrafilter on $\omega$ with $\pi$-character at least $cf(\mathfrak{c})$.*

Of course $\mathcal{F}$ could be an ultrafilter, in which case $Pr(\mathcal{F})$ will destroy each splitting family. But for selective ultrafilters something very special happens.

**Lemma 36.** *If $\mathcal{U}$ is a selective ultrafilter, then $Pr(\mathcal{U})$ has the pure decision property. That is, for each $\tau$ and each $(a, B) \in Pr(\mathcal{U})$, there is a $A \in \mathcal{U}$ such that $(a, A)$ decides the forcing statement "$\tau = \check{0}$" (or any other).*

*Proof.* For simplicity we start with $(\emptyset, B)$ and we show there is an $A \subset B$. Define $A_\ell \in \mathcal{U}$ so that for all $s \subset \ell$ if there is any $U \in \mathcal{U}$, such that $(s, U)$ forces a decision on $\tau = \check{0}$, then $(s, A_\ell)$ does as well.

Similarly, for each $s \subset \ell$, there is some member $U(\ell, s) \in \mathcal{U}$ such that for all $u \in U(\ell, s)$, the status is constant for the statement about if there is some $U \in \mathcal{U}$ such that $(s \cup \{u\}, U)$ decides $\tau = \check{0}$, and so we may assume that is the same for all $u \in A_\ell$. (just intersecting the above $A_\ell$ with more member of $\mathcal{U}$).

Apply the fact that $\mathcal{U}$ is selective and obtain a set $U \in \mathcal{U}$ with the property that $U$ meets each $A_\ell \setminus A_{\ell+1}$ in at most a singleton. Let $\{u_\ell : \ell \in \omega\}$ be the enumeration of $U$ and define $g(m)$ (for each $m$) so large that $u_{g(m)} < g(m+1)$ – hence $\{u_\ell : \ell \geq g(m+1)\}$ is contained in $A_{u_{g(m)}}$. We may assume that $W = \bigcup\{[g(2m), g(2m+1)) : m \in \omega\} \in \mathcal{U}$, and then apply selectivity to choose $A = \{a_m : m \in \omega\} \subset W$ in $\mathcal{U}$ so that for each $m$, $a_m \in [g(2m), g(2m+1))$.

Now, let us choose $s \subset \{a_m : m \in \omega\}$ with minimal cardinality so that there is some $U \in \mathcal{U}$ which forces (wlog) that $\tau = \check{0}$ – just decides the truth. We want to show that $s = \emptyset$, so let $m$ be chosen so that $a_m = \max(s)$. Now it follows that $(s, A \setminus (a_m{+}1))$ already forces that $\tau = \check{0}$, and moreover that for all $\ell \geq m$, $(\{a_\ell\} \cup s \setminus \{a_m\}, A \setminus (a_\ell{+}1))$ also forces that $\tau = \check{0}$. But this means that $(s \setminus \{a_m\}, A \setminus a_m)$ forces that $\tau = \check{0}$, contradicting the minimality of $|s|$. $\square$

**Definition 37.** A set of reals $X$ is a $\gamma$-set if for each $\omega$-cover by open sets (every finite subset of $X$ is covered by a member of the $\omega$-cover) has a $\gamma$-subcover $\{U_n : n \in \omega\}$ (every member of $X$ is in all but finitely many).

For an increasing sequence $\vec{k} = \langle k_\ell : \ell \in \omega \rangle$ (($\ell < k_\ell$), define the $\omega$-cover $\mathcal{C}_{\vec{k}} = \bigcup_\ell \mathcal{C}_{\vec{k},\ell}$ of $2^\omega$ by $C \in \mathcal{C}_{\vec{k},\ell}$ if there is a set $\rho_C \in [2^{k_\ell}]^\ell$ such that $C = \bigcup\{[s] : s \in \rho_C\}$.

This next result was discovered for Laver forcing by Laver when proving the consistency of the Borel conjecture, and adapted to Mathias forcing by Baumgartner.

**Lemma 38.** *If $\langle k_\ell : \ell \in \omega \rangle$ is the real added by $Pr(\mathcal{U})$, and $C_\ell \in \mathcal{C}_{\vec{k},\ell}$ for each $\ell$, then $\{x \in 2^\omega \cap V : (\exists^\infty \ell) x \in C_{\vec{k},\ell}\}$ is countable. Denote this forcing extension as $V[G_{\mathcal{U}}]$.*

*Proof.* Assume that $\{\dot{C}_j : j \in \omega\}$ is forced to be a subsequence of $\mathcal{C}_{\vec{k}}$ for the generic sequence such that for each $j$, $\dot{C}_j \in \mathcal{C}_{\vec{k},j}$.

