Metastable States and the Navier-Stokes Equations Fields Institute

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Abstract

The study of stable, or stationary, states of a physical system is a well established field of applied mathematics. Less well known or understood are “metastable” states. Such states are a signal that multiple time scales are important in the problem - for instance, one associated with the emergence of the metastable state, one associated with the evolution along the family of such states, and one associated with the emergence of the asymptotic states.

This is joint work with Margaret Beck.

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Moral

Small viscosity can have an outsized effect in determining which structures dominate a disordered flow on intermediate time scales.

Metastable States

Much progress has been made in understanding the existence and stability of stationary states of PDE's and in extending ideas like invariant manifolds which have been useful in studying the properties of stationary states of ODE's to this infinite dimensional context.

However, today, I want to focus on metastable states. These are a family of states of an infinite system with the properties:

1. They are not stationary states of the system.
2. They typically occur in families.
3. The system approaches this family quickly.
4. It then moves along the family very slowly, before
5. Finally converging to the asymptotic state of the system (which typically is a stationary solution.)

Metastable States

A few notes:

1. The presence of metastable states indicates the existence of multiple time-scales in a problem.
2. A slightly different (though related) meaning is associated with the term metastable state in statistical physics.
3. Metastable states occur in a variety of interesting physical systems.

Metastable states in two-dimensional fluids

Another circumstance where metastable states occur are in two-dimensional fluid flows. Such flows are governed by the Navier-Stokes equations:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \Delta \mathbf{u} - \nabla p \,,$$
$$\nabla \cdot \mathbf{u} = 0 \,,$$

where ν is the kinematic viscosity of the fluid (which we will assume is constant, and very small), ρ is the fluid density (which is constant due to the incompressibility condition) and p is the pressure in the fluid.

For analyzing such flows it is convenient to work with the vorticity,

$$\omega(x,t) = \partial_x u_2 - \partial_y u_1 \,.$$

Metastable states in two-dimensional fluids

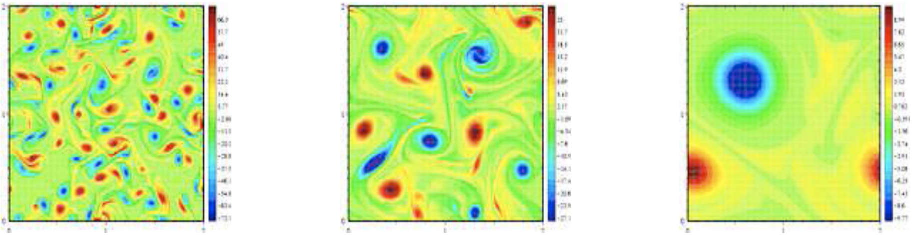


Figure: A numerical simulation of a two-dimensional turbulent flow. The figures display the vorticity field (with blue and red representing fluid “swir ling in opposite directions) at successively later and later times and clearly indicate the tendency of regions of vorticity of like sign to coalesce into a smaller and smaller number of larger vortices. From the Technical University of Eindhoven; Fluid mechanics lab

Metastable states in two-dimensional fluids

The emergence of these large vortex pairs is a characteristic and long-lived feature of such flows. The time scales on which they emerge can be seen in the following numerical experiments:

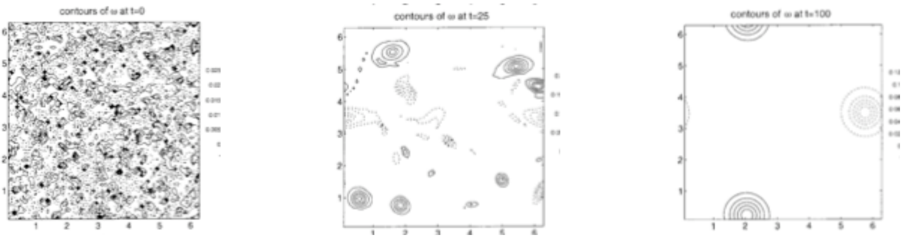


Figure: From Yin, Montgomery and Clercx, Phys. Fluids, vol. 15, p. 1397 (2003). The times of the three figures correspond to $t = 0$, $t = 25$ and $t = 100$. Note that the diffusive time scale in these experiments is $t_D \sim 5000$.

