Traces and Ultrapowers

Tristan Bice (joint work with Ilijas Farah)

September 13, 2012

General Situation

- Given a C^*-algebra A and free ultrafilter $\mathcal{U}$ on $\mathbb{N}$.

$$\begin{aligned} l^\infty(A) &= \{(a_n) \subseteq A : \sup ||a_n|| < \infty\} \\ c_\mathcal{U} &= \{(a_n) \in l^\infty(A) : \lim_{n\to\mathcal{U}} ||a_n|| = 0\} \\ A^\mathcal{U} &= l^\infty(A)/c_\mathcal{U} \end{aligned}$$

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- Given a trace τ on A, i.e. $\tau \in A^*_+$ & $\forall a, b \in A\ \tau(ab) = \tau(ba)$,

$$\tau^\mathcal{U}((a_n)) = \lim_{n\to\mathcal{U}} \tau(a_n)$$

defines a trace on $A^\mathcal{U}$. More generally, if $(\tau_n) \subseteq \mathcal{T}(A)$,

$$(\tau_n)^\mathcal{U}((a_n)) = \lim_{n\to\mathcal{U}} \tau_n(a_n) \in \mathcal{T}(A^\mathcal{U}).$$

With CH

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- For $f \in 2^{\omega_1}$ define $\tau^f = \bigcup_{\alpha<\omega_1} (\tau_n^{f\restriction\alpha})^{\mathcal{U}} \restriction A_\alpha$. Then

$$|\{\tau^f : f \in 2^{\omega_1}\}| = 2^{\aleph_1} > 2^{\aleph_0} = \{(\tau_n)^{\mathcal{U}} : (\tau_n) \subseteq \mathcal{T}(A)\}.$$

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- For $(a_n) = (\lambda_n^m) \in l^\infty(A)$ define

$$\tau((\lambda_n^m)) = \lim_{(m,n)\to\mathcal{V}} \lambda_n^m,$$

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Without CH (cont.)

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- Apply the same argument as before.

Trace Out of Nowhere

Question

Can we have $\tau^{\mathcal{U}} \in \mathcal{T}(A^{\mathcal{U}})$ and $(a_n) \in l^{\infty}(A)$ with $\tau^{\mathcal{U}}((a_n)) \neq 0$ even though $\tau(a_n) = 0$ for all $\tau \in \mathcal{T}(A)$ and $n \in \mathbb{N}$?

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Definitions

Given a C^*-algebra A, $a, b \in A_+$ are *CP-equivalent* if there exist $(c_n) \subseteq A$ such that $a = \sum c_n c_n^*$ and $b = \sum c_n^* c_n$. We define

$$A_0 = \{a - b : a, b \in A \text{ and } a \text{ and } b \text{ are CP-equivalent}\}$$

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Theorem [Cuntz-Pedersen (1979)]

$$A_0 = \{a \in A_{\mathrm{sa}} : \forall \tau \in \mathcal{T}(A)(\tau(a) = 0)\}.$$

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- Example: in $A = M_2$, e_{11} and $\frac{1}{2}(e_{11} + e_{22})$ are CP-equivalent, as witnessed by $\frac{1}{\sqrt{2}}e_{11}$ and $\frac{1}{\sqrt{2}}e_{12}$, while membership of $\frac{1}{2}(e_{11} - e_{22})$ in A_0 is witnessed by just $\frac{1}{\sqrt{2}}e_{12}$.

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- Example 2: in $A = C(S^2, M^2)$, the Bott projection P and trivial projection $Q(= e_{11}$ everywhere) are CP-equivalent, requires 2 operators to witness, while $P - Q \in A_0$ requires just one.

Vector Bundle Solution

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$$B = \{(\begin{bmatrix} a & \mathbf{b} \\ \mathbf{c} & d \end{bmatrix}, x) : x \in \mathbb{C}P^n(\subseteq \mathbb{C}^{n+1}); a, d \in \mathbb{C}; \mathbf{b}, \overline{\mathbf{c}} \in x\}.$$

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- Multiplication is defined pointwise by

$$\begin{bmatrix} a & \mathbf{b} \\ \mathbf{c} & d \end{bmatrix} \begin{bmatrix} a' & \mathbf{b}' \\ \mathbf{c}' & d' \end{bmatrix} = \begin{bmatrix} aa' + \mathbf{b} \cdot \mathbf{c}' & a\mathbf{b}' + d\mathbf{b} \\ a'\mathbf{c} + d\mathbf{c}' & dd' + \mathbf{b}' \cdot \mathbf{c} \end{bmatrix}.$$

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- In particular,

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- For each i define $f_i : S^{2n+1} \to \mathbb{C}$ by $f_i(\mathbf{e})\mathbf{e} = \mathbf{b}_i(\mathbb{C}\mathbf{e}) - \overline{\mathbf{c}}_i(\mathbb{C}\mathbf{e})$.

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$$\begin{bmatrix} a & \mathbf{b} \\ \mathbf{c} & d \end{bmatrix} \begin{bmatrix} \overline{a} & \overline{\mathbf{c}} \\ \overline{\mathbf{b}} & \overline{d} \end{bmatrix} - \begin{bmatrix} \overline{a} & \overline{\mathbf{c}} \\ \overline{\mathbf{b}} & \overline{d} \end{bmatrix} \begin{bmatrix} a & \mathbf{b} \\ \mathbf{c} & d \end{bmatrix} = \begin{bmatrix} |\mathbf{b}|^2 - |\mathbf{c}|^2 & \dots \\ \dots & |\mathbf{c}|^2 - |\mathbf{b}|^2 \end{bmatrix}.$$

- Take $m_1, \dots, m_k \in A$, i.e. $m_i(x) = \begin{bmatrix} a_i(x) & \mathbf{b}_i(x) \\ \mathbf{c}_i(x) & d_i(x) \end{bmatrix}$
- Assume $|| \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \sum(m_i m_i^* - m_i^* m_i)|| < \epsilon$, for small $\epsilon > 0$.
- For each i define $f_i : S^{2n+1} \to \mathbb{C}$ by $f_i(\mathbf{e})\mathbf{e} = \mathbf{b}_i(\mathbb{C}\mathbf{e}) - \overline{\mathbf{c}}_i(\mathbb{C}\mathbf{e})$.
- Then $f = (f_1, \dots, f_k)$ defines a map from S^{2n+1} to a subset of $\mathbb{C}^k$ with $f(-\mathbf{e}) = -f(\mathbf{e})$ avoiding 0.

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- Wlog $f : S^{2n+1} \to S^{2k-1}$. By Borsuk-Ulam, $k > n$.

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Question

Does there exist A with a unique trace (and, say, separable, nuclear, simple, etc.) s.t. $A^{\mathcal{U}}$ does not have a unique trace?

Projections and Tensor Products

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- Yes for a non-minimal tensor product of $C_r^*(\mathbb{F}_2)$ with itself [Ackemann and Ostrand (1976)].

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Problem

Find a vector bundle V with no non-trivial subbundles but such that $V \otimes V$ contains non-trivial subbundles.

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- The section $v_1 \otimes v_1 + v_2 \otimes v_2 + v_3 \otimes v_3$ in $V \otimes V$ is never 0.
- Note that $v_1 \otimes v_1 + v_2 \otimes v_2$ would not do: there exists $x, y \in \mathbb{C}S^2$ such that $v_1(x) = v_2(x)$ and $v_1(y) = -v_2(y)$ and hence $v_1(x) \otimes v_1(y) + v_2(x) \otimes v_2(y) = 0$.