Step 1 is to find a sequence $U = \{u_m : m \in \omega\} \in \mathcal{U}$ so that the following holds for each $m$. For each $\ell \leq m$, and each $u_m \in s \subset \{u_i : i \leq m\}$ with $\ell = |s|$, notice that $(s, U \setminus (u_m{+}1))$ forces that $\vec{k}(\ell) = u_m$. We can arrange that $(s, U \setminus u_m + 1)$ decides the value of $\rho_\ell \in [2^{u_m}]^\ell$ so that $\dot{C}_\ell$ is determined by $\rho_\ell$. This uses the decision property $2^m$ times, and then uses that $\mathcal{U}$ is selective to get one set $U$ to work in this way for all $m$.

That is, for each $\ell$ and each $s \in [U]^{\ell-1}$, there is a set $\rho_\ell^{s,m} \in [2^{u_m}]^\ell$ for large enough $m$ such that $(s \cup \{u_m\}, U \setminus (u_m + 1))$ forces that the clopen set $C_\ell^{s,m} = \bigcup\{[\psi] : \psi \in \rho_\ell^{s,m}\}$ is equal to $\dot{C}_\ell$.

For this next consequence, let $\otimes C_\ell^{s,m}$ denote the following clopen subset of $(2^\omega)^\ell$. Let $\{\psi_0^{s,m,\ell}, \ldots, \psi_{\ell-1}^{s,m,\ell}\}$ be $\rho_\ell^{s,m}$ ordered lexicographically, and set $\otimes C_\ell^{s,m}$ equal the product $[\psi_0^{s,m,\ell}] \times \cdots [\psi_{\ell-1}^{s,m,\ell}]$. This is all just so that we can identify the unique $\ell$-tuple $\vec{x}_\ell^s \in [2^\omega]^\ell$ which is the $\mathcal{U}$-limit of this sequence of clopen sets. That is, $\vec{x}_\ell^s$ is the unique point of $(2^\omega)^\ell$ which is in the closure of $\bigcup\{C_\ell^{s,m} : m \in W\}$ for all $W \in \mathcal{U}$. If $x$ is any point not equal to a coordinate value of $\vec{x}_\ell^s$, then there is a set $W_{x,\ell} \in \mathcal{U}$ such that $x \notin C_\ell^{s,m}$ for all $m \in W_{x,\ell}$.

Suppose that $x \in 2^\omega$ and that $x$ is not a member of $\vec{x}_\ell^s$ for any $s, \ell$. We show (for simplicity) that an arbitrary $(\emptyset, U')$ has an extension forcing that $x$ is in only finitely many of the $\dot{C}_j$ rather than working below some given $(s', U')$. For each $j$, there is a $U_j \in \mathcal{U}$ such that $x \notin C_\ell^{s,m}$ for all $u_m \in U_j$ and $s \in [\{u_i : i \leq j\}]^{\ell-1}$. Apply selectivity to get $U_x$ so that $(\emptyset, U_x) \Vdash \check{x} \notin \dot{C}_\ell$ for all $\ell$. Of course $(\emptyset, U_x \cap U')$ does the job. $\square$

**Theorem 39** (Miller)**.** *In the Hechler model, every set of reals of size $\omega_1$ has strong measure zero (because of the Cohen reals) but none are $\gamma$-sets.*

Here are the steps. We start with formulating a weaker statement than that in Lemma 38 which is preserved by Cohen forcing.

**Corollary 40.** *In the model obtained by forcing with $\omega^{<\omega\uparrow}$ (adding the Cohen real $c$) over the model $V[G_{\mathcal{U}}]$ from Lemma 38, the sequence $\vec{k}' = \{k'_\ell = k_\ell + c(\ell) : \ell \in \omega\}$ has the property that $V \cap \bigcap\{C_j : j \in \omega\}$ is countable whenever $\{C_j : j \in \omega\}$ is an infinite subset of $\mathcal{C}_{\vec{k}'}$.*

*Proof.* Fix any sequence of names $\dot{C}_j$ ($j \in \omega$) which are forced to be distinct members of $\mathcal{C}_{\vec{k}'}$. For each $x \in V \cap 2^\omega$, choose if possible $p_x \in \omega^{<\omega\uparrow}$ so that $p_x \Vdash \check{x} \in \dot{C}_j$ for all $j$. Assume there is an uncountable set $X \subset 2^\omega \cap V$ so that $p_x = p$ for each $x \in X$. Choose any infinite set $\{D_\ell : \ell \in \omega\}$ such that, for each $\ell$ there is a $p_\ell \supset p$ and a $j_\ell \geq \ell$ such that $p_\ell \Vdash \check{D}_\ell = \dot{C}_{j_\ell}$. By thinning out to a subsequence (and re-indexing) we can assume that $\ell < \ell'$ implies that $j_\ell < j_{\ell'}$. For each $\ell$ then there is a $C_\ell \in \mathcal{C}_{\vec{k},j_\ell}$ such that $D_\ell \subset C_\ell$. It follows that $X \subset C_\ell$ for all $\ell$, which contradicts Lemma 38. □

Miller uses a very clever proof in which he first adds a Laver real (similar argument as for our $Pr(\mathcal{U})$) and then shows Corollary 40 – hence, by Truss' theorem, it holds in the model obtained by adding a single Hechler real. Then he connects the tower preservation property to preserving the property in Corollary 40.