Metastable states in two-dimensional fluids

The dipole pairs are not the only metastable states observed in these numerics. One also sees states that the authors call “ bar states”:

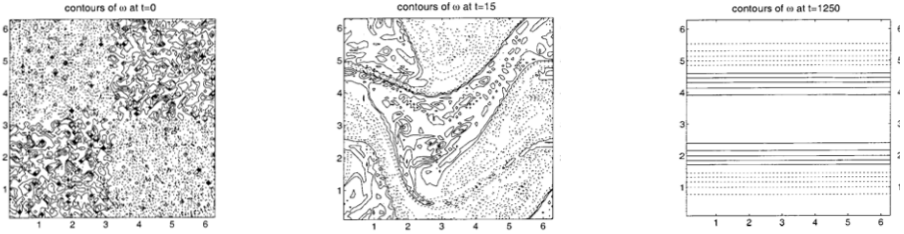


Figure: From Yin, Montgomery and Clercx, Phys. Fluids, vol. 15, p. 1397 (2003). The times of the three figures correspond to $t = 0$, $t = 15$ and $t = 1250$. Note that the diffusive time scale in these experiments is $t_D \sim 5000$.

Metastable states in two-dimensional fluids

Our results so far are concerned with analyzing the dynamics near the bar states, which are somewhat easier to treat mathematically than the dipole states. While the bar states are observed less frequently than the dipoles on a square domain, for rectangular domains with one side longer than the other, the bar states are sometimes observed to be the dominant metastable states.

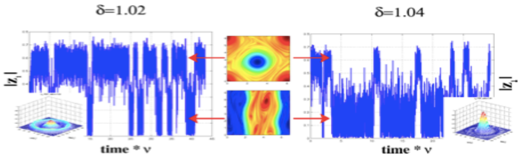


Figure 1: Time series and probability density functions (PDFs) for the order parameter z_1 (see page 4) illustrating random changes between dipoles and unidirectional flows.

Figure: From Bouchet and Simmonet, PRL, vol. 102, (2009)

Metastable states in Burgers equation [1]

In recent years progress has been made in understanding the metastable behavior that occurs in the viscous Burgers equation, which originated as a simple model of turbulent fluids:

$$\partial_t u = \nu \partial_x^2 u - u \partial_x u \ .$$

(Could also serve as a model for the experimentally observed metastable states in traffic flow?)

Note that even though Burgers equation can be "solved" via the Cole-Hopf transformation the resulting form of the solution is not very transparent for general types of initial data.

[1] Beck and Wayne, SIAM Review, vol. 53, pp. 129-153 (2011)

Metastability in Burgers equation

Let's begin by looking for the long-time asymptotic states of the system.

Motivated by the similarity with the heat equation we introduce new dependent and independent variables:

$$\xi = \frac{x}{\sqrt{1+t}}\ , \quad \tau = \log(1+t)$$
$$u(x,t) = \frac{1}{\sqrt{1+t}} w\Big(\frac{x}{\sqrt{1+t}}, \log(1+t)\Big)$$

Metastability in Burgers equation

In terms of these new variables the equation takes the form:

$$\partial_\tau w = \mathcal{L}_\nu w + w\partial_\xi w \ ,$$

where

$$\mathcal{L}_\nu w = \nu \partial_\nu^2 w + \frac{1}{2}\partial_\xi(\xi w)$$

At first glance, we've made things worse, rather than better since the linear part of the equation looks more complicated than before. However, the operator $\mathcal{L}_\nu$ is just the quantum mechanical harmonic oscillator in disguise!

Stationary states for Burgers equation

In these new variables, the equation has a family of (self-similar) fixed points:

$$\partial_\tau w = \nu \partial_\xi^2 w + \frac{1}{2}\partial_\xi(\xi w) - \frac{1}{2}\partial_\xi w^2$$

Note that the stationary states of Burger's equation (in these new variables) satisfy the nonlinear ODE

$$\partial_\xi \left(\nu \partial_\xi w + \frac{1}{2}(\xi w) - \frac{1}{2} w^2 \right) = 0 \, .$$

This is an ODE we can solve explicitly!

Stationary states for Burgers equation

We find a family of stationary points:

$$A_M(\xi) = \frac{\alpha(M)e^{-\xi^2/(4\nu)}}{1 - \frac{\alpha(M)}{2\nu}\int_{-\infty}^{\xi} e^{-\nu^2/(4\nu)}d\eta} ,$$

where M refers to the total "mass" of the solutions (which is conserved) and is related to the parameter α by the formula

$$\alpha = \sqrt{\frac{\nu}{\pi}}(1 - e^{-M/(2\nu)})$$

Important note: As M (or α) goes to zero, $A_M(\xi) \sim \alpha(M)e^{-\xi^2/(4\nu)}$.

