FINITE GENERATORS FOR COUNTABLE GROUP ACTIONS

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Introduction

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Easy fact: $\mathcal{P}$ is a generator if and only if $f_\mathcal{P}$ is injective.

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If X is an invariant Borel subset of the shift k^G, then letting $V_i = \{x \in k^G : x(1_G) = i\}$, $i < k$, we get that $\mathcal{P} = \{V_i\}_{i<k}$ is a k-generator.

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Observation

For $k \leq \infty$, X admits a k-generator if and only if there is a Borel G-embedding of X into k^G.

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In this talk, we are concerned with the existence of finite generators.

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and it could be finite or infinite. When is this supremum achieved?

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Theorem (Krieger, '70)

Suppose (X, μ, T) is ergodic. If $h_\mu(T) < \log k$, for some $k \geq 2$, then there is a k-generator modulo μ-NULL.

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Thus we focus on Weiss's question for arbitrary group G.

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Measure theoretic context: the Krengel-Kuntz theorem

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The proof of this uses the Kuratowski-Ulam method introduced in the proofs of generic hyperfiniteness and generic compressibility by Kechris and Miller.

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We will spend the remaining time discussing the idea of the proof of the above theorem.

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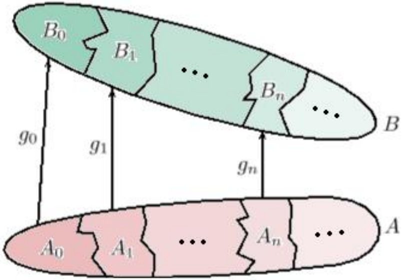

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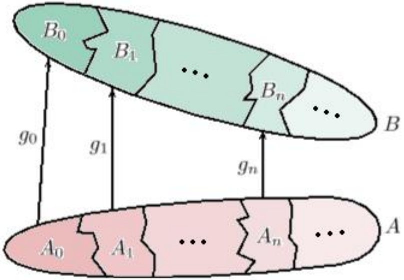

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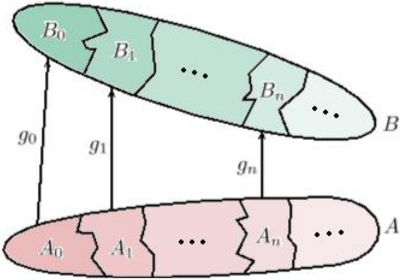

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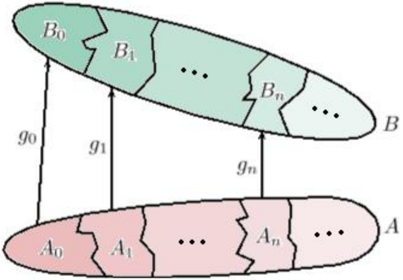

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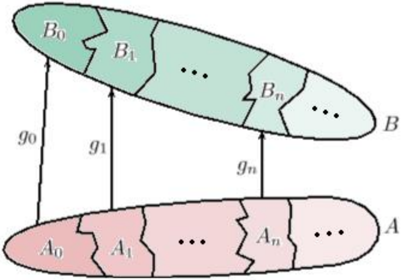

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Theorem (Nadkarni, '91)

There is no invariant probability measure on X if and only if X is compressible.

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No finite generators $\searrow$ X is not "small" $=$ X is "large" $\nearrow$ $\exists$ an invariant probability measure

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In other words, $A \sim_i B$ if i-many Borel sets are enough to generate a G-invariant σ-algebra that is sufficiently fine to carve out partitions $\{A_n\}_{n \in \mathbb{N}}$ and $\{B_n\}_{n \in \mathbb{N}}$ witnessing $A \sim B$.

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(1) $\searrow$ $\quad\quad\quad\quad$ $\nearrow$ (2)

X is not 4-compressible

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Remark: It is not hard to see that i-compressibility is necessary for the existence of a finite generator under the assumption that X is compressible.

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The main result

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Theorem (Ts.)

Let X be a Borel G-space that admits a σ-compact realization. If there is no invariant probability measure on X, then X admits a 32-generator.

THANK YOU