**Corollary 41.** *In the model obtained by adding a Hechler real over V, we have the Hechler sequence $\langle k_\ell : \ell \in \omega\rangle$ satisfies that if $\{C_n : n \in \omega\}$ enumerates $\mathcal{C}_{\vec{k}}$, and if, for $x \in 2^\omega \cap V$, we let $a_x = \{n : x \notin C_n\}$, then every uncountable subset of the collection $\mathcal{A} = \{a_x : x \in 2^\omega \cap V\}$ is $\omega$-hitting.*

*Proof.* This is a restatement of Corollary 40. Let us check. Let $B_n : n \in \omega$ be infinitely many infinite subsets of $\omega$. For each $n$, the set of $x$ such that $a_x$ is almost disjoint from $B_n$ is countable (i.e the set of $x$ which is in $C_\ell$ for all but finitely many $\ell \in B_n$ is countable). Therefore, except for those countably many $x$, we have that $a_x$ hits each $B_n$ infinitely. □

*Proof of Theorem 39.* Let $G_{\omega_2}$ be a generic filter for $\mathbb{P}_{\omega_2}$ which is the FS-iteration of $\mathcal{H}$. Let $X \subset 2^\omega$ have cardinality $\omega_1$, and we show that $X$ is not a $\gamma$-set. By a standard reflection – but the best kind is to fix a name $\dot{X}$ for $X$ and a condition $p \in G_{\omega_2}$ to show that $p$ doesn't force that $\dot{X}$ is a $\gamma$-set, and to then take an $M \prec H(\theta)$ with $M^\omega \subset M$ and $|M| = \omega_1$, and finally $\lambda = M \cap \omega_2$ – we have that $X \in V[G_\lambda]$. Let $\vec{k}$ be the Hechler real added by $G_{\lambda+1}$. $\mathcal{C}_{\vec{k}}$ is an $\omega$-cover of $X$. Fix an enumeration $\{C_n : n \in \omega\}$ for $\mathcal{C}_{\vec{k}}$ in $V[G_{\lambda+1}]$. We work in the model $V[G_{\lambda+1}]$ and note that the final model is a FS-iteration Hechler forcing extension. Suppose that $\dot{b}$ is a name of a subset of $\omega$. Choose a countable elementary submodel $N$ with $\dot{b}$ (etc.) in $N$. By Corollary 32, every member of the family $\{a_x : x \in X\} \setminus N$ will hit $val_G(\dot{b})$ (we just need one). It follows that each such $x$ is not in infinitely many of $\{C_n : n \in \dot{b}\}$ – hence there is no $\gamma$-subcover. □

By similar methods using the decision property, one can prove:

**Lemma 42.** *If $\mathcal{U}$ is selective, and $\tau$ is a name which is forced to be an infinite subset of $\omega$ (by $(\emptyset, B)$). Then there is an $A = \{a_\ell : \ell \in \omega\} \in \mathcal{U}$, and a sequence $\langle S_\ell : \ell \in \omega\rangle$ so that $a_\ell \leq \max(S_\ell) < \min(S_{\ell+1}) \leq a_{\ell+1}$ such that for each $\ell$ and each $s \subset \{a_j : j \leq \ell\}$, if $(s, A \setminus a_\ell + 1)$ decides a value in $\tau \setminus \max(S_{\ell-1})$ then it forces $\tau \cap S_\ell \neq \emptyset$.*

**Corollary 43.** *If $\mathcal{U}$ is selective and if $Pr(\mathcal{U})$ diagonalizes an ultrafilter $\mathcal{V}$, then $\mathcal{V}$ is RK-equivalent to $\mathcal{U}$.*

**Definition 44.** $\mathfrak{h}$ is the minimum height of a dense subtree $\mathcal{T}$ of $[\omega]^\omega/fin$. It exists and is equal to the distributivity degree of the poset $[\omega]^\omega/fin$. It is known that $\mathfrak{t} \leq \mathfrak{h} \leq \min\{\mathfrak{b}, \mathfrak{s}\}$.