Stationary states for Burgers equation

We can now update our dynamical systems picture to include the features of Burgers equation, rather than just the heat equation and we find:

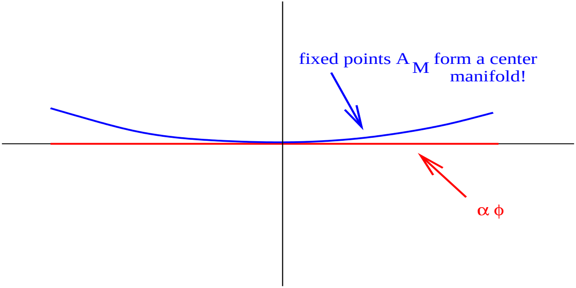


Figure: The "dynamical systems" view of the Burgers equation.

Stability of stationary states for Burgers equation

Can we say anything about the stability of these stationary states?

1. If we linearize Burgers equation about the solution A_M we obtain the linear equation
$$\partial_\tau w = \nu \partial_\xi^2 w + \frac{1}{2}\partial_\xi(\xi w) - \partial_\xi(A_M(\xi) w) \ .$$
2. Using the Cole-Hopf transformation we can compute the spectrum of the linear operator on the right hand side explicitly. In fact, the spectrum is exactly the same as the operator $\mathcal{L}_\nu$ we discussed earlier.
3. In particular, there is an eigenvalue $\lambda = 0$ which corresponds to moving along the center manifold and all other points in the spectrum lie in the left half plane.
4. The largest, negative eigenvalue is $\lambda = -1/2$.

Stationary states for Burgers equation

Including this information in our picture, we find:

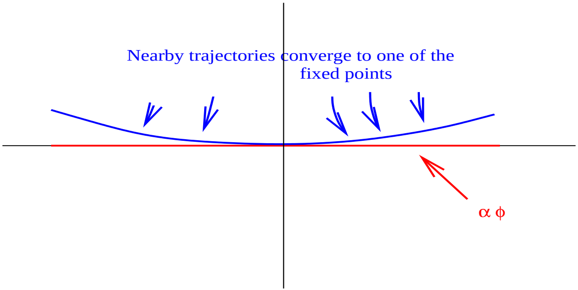


Figure: The “dynamical systems” view of the Burgers equation (updated).

Stationary states for Burgers equation

Summarizing:

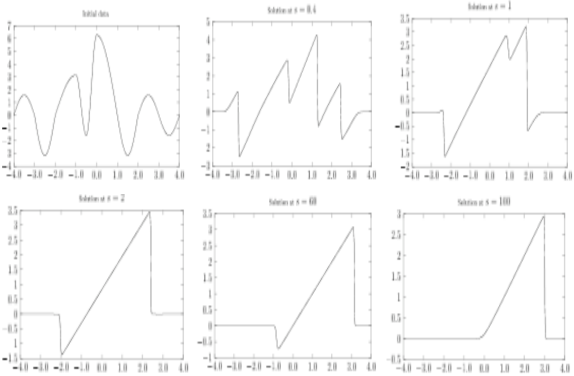
1. We have shown that in similarity variables, Burgers equation has a family of locally attractive stationary solutions which govern the long-time asymptotics of nearby solutions.
2. If we revert to the original variables we find these fixed points correspond to a family of self-similar solutions

$$\mathcal{A}_M(x,t) = \frac{\alpha(M) e^{-x^2/(4\nu(1+t))}}{1 - \frac{\alpha(M)}{2\nu\sqrt{1+t}} \int_{-\infty}^{x/\sqrt{1+t}} e^{-y^2/(4\nu(1+t))} dy},$$

3. In fact, using Lyapunov functionals, DiFrancesco showed that these self-similar solutions are actually globally stable - i.e. any solution will eventually approach some member of this family.

Metastable states for Burgers equation

So far we have only been discussing long-time stable states of Burgers equation - what about metastable states? Kim and Tsvaras (SIMA vol. 33, pp 607-633 (2001)) made the following numerical observation:

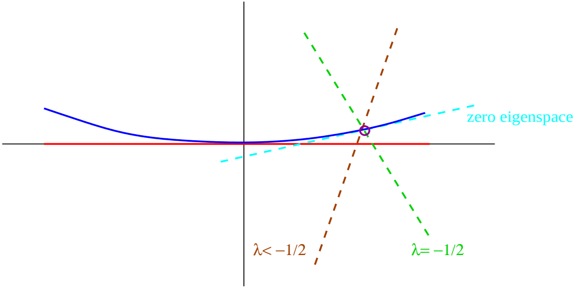


Kim and Tsvaras

1. Kim and Tsvaras define the very long lived states in the first figure in the second row to be "diffusive N-waves", in light of their similarity to the N-waves that govern the asymptotic behavior of the inviscid Burgers equation.
2. The diffusive time scale in this problem is $t \sim 100$.
3. Clearly the diffusive N-waves appear on a much shorter time scale.
4. Kim and Tsvaras used the Cole-Hopf transformation to find an explicit formula for these solutions

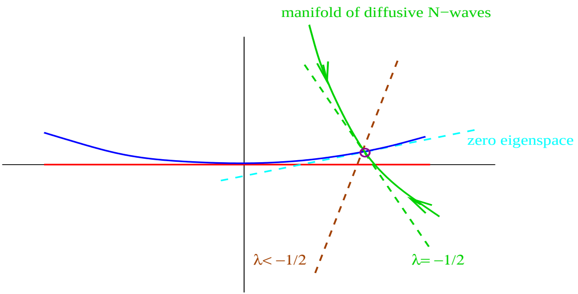
Metastable manifolds

How does this fit into the dynamical systems picture we outlined earlier? In a neighborhood of any of the fixed points corresponding to the long-time asymptotic states we have the following picture at the linear level.



Metastable manifolds

We expect that the approach to the asymptotic state will be governed by the eigenspace corresponding to eigenvalue $\lambda = -1/2$. Beck and I show that one has the following picture.



In fact, the manifold of diffusive N-waves is constructed by applying the Cole-Hope transformation to a linear combination of the first two eigenfunctions of the operator $\mathcal{L}_\nu$.

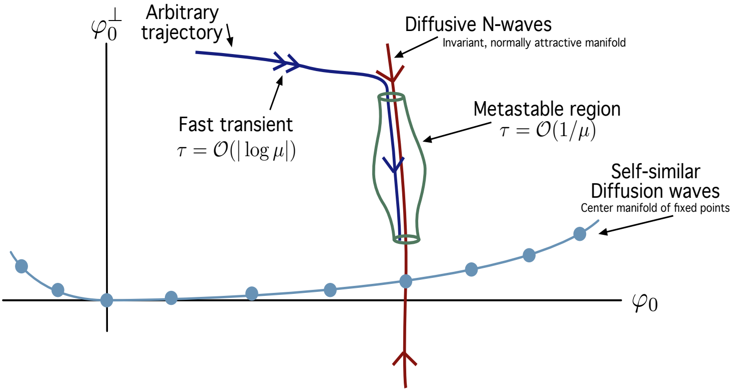
Metastable manifolds

Thus far, this picture is only “local” - i.e. it holds only hear the asymptotic state. However, Beck and I then proved the following results:

1. The local N-wave (or metastable) manifold can be extended globally. (This essentially follows from the results of Kim and Tsvaras.)
2. The metastable manifold is locally attractive - i.e. if one starts near it, one stays near it for all time.
3. “Every” solution gets close to one of the metastable manifolds after a “short” time. More precisely, given any initial condition in an appropriate function space, the corresponding solution will be in a small neighborhood of one of the metastable manifolds in a time $\tau \sim \mathcal{O}(|log \nu|)$.

Global Dynamics of Burgers Equation

Thus, we have the following global picture:



Global Dynamics of Burgers Equation

A few notes on the proof. There are essentially two new results:

1. An estimate on the initial transient: We use the fact that one knows that solutions of the *inviscid* Burgers equation approach an inviscid N-wave, plus the fact that the functional form of the diffusive N-wave is close to that of the inviscid N-wave, plus the fact that for small viscosity the solutions of the viscous and inviscid equations remain close for some time to show that after a relatively short time (i.e. $\mathcal{O}(|log\nu|)$) we are close to one of the diffusive N-wave manifolds.