**Question 8.** *Can we get a model of* $\mathfrak{t} < \mathfrak{h} < \min\{\mathfrak{b}, \mathfrak{s}\}$ *(and how about also distinguishing* $\max\{\mathfrak{b}, \mathfrak{s}\}$*)?*

We have seen how to sneakily get $\mathfrak{t} = \omega_1$. We have seen that forcing with $Pr(\mathcal{F})$ can be used to fill branches through $\mathcal{T}$. We have seen that forcing with $\mathcal{H}$ can be used to raise $\mathfrak{b}$ without raising $\mathfrak{h}$. And with considerable care we can add Cohen reals and create selective ultrafilters $\mathcal{U}$ so that forcing with $Pr(\mathcal{U})$ will increase $\mathfrak{s}$ without increasing $\mathfrak{h}$.

## 2. Martin's Axiom, gaps and such

Martin and Solovay showed that there is a FS-iteration of ccc posets (which they showed will be ccc – and there are many such posets) so that

(1) $\mathrm{MA}_{\mathcal{C}_1}(\omega_1)$ holds, where $\mathcal{C}_1$ is the family of ccc posets of cardinality at most $\omega_1$, and
(2) $\mathrm{MA}_{\mathcal{C}_1}(\omega_1)$ implies $\mathrm{MA}(\omega_1)$ (where no subscript means the family of ccc posets).

Breaking into these two steps is important to note because, looking ahead, the second fails for proper posets.

Full MA means $\mathrm{MA}(< \mathfrak{c})$ and implies that $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$, hence we must arrange in $V$ to have an uncountable cardinal $\kappa$ so that $\kappa^{<\kappa} = \kappa$. This shows up in the proof as follows:

**Proposition 45** ($\kappa^{<\kappa} = \kappa$)**.** *There is a ccc FS-iteration sequence* $\{P_\alpha, \dot{Q}_\alpha : \alpha < \kappa\}$ *so that for each* $P_\kappa$*-name,* $\dot{Q} \in H(\kappa)$*, and each* $p \in P_\kappa$ *so that* $p \Vdash \dot{Q}$ *is ccc, there are cofinally many* $\alpha \in \kappa$ *so that* $p \Vdash_{P_\alpha} \dot{Q}_\alpha = \dot{Q}$*.*

There are many ways to make the choice of the various $\dot{Q}_\alpha$'s and in this way we can get results that are independent of Martin's Axiom. We start with gaps.

**Definition 46.** A pregap will be a pair $(\mathcal{A}, \mathcal{B})$ of ideals on $\omega$ so that $\mathcal{A} \perp \mathcal{B}$. The pregap is a $(\kappa, \lambda)$-pregap if $\mathcal{A}$ has a cofinal $\kappa$-chain, and $\mathcal{B}$ has a cofinal $\lambda$-chain. Usually we just write $(\{a_\alpha : \alpha \in \kappa\}, \{b_\beta : \beta \in \lambda\})$. A pregap is a gap if there is no $Y \subset \omega$ splitting the gap ($a \subset^* Y$ and $Y \cap b =^* \emptyset$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$). A forcing is said to preserve or split the gap according to whether there is a $Y$ in the extension.

**Definition 47.** A Hausdorff gap is a certain kind of $(\omega_1, \omega_1)$-gap which remains a gap in all $\omega_1$-preserving extensions; in fact, it has the property that for each $\alpha \in \omega_1$ and each $n \in \omega$, the set $\{\beta < \alpha : a_\beta \cap b_\alpha \subset n\}$ is finite.

**Definition 48.** A Lusin gap will be a family $\{(a_\alpha, b_\alpha) : \alpha \in \omega_1\}$ so that for each $\alpha \neq \beta$, $a_\alpha \cap b_\alpha = \emptyset$ and $\emptyset \neq a_\alpha \cap b_\beta =^* \emptyset$ (usually the entire family is an almost disjoint family).

**Theorem 49.** *There is a Hausdorff gap and there is a Lusin gap. As a Boolean algebra,* $\mathcal{P}(\omega)/fin$ *is* $\omega_1$*-saturated but it is not* $\omega_2$*-saturated.*

**Proposition 50.** *A ccc poset of cardinality less than* $\kappa$ *does not split a* $(\kappa, \lambda)$*-gap for any* $\lambda$*.*

*Proof.* We leave this an exercise if $\kappa = \omega$. Suppose that $\langle \{a_\alpha : \alpha \in \kappa\}, \{b_\beta : \beta \in \lambda\} \rangle$ is a gap and that $\tau$ is a $P$-name with $|P| < \kappa$. For each $\alpha \in \kappa$, choose $p_\alpha \in P$ and integer $n_\alpha$ such that $p_\alpha \Vdash a_\alpha \setminus \tau \subset n_\alpha$. There is a pair $p, n$ and a cofinal set