2. An estimate on the local attractivity of the metastable manifold: Here we use the explicit form of the diffusive N-waves, plus the Cole-Hopf transformation to prove that if we start near a diffusive N-wave we never drift too far away from it.

Extension to the Navier-Stokes equations

We now turn to the question of whether or not we can extend these results to the Navier-Stokes equations. We consider the two-dimensional, incompressible Navier-Stokes equation, with periodic boundary conditions on a square:

$$\partial_t \mathbf{u} = \nu \Delta \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u} - \nabla p, \qquad \nabla \cdot \mathbf{u} = 0,$$

where $-\pi < x < \pi$, $-\pi < y < \pi$, $\mathbf{u} : \mathbb{T}^2 \to \mathbb{R}^2$, $\mathbf{u}(-\pi, y, t) = \mathbf{u}(\pi, y, t)$ and $\mathbf{u}(x, -\pi, t) = \mathbf{u}(x, \pi, t)$ for all $t \geqslant 0$.

Extension to the Navier-Stokes equations

It's convenient to work not with the velocity itself, but instead with the vorticity:

$$\omega(x, y, t) = \partial_x u_2 - \partial_y u_1 \ . \tag{1}$$

Then:

$$\partial_t \omega = \nu \Delta \omega - \mathbf{u} \cdot \nabla \omega, \tag{2}$$

One can recover the velocity from the vorticity via the Biot-Savart law:

$$\hat{\mathbf{u}}(k, l) = \frac{i}{k^2 + l^2}(l, -k)\hat{\omega}(k, l), \qquad (k, l) \neq (0, 0).$$

Extension to the Navier-Stokes equations

Note the similarities to Burgers equation:

- The linear part of the equation is just the heat equation.
- The nonlinear term is quadratic, and semilinear in ω.

Our approach to Burgers equation suggests looking at the invariant manifolds tangent to the "least unstable" eigenspaces of the long-time asymptotic state of the system. However, the questions are:

- What play's the role of the metastable manifold in Burgers equation?
- Can we explain the origin of the "fast" time-scale on which solutions approach the dipole or bar states?

Extension to the Navier-Stokes equations

- In this case, all solutions tend toward zero, so the only long-time asymptotic state is the trivial solution.
- If we linearize about zero, the eigenspaces with largest non-zero eigenvalues are $-1/2$, with eigenfunctions $A\sin x + B\cos x$ or $A\sin y + B\cos y$ (For simplicity we consider the solutions $\sin x$.)
- Is there a family of solutions of the Navier-Stokes equations tangent at the zero solution to $\sin x$?
  - Yes - an explicit and extremely simple solution!

$$\tilde{\omega}^b(x,y,t) = e^{-\nu t}\sin(x),$$

Extension to the Navier-Stokes equations

- We refer to these as "bar" states, in keeping with the terminology of the numerical paper of Yin, et al. referred to earlier. They are also referred to as the "Kolmogorov flow" some places in the literature.
- There are also simple, explicit solutions that appear qualitatively similar to the metastable dipole states observed in Yin, et al's numerics. They are often called the "Taylor-Green vortex". We focus on the bar states here because they are easier to analyze mathematically.
- One can easily compute the velocity field associated with the bar states and one finds:

$$\tilde{\mathbf{u}}^b(x,y,t) = -e^{-\nu t} \begin{pmatrix} 0 \\ \cos x \end{pmatrix} .$$

Extension to the Navier-Stokes equations

- To show that the bar states are metastable in the sense of the diffusive N-waves are in Burgers equation we would like to show that solutions approach the bar states "quickly".
- The "natural" time scale in this problem is the diffusive time scale $\tau \sim \nu^{-1}$, which is very long when ν is small and is also the time scale on which the bar states evolve.
- The numerics shown earlier indicates that solutions approach the bar or dipole states on a much shorter time-scale than this.
- Our approach is to linearize about the bar states and examine the evolution of nearby solutions to see if they give any hint as to how these shorter time scales arise.

Extension to the Navier-Stokes equations

Linearizing about the bar state $\tilde{\omega}^b$, leads to the linear equation

$$\begin{aligned}\partial_t w &= = \nu\Delta w - \tilde{\mathbf{u}}^b \cdot \nabla w - \mathbf{u} \cdot \nabla \tilde{\omega}^b = \mathcal{L}(t)w \\ \mathcal{L}(t)w &= \nu\Delta w - ae^{-\nu t}[\sin x \partial_y (1+\Delta^{-1})]w,\end{aligned}$$

where $\mathbf{u}$ is the velocity corresponding to w via the Biot-Savart law, and the second line has used the Biot-Savart law to rewrite the term $\mathbf{u} \cdot \nabla \tilde{\omega}^b$ in terms of w.

The key fact is that $\mathcal{L}(t)$ consists (essentially) of a "small", dissipative, symmetric piece and a "large" anti-symmetric piece.

Extension to the Navier-Stokes equations

- The linearized equation is non-autonomous, but as a first attempt to understand the dynamics generated by $\mathcal{L}(t)$ we can compute the spectrum of this operator for fixed t.
- We're particularly interested in the behavior of solutions as ν gets very small.
- To see if there is any scaling behavior, we plot the logarithm of the (real part of the) eigenvalue as a function of the logarithm of ν.
- This computation is simplified by the fact that if we expand $\omega(x,y) = \sum_{k,\ell} \hat{\omega}(k,\ell) e^{i(kx+\ell y)}$, then $\mathcal{L}(t)$ does not mix terms with different values of ℓ, and thus we can study the restricted operator

$$\mathcal{L}^{\ell}(t)\hat{w}_{\ell} = \nu(\partial_x^2 - \ell^2)\hat{w}_{\ell} - iae^{-\nu t} \sin x (1 + \Delta_{\ell}^{-1})\hat{w}_{\ell}$$

The first numerical results

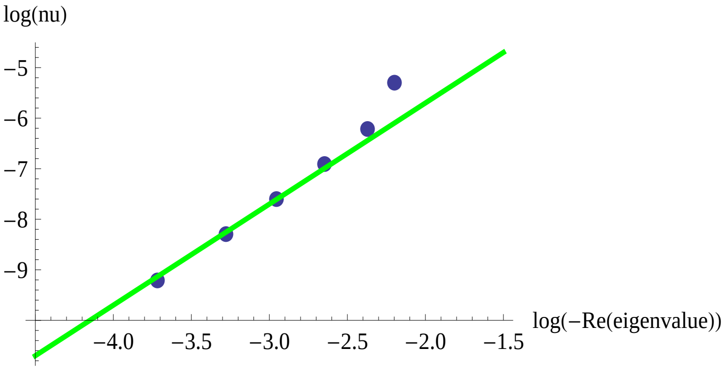


Figure: The computed value of the real part of the eigenvalue of the linearization about a bar state, compared with a line which represents a scaling proportional to $\sqrt{\nu}$.

Additional numerical results

To further illustrate this scaling, we next plot, for different values of ν, the real part of the first 30 eigenvalues of $\mathcal{L}^\ell$ (with $\ell = 2$), divided by $\sqrt{\nu}$ for several different values of ν. If the eigenvalues scale with the square root of viscosity, then these plots should fall on top of one another.

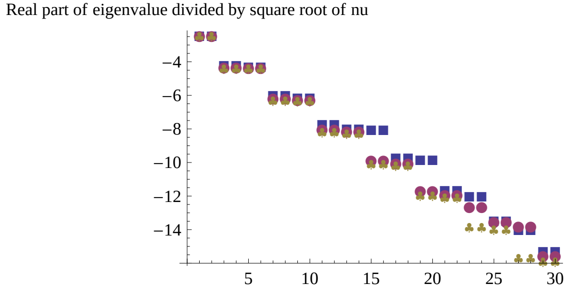


Figure: The scaling behavior of the first thirty eigenvalues of the linearization about the bar states, with $l = 2$. In this figure, solid squares correspond to the eigenvalues with $\nu = 0.00025$, solid circles correspond to $\nu = 0.0001$ and clubs correspond to $\nu = 0.00005$.

Metastability of bar states

- Note that these results suggest that the bar states should attract nearby solutions on a time scale $\tau \sim 1/\sqrt{\nu}$, which is much faster than the diffusive time scale. For numerics of Yin, et al. shown earlier, the diffusive time scale $\tau_D \sim 5000$, which by this reasoning would lead to a "metastable" timescale of $\tau \sim 70$, which is in qualitative agreement with the time scales over which the bar and dipole states appear.
- This accelerated dissipation rate is a result of the interplay between the large off-diagonal piece and the dissipative, symmetric piece. These effects have been exploited by a number of authors (e.g. Eckmann and Hairer) and systematized by Villani under the name of hypocoercivity.
- The applicability of these idea in a fluid mechanics context was first pointed out by Gallagher, Gallay and Nier, in a model for the linearization of the two-dimensional vorticity equation about the Oseen vortex, and then, roughly contemporaneously with this work by Wen Deng, who studied the actual linearization about the Oseen vortex.