$\Lambda \subset \kappa$ so that $(p_\alpha, n_\alpha) = (p, n)$ for all $\alpha \in \Lambda$. Let $Y = \bigcup\{a_\alpha \setminus n : \alpha \in \Lambda\}$. Notice that there is some $\beta < \lambda$ such that $Y \cap b_\beta$ is infinite. It follows that $p \Vdash \tau \cap b_\beta$ is infinite. □

**Proposition 51** (Kunen)**.** *If $(\mathcal{A}, \mathcal{B})$ is an $(\omega_1, \omega_1)$-gap, there is a ccc poset $Q_{\mathcal{A},\mathcal{B}}$ which forces it to contain a Hausdorff gap (we say it freezes the gap).*

*Proof.* There are other ways to freeze the gap, we use the technique explained by Todorcevic. We may assume that $a_\alpha \cap b_\alpha = \emptyset$ for all $\alpha \in \omega_1$, and that $\mathcal{A} = \{a_\alpha : \alpha \in \omega_1\}$ and $\mathcal{B} = \{b_\alpha : \alpha \in \omega_1\}$. Choose any countable elementary submodel $M$ with $\mathcal{A}, \mathcal{B}$ in $M$ – just to pick $\delta = M \cap \omega_1$. Observe that for all $\alpha \geq \delta$ and $j \in a_\alpha$, the set $I_j = \{\gamma : j \in a_\gamma\}$ is uncountable. Similarly for $j$ in any such $b_\alpha$, we have that $J_j = \{\gamma : j \in b_\gamma\}$ is uncountable.

The poset $Q$ consists of finite sets $q \subset \omega_1 \setminus \delta$ so that for $\alpha \neq \beta \in q$, $a_\alpha \cap b_\beta \neq \emptyset$. We show that $Q$ is ccc (and simple density implies that there is a generic for $Q$ which produces a Luzin subgap). Suppose that $\{q_\xi : \xi \in \omega_1\} \subset Q$ is a $\Delta$-system of finite sets with root $q$. For each $\xi$, set $A_\xi = \bigcap\{a_\alpha : \alpha \in q_\xi \setminus q\}$ and $B_\xi = \bigcap\{b_\beta : \beta \in q_\xi \setminus q\}$. It is immediate that there is some $\xi, \zeta$ such that $A_\xi \cap B_\zeta$ is non-empty. Using the same trick as above, we can assume that for each $j$ that appears in any $A_\xi$ (respectively) $B_\zeta$ appears in uncountably many. Again call these sets $I_j$ and $J_j$ as before. Choose any $j$ in some $A_\xi \cap B_\zeta$ and notice that $j \in a_\alpha \cap b_\beta$ for all $\alpha \in q_\xi \setminus q$ and $\beta \in q_\zeta \setminus q$. Next, the set $\bigcup\{A_\zeta : \zeta \in J_j\}$ must meet the set $\bigcup\{B_\xi : \xi \in I_j\}$, and so now we choose any $\zeta \in J_j$ and $\xi \in I_j$. We have that $j \in A_\xi \cap B_\zeta$ by the above, and now we have $i \in A_\zeta \cap B_\xi$. It is easily checked now that $a_\alpha \cap b_\beta \neq \emptyset$ for any distinct $\alpha, \beta \in q_\xi \cup q_\zeta$. □

**Proposition 52.** *If $(\mathcal{A}, \mathcal{B})$ is a $(\kappa, \lambda)$-gap with $\kappa > \omega_1$ regular, then there is a ccc poset which splits the gap.*

*Proof.* As usual, $\mathcal{A} = \{a_\alpha : \alpha \in \kappa\}$ and $\mathcal{B} = \{b_\beta : \beta \in \lambda\}$. A condition $q \in Q$ is a triple $(A_q, B_q, F_q)$ where $F_q \in [\kappa \cup \lambda]^{<\omega}$, $A_q =^* \bigcup\{a_\alpha : \alpha \in F_q \cap \kappa\}$ is disjoint from $B_q =^* \bigcup\{b_\beta : \beta \in F_q \cap \lambda\}$. We define $p < q$ providing $A_p \supset A_q$, $B_p \supset B_q$, $F_p \supset F_q$. Assume that $\{q_\xi : \xi \in \omega_1\} \subset Q$ and that the family $\{F_\xi = F_{q_\xi} : \xi \in \omega_1\}$ forms a $\Delta$-system with root $F$. Choose any $\delta \in \kappa$ so that $F_\xi \subset \delta$ for all $\xi$. Thin out the sequence so that there is an $n$ so that $A_\xi \setminus n \subset a_\delta$ and $B_\xi \cap a_\delta \subset n$ for all $\xi$. Additionally, we may assume that $A_{q_\xi} \cap n$ and $B_{q_\xi} \cap n$ are independent of $\xi$. Now we have that the described subsequence $\{q_\xi : \xi \in \omega_1\}$ are pairwise compatible. □