Theoretical Results

Villani's method applies to operators of the form $H = A^*A + B$, with B skew-symmetric, and $[A, B] \neq 0$. In the case of our operator

$$\mathcal{L}^\ell(t)\hat{w}_\ell = \nu(\partial_x^2 - \ell^2)\hat{w}_\ell - iae^{-\nu t}\sin x(1 + \Delta_\ell^{-1})\hat{w}_\ell$$

the term $-iae^{-\nu t}(1 + \Delta_\ell^{-1})\hat{w}_\ell$ is not skew-symmetric (at least not with respect to the ordinary inner product) so let's first consider the approximate evolution

$$\begin{aligned} \partial_t \hat{w} &= \mathcal{L}^{\ell,a}(t)\hat{w} \\ \mathcal{L}^{\ell,a}(t)\hat{w} &= \nu(\partial_x^2 - \ell^2)\hat{w} - iae^{-\nu t}(\sin x)\hat{w} \end{aligned}$$

Note that the term omitted in this approximation is a compact operator (with small norm when ℓ is large) so we can hope that $\mathcal{L}^{\ell,a}(t)$ captures the qualitatively important aspects of the evolution. We also note that a similar approximation was used in the first studies of the linearization about the Oseen vortex.

Theoretical Results

Following closely Villani's approach,
we define operators,

$$
\begin{aligned}
B^{\ell}\hat{w}_{\ell} &= -ia\ell e^{-\nu t}(\sin x)\hat{w}_{\ell} \\
C^{\ell}\hat{w}_{\ell} &= [\partial_x, B^{\ell}]\hat{w}_{\ell} = -ia\ell e^{-\nu t}(\cos x)\hat{w}_{\ell}\ .
\end{aligned} \tag{3}
$$

and analyze the evolution of $\mathcal{L}^{\ell,a}(t)$ with the aid of the functional

$$
\Phi^{\ell}(t) = \|\hat{w}_{\ell}\|^2 + \alpha\|\partial_x\hat{w}_{\ell}\|^2 - 2\beta\Re(\partial_x\hat{w}_{\ell}, C^{\ell}\hat{w}_{\ell}) + \gamma(C^{\ell}\hat{w}_{\ell}, C^{\ell}\hat{w}_{\ell}).
$$

Note that if $\beta^2 < \alpha\gamma/4$ we have

$$
\|\hat{w}_{\ell}\|^2 + \frac{\alpha}{2}\|\partial_x\hat{w}_{\ell}\|^2 + \frac{\gamma}{2}\|C^{\ell}\hat{w}_{\ell}\|^2 \leqslant \Phi^{\ell}(t) \leqslant \|\hat{w}_{\ell}\|^2 + \frac{3\alpha}{2}\|\partial_x\hat{w}_{\ell}\|^2 + \frac{3\gamma}{2}\|C^{\ell}\hat{w}_{\ell}\|^2
$$

Theoretical Results

If we now consider the time evolution of Φ^ℓ we find for a fixed value of ℓ, so we suppress the subscript ℓ on the functions $\hat{w}_\ell$ to avoid overburdening the notation.

$$\begin{aligned}\frac{d}{dt}\Phi^\ell(t) = ((\hat{w}_t,\hat{w}) + (\hat{w},\hat{w}_t)) + \alpha\left((\partial_x\hat{w}_t,\partial_x\hat{w}) + (\partial_x\hat{w},\partial_x\hat{w}_t)\right) \\ -2\beta\Re\left((\partial_x\hat{w}_t, C^\ell\hat{w}) + (\partial_x\hat{w}, C^\ell\hat{w}_t)\right) \\ +\gamma\left((C^\ell\hat{w}_t, C^\ell\hat{w}) + (C^\ell\hat{w}, C^\ell\hat{w}_t)\right) \\ -2\beta\Re(\partial_x\hat{w}, \frac{dC^\ell}{dt}\hat{w}) + \gamma\left((\frac{dC^\ell}{dt}\hat{w}, C^\ell\hat{w}) + (C^\ell\hat{w}, \frac{dC^\ell}{dt}\hat{w})\right).\end{aligned}$$

Most of these terms are easily treated. For simplicity (and to make comparison with the work of Gallagher, Gallay and Nier easier) we rescale time to that

$$\partial_t\hat{w} = (\partial_x^2 - \ell^2)\hat{w} + \frac{B^\ell}{\nu}\hat{w}\,.$$

Theoretical Results

Then, for instance:

$$
\begin{aligned}
((\hat{w}_t, \hat{w}) + (\hat{w}, \hat{w}_t)) \quad &= \quad \Big(((-\ell^2 + \partial_x^2 + \frac{1}{\nu} B^\ell)\hat{w}, \hat{w}) + (\hat{w}, (-\ell^2 + \partial_x^2 + \frac{1}{\nu} B^\ell)\hat{w}) \Big) \\
&= -2\ell^2 \|\hat{w}\|^2 - 2\|\hat{w}_x\|^2,
\end{aligned}
$$

where the terms involving B vanish by anti-symmetry.
The interesting terms are those proportional to β:

$$
\begin{aligned}
(\partial_x \hat{w}_t, C^\ell \hat{w}) + (\partial_x \hat{w}, C^\ell \hat{w}_t) &= (\partial_x(-\ell^2 + \partial_x^2 + \frac{1}{\nu} B^\ell)\hat{w}, C^\ell \hat{w}) \\
&\quad + (\partial_x \hat{w}, C^\ell (-\ell^2 + \partial_x^2 + \frac{1}{\nu} B^\ell)\hat{w}) \\
&= -2\ell^2 \mathrm{Re}(\partial_x \hat{w}, C^\ell \hat{w}) + [(\hat{w}_{xxx}, C^\ell \hat{w}) + (\hat{w}_x, C^\ell \hat{w}_{xx})] \\
&\quad + \frac{1}{\nu}[(\partial_x(B^\ell \hat{w}), C^\ell \hat{w}) + (\hat{w}_x, C^\ell (B^\ell \hat{w}))]
\end{aligned}
$$

Theoretical Results

The terms involving derivatives of $\hat{w}$ can be treated by integration by parts, and then absorbed in other terms. The interesting part of this expression are the terms involving B^ℓ which we can rewrite as:

$$\begin{aligned}\frac{1}{\nu}(\partial_x(B^\ell \hat{w}), C^\ell \hat{w}) + \frac{1}{\nu}(\hat{w}_x, C^\ell(B^\ell \hat{w})) &= \frac{1}{\nu}\Big((B^\ell \hat{w}_x, C^\ell \hat{w}) + (C^\ell \hat{w}, C^\ell \hat{w}) \\ &\quad + (\hat{w}_x, B^\ell C^\ell \hat{w}) + (\hat{w}_x, [C^\ell, B^\ell]\hat{w})\Big) \\ &= \frac{1}{\nu}\|C^\ell \hat{w}\|^2 + \frac{1}{\nu}(\hat{w}_x, [C^\ell, B^\ell]\hat{w}) = \frac{1}{\nu}\|C^\ell \hat{w}\|^2 \,.\end{aligned}$$

Taking into account the fact that this term has a negative coefficient, this gives a large, negative contribution to the evolution of Φ^ℓ and is really responsible for our accelerated convergence rate.
Treating carefully the other terms, one can show that there exists a constants $M, K > 0$ (independent of ν) such that

$$\Phi^\ell(t) \leqslant K e^{-M\sqrt{\nu} t} \Phi^\ell(0) \,.$$

Summary of the theoretical results

Given a linear operator consisting of a small, symmetric dissipative piece, and a large, anti-symmetric piece, the interplay between the two can lead to unexpectedly large dissipation.

So far, we have only proven that this occurs for the simplification of the linearization about the bar states. We hope:

- to show that this effect persists for the nonlocal operator that arises in linearizing about the bar states.
- extend this linear analysis to the full nonlinear equation.
- include stochastic effects

Conclusions

- We can use dynamical systems methods to understand metastability in terms of attractive invariant manifolds.
- Linear operators consisting of a small, symmetric, dissipative piece and a large anti-symmetric piece, which arise often in fluid mechanics can exhibit unexpected scaling properties.
- At least in some circumstances, the origin of intermediate time scales, much shorter than the time scale associated with the long-time asymptotics of the problem can be understood with the aid of the hypercoercivity methods.