**Lemma 53.** *Let $P_\lambda$ be a FS-support iteration and, let $\{\alpha_\xi : \xi < cf(\lambda)\}$ be cofinal in $\lambda$ so that for each $\xi < cf(\lambda)$, $\dot{a}_{\alpha_\xi}, \dot{b}_{\alpha_\xi}$ is added by the poset $\dot{Q}_{\alpha_\xi}$ which generically splits the gap $\{\dot{a}_{\alpha_\mu}, \dot{b}_{\alpha_\mu} : \mu < \xi\}$. Then $\dot{Q}_\lambda$ (generically splitting the gap) is ccc.*

*Proof.* For a gap $\{a_\beta, b_\beta : \beta \in \gamma\}$, the elements $q$ of the gap splitting posets are of the form $q = (c_q, d_q, n_q, F_q)$ where $n_q \in \omega$, $c_q \setminus n_q = \bigcup\{a_\beta \setminus n_q : \beta \in F_q \in [\gamma]^{<\omega}\}$ is disjoint from $d_q \setminus n_q = \bigcup\{b_\beta \setminus n_q : \beta \in F_q \in [\gamma]^{<\omega}\}$ – hence $a_\beta \cap b_\delta \subset n_q$ for $\beta, \delta \in F_q$.

This poset is not, in general, ccc. Assume that $p_\beta \in P_\lambda$ is such that $p_\beta \Vdash (\check{c}_\beta, \check{d}_\beta, \check{n}_\beta, \check{F}_\beta) \in \dot{Q}_\lambda$ for $\beta \in \omega_1$. Assume, by induction, that $P_\lambda$ is ccc. Check that we may assume that, for each $\beta$, $F_\beta \subset dom(p_\beta)$, and each $\xi$ such that $\alpha_\xi \in dom(p_\beta)$, then $p_\beta \restriction \alpha_\xi$ forces a value on $(c^\alpha_\xi, d^\alpha_\xi, n^\alpha_\xi, F^\alpha_\xi)$ in which $F^\alpha_\xi \subset dom(p_\beta)$.

By the above results, this is only hard if $cf(\lambda) = \omega_1$, so we assume that this is the case we are looking at. We arrange that for each $\beta$ and each $\mu < \xi$ such that $\{\alpha_\mu, \alpha_\xi\} \subset dom(p_\beta)$, that we also have $\alpha_\mu \in F^\beta_{\alpha_\xi}$. We may also arrange that there is a single $n$ so that for all $\beta$ and all $\xi$ such that $\alpha_\xi \in dom(p_\beta)$, we have that $n^\beta_{\alpha_\xi} = n$.

We apply the $\Delta$-system lemma to the domains of the $p_\beta$'s, and we obtain a $\delta < \omega_1$ so that the root is contained in $\alpha_\delta$. We pass to the generic extension by $G_{\alpha_\delta}$. We must arrange that there is an uncountable set (and we then assume all) of $\beta$ such that $p_\beta \restriction \alpha_\delta \in G_{\alpha_\delta}$.

Now choose any $\beta < \gamma$, and enlarge $F^\gamma_{\alpha_\xi}$ for each $\alpha_\xi \in dom(p_\gamma)$, by adding $dom(p_\beta) \cap \{\alpha_\mu : \mu \in \omega_1\}$. We claim that $p_\beta$ union this enlarged version of $p_\gamma$ is an extension of $p_\gamma$. Call this extension $\bar{p}$

**Exercise 9.** *Prove the claim about $\bar{p}$.*

In addition, $\bar{p}$ forces that $q_\beta$ and $q_\gamma$ are compatible in $\dot{Q}$. The reason is that for $\alpha_\mu, \alpha_\xi$ in $F_\beta \cup F_\gamma$, we will have that $\bar{p}$ forces that $a_{\alpha_\mu} \cap b_{\alpha_\xi}$ is contained in $n$. $\square$

**Corollary 54.** *We can force there to be an $(\omega_2, \omega_2)$-gap. And there is a model of MA in which $\mathfrak{c} = \omega_2$ and there are $(\omega_2, \omega_2)$-gaps and $(\omega_2, \omega_1)$-gaps.*

**Remark 55.** *One could add such gaps with a single ccc poset of finite conditions.*

**Theorem 56** (CH + $\diamondsuit_{S^2_1}$)**.** *There is a model of MA in which there are no $(\mathfrak{c}, \mathfrak{c})$-gaps and no $(\mathfrak{c}, \omega_1)$-gaps.*

*Proof.* The statement $\diamondsuit_{S^2_1}$ is the assertion that there is a sequence $\{A_\alpha : \alpha \in \omega_2\}$ such that for all $X \subset \omega_2$ there is a stationary set $S$ of limit ordinals of cofinality $\omega_1$ ( $S^2_1$ denotes the set of all such limit ordinals) with the property that $X \cap \lambda = A_\lambda$ for all $\lambda \in S$.

It is a standard coding technique to use such a diamond sequence to "predict" more general subsets of $H(\omega_2)$ such as sequences of names of subsets of $\omega$. In fact, we may choose any function $e : H(\omega_2) \to \omega_2$ and in this way, if $X \subset H(\omega_2)$, we can use the diamond sequence to predict $e[X]$.

We define our FS-supported iteration $\{P_\alpha, \dot{Q}_\alpha : \alpha \in \omega_2\}$ as follows. As usual $P_0$ is the trivial poset, and we let $\dot{Q}_0$ be the $P_0$-name of the poset for adding a single Cohen real. Our inductive assumption on $P_\alpha$ is that it is a ccc poset which is a member of $H(\omega_2)$ (and so has cardinality at most $\omega_1$). Having defined $P_\alpha$, have a "look at" $e^{-1}[A_\alpha]$. If $e^{-1}[A_\alpha]$ is a $P_\alpha$-name (as unlikely as that may be) of a ccc poset, then let $\dot{Q}_\alpha$ be this name. If $e^{-1}[A_\alpha]$ happens to be the $P_\alpha$-name of gap $(\mathcal{A}, \mathcal{B})$ which contains an $(\omega_1, \omega_1)$-subgap, then let $\dot{Q}_\alpha$ be the $P_\alpha$-name of $Q_{\mathcal{A},\mathcal{B}}$ (the gap freezing poset discussed above). If neither of these two situations occurs, then again let $\dot{Q}_\alpha$ be the $P_\alpha$-name of the poset for adding a single Cohen real.

Let $G \subset P_{\omega_2}$ be a generic filter, and for each $\lambda < \omega_2$, let $G_\lambda = G \cap P_\lambda$.

**Claim 1.** Martin's Axiom holds in $V[G]$.

*Proof of Claim 2.* Let $\dot{Q}$ be any $P_{\omega_2}$-name so that there is a $p \in G$ forcing that $\dot{Q}$ is a ccc poset of size $\aleph_1$. We may suppose that 1 forces that $\dot{Q}$ is a ccc poset (by slightly changing $\dot{Q}$ without changing what it is forced to be by $p$) and that $\dot{Q}$ is in $H(\omega_2)$. Also let $\dot{\mathcal{D}}$ be any $P_{\omega_2}$-name of a family of at most $\aleph_1$-many maximal antichains of $\dot{Q}$. Let $S$ be the stationary set of $\alpha$ satisfying that $e[\dot{Q}] = A_\alpha$. Choose any $\alpha \in S$ so large that $\dot{\mathcal{D}}$ is a $P_\alpha$-name and $p \in G_\alpha$. It should be clear that $G_{\alpha+1}$ is such that $V[G_{\alpha+1}]$ includes a filter on $val_{G_\alpha}[\dot{Q}]$ which is $val_{G_\alpha}(\dot{\mathcal{D}})$-generic. $\square$

**Claim 2.** There are no $(\omega_2, \omega_2)$-gaps in $V[G]$.

*Proof of Claim 2.* We consider a $P_{\omega_2}$-name $\tau$ for an $(\omega_2, \omega_2)$-pregap and we assume some condition $p_0$ forces that it is a gap. Recall that $\emptyset$ is the maximal element of $P_{\omega_2}$. It is an elementary argument that we can assume that $\tau = \{(\sigma_\alpha, \emptyset) : \alpha \in \omega_2\}$ (as a set – hence name, not an $\omega_2$-sequence) where each $\sigma_\alpha$ is a $P_{\omega_2}$-name of a disjoint pair, $(\dot{a}_\alpha, \dot{b}_\alpha)$, of subsets of $\omega$.

Choose any sufficiently large regular $\theta > \omega_2$ and let $\{p_0, P_{\omega_2}, \tau\} \in M \prec H(\theta)$ with $M^\omega \subset M$ and $|M| = \omega_1$. There is a $\lambda \in \omega_2$ such that $M \cap \omega_2 = \lambda \in S^2_1$.

**Exercise 10.** *Show that in $V[G_\lambda]$, $\{(a_\alpha, b_\alpha) : \alpha \in \lambda\}$ is a gap; where, for all $\alpha \in \lambda$, $\sigma_\alpha$ is a $P_\lambda$-name and $val_{G_\lambda}(\sigma_\alpha)$ is the pair $(a_\alpha, b_\alpha)$.*

**Exercise 11.** *Show that there is such a $\lambda \in S^2_1$ as above so that $e[A_\lambda] = \{\sigma_\alpha : \alpha \in \lambda\}$*

By the two exercises we obtain a contradiction since we know there will be a $\lambda$ as in Exercise 11, so that $V[G_{\lambda+1}]$ will be a model in which $\{(a_\alpha, b_\alpha) : \alpha \in \lambda\}$ contains a Lusin (frozen) gap. $\square$

The proof that there are no $(\omega_2, \omega_1)$-gaps is handled very similarly. $\square$

**Definition 57.** The space $\beta\mathbb{N}$ can be taken to be the Stone space of $\mathcal{P}(\mathbb{N})$. As a set, we treat $\mathbb{N}$ as a dense subset, and the remaining set of points are the free ultrafilters on $\mathbb{N}$. Notice that for each $a \subset \mathbb{N}$, $a$ and $\mathbb{N} \setminus a$ have disjoint (hence clopen) closures in $\beta\mathbb{N}$. For a subset $a$ of $\mathbb{N}$ (including $a = \mathbb{N}$), we let $a^*$ denote the subset of $\beta\mathbb{N}$ given by $\overline{a} \setminus \mathbb{N}$.

In a space $X$, a subspace $Y$ is said to be $C^*$-embedded if every bounded continuous real-valued function on $Y$ has a continuous extension to all of $X$.

**Corollary 58.** *It is independent of MA + $\neg$CH if $\mathbb{N}^* \setminus \{p\}$ is $C^*$-embedded in $\mathbb{N}^*$ for all $p$.*

*Proof.* Using Corollary 54 we can check that the gap is split by a single point $p$, and so $\mathbb{N}^* \setminus \{p\}$ has a non-extending 2-valued function.

Now work in the no gap model and fix any $p \in \mathbb{N}^*$. Let $g : \mathbb{N}^* \setminus \{p\} \mapsto [0, 1]$. If $g$ does not extend continuously to $p$, then there must be $0 \leq r < s \leq 1$ so that for each $U \in p$, so that $g[U^*]$ meets both $[0, r]$ and $[s, 1]$. If $p$ is not a $P_{\omega_2}$-point choose a $\leq\aleph_1$-sized family $\mathcal{C} \subset [\omega]^\omega \setminus p$ witnessing. If $p$ is a $P_{\omega_2}$-point, this family is any $\leq\aleph_1$-sized subset of $[\omega]^\omega \setminus p$. We build an $(\omega_2, \omega_2)$-gap by induction, consisting of pairs $a_\alpha, b_\alpha$ from $[\omega]^\omega \setminus p$ which are almost disjoint from each member of $\mathcal{C}$. Let $\{U_\alpha : \alpha \in \omega_2\}$ be an enumeration of $p$. At successor steps $\alpha + 1$, first choose $\gamma_\alpha \in \omega_2 \setminus \alpha$ so that $U_{\gamma_\alpha}$ is almost disjoint from $a_\alpha \cup b_\alpha$. Observe that $\bigcap_{c \in \mathcal{C}} \bigcap_{\gamma \leq \gamma_\alpha} (U_\gamma \setminus c)^* \cap g^{-1}[0, r]$ is a non-empty $G_{\omega_1}$ – as is $\bigcap_{c \in \mathcal{C}} \bigcap_{\gamma \leq \gamma_\alpha} (U_\gamma \setminus c)^* \cap g^{-1}[s, 1]$. Thus we can choose $a_{\alpha+1} \supset a_\alpha$ so that $a_{\alpha+1} \setminus a_\alpha \subset^* U_\gamma$ for all $\gamma \leq \gamma_\alpha$ and $g[(a_{\alpha+1} \setminus a_\alpha)^*] \subset [0, r]$. Similarly, choose $b_{\alpha+1}$ so that $g[(b_{\alpha+1} \setminus b_\alpha)^*] \subset [s, 1]$.

For limit step $\lambda < \omega_2$: We have built $\{a_\alpha, b_\alpha : \alpha < \lambda\}$ to be an $(\omega_1, \omega_1)$-pregap contained in $[\omega]^\omega \setminus p$ and inside of $\mathcal{C}^\perp$. Since there are no $(\omega_1, \omega_1)$-gaps and since $\mathfrak{b} > \omega_1$ (i.e. there are no $(\omega_1, \omega)$-gaps), we can choose $a_\lambda, b_\lambda$ as required to continue.

Since there are no $(\omega_2, \omega_2)$-gaps, we have achieved a contradiction. $\square$