Relaxed Variational Principle for Water Wave Modeling

Denys Dutykh[1]
Senior Research Fellow UCD & Chargé de Recherche CNRS

[1]University College Dublin
School of Mathematical Sciences

Workshop on Ocean Wave Dynamics

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COLLABORATOR:

- Didier Clamond: Professor, LJAD, Université de Nice Sophia Antipolis, France

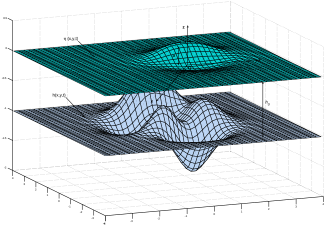
TALK IS MAINLY BASED ON:

Clamond, D., Dutykh, D. (2012). *Practical use of variational principles for modeling water waves*. Physica D: Nonlinear Phenomena, 241(1), 25-36.

Water wave problem - I

Physical assumptions:

- Fluid is ideal
- Flow is incompressible
- . . . and irrotational, i.e. $\boldsymbol{u} = \nabla\phi$
- Free surface is a graph
- Above free surface there is void
- Atmospheric pressure is constant



Surface tension can be also taken into account

Water wave problem - II

- Continuity equation

$$\nabla^2_{\boldsymbol{x},y}\phi = 0, \quad (\boldsymbol{x}, y) \in \Omega \times [-d, \eta],$$

- Kinematic bottom condition

$$\frac{\partial \phi}{\partial y} + \nabla\phi \cdot \nabla d = 0, \quad y = -d,$$

- Kinematic free surface condition

$$\frac{\partial \eta}{\partial t} + \nabla\phi \cdot \nabla\eta = \frac{\partial \phi}{\partial y}, \quad y = \eta(\boldsymbol{x}, t),$$

- Dynamic free surface condition

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla_{\boldsymbol{x},y}\phi|^2 + g\eta + \sigma\nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1 + |\nabla\eta|^2}}\right) = 0, \quad y = \eta(\boldsymbol{x}, t).$$

Hamiltonian structure

Zakharov (1968) [Zak68]; Craig & Sulem (1993) [CS93]

Canonical variables:

$\eta(\boldsymbol{x}, t)$: free surface elevation

$\tilde{\phi}(\boldsymbol{x}, t)$: velocity potential at the free surface
$\tilde{\phi}(\boldsymbol{x}, t) := \phi(\boldsymbol{x}, y = \eta(\boldsymbol{x}, t), t)$

- Evolution equations:

$$\rho \frac{\partial \eta}{\partial t} = \frac{\delta \mathscr{H}}{\delta \tilde{\phi}}, \qquad \rho \frac{\partial \tilde{\phi}}{\partial t} = -\frac{\delta \mathscr{H}}{\delta \eta},$$

- Hamiltonian:

$$\mathscr{H} = \int_{-d}^{\eta} \frac{1}{2} |\nabla_{\boldsymbol{x},y} \phi|^2 \, \mathrm{d}y + \frac{1}{2} g \eta^2 + \sigma \left(\sqrt{1 + |\nabla \eta|^2} - 1 \right)$$

LUKE'S VARIATIONAL PRINCIPLES

J.C. LUKE, JFM (1967) [LUK67]

- First improvement of the classical Lagrangian $\mathscr{L} := K + \Pi$:

$$\mathcal{L} = \int_{t_1}^{t_2} \int_{\Omega} \rho \mathscr{L} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t, \qquad \mathscr{L} := \int_{-d}^{\eta} \left(\phi_t + \frac{1}{2} |\boldsymbol{\nabla}_{\boldsymbol{x},y} \phi|^2 + gy \right) \mathrm{d}y$$

$$\begin{aligned}
\delta\phi &: \ \Delta\phi = 0, \quad (\boldsymbol{x}, y) \in \Omega \times [-d, \eta], \\
\delta\phi|_{y=-d} &: \ \frac{\partial\phi}{\partial y} + \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla} d = 0, \quad y = -d, \\
\delta\phi|_{y=\eta} &: \ \frac{\partial\eta}{\partial t} + \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\eta - \frac{\partial\phi}{\partial y} = 0, \quad y = \eta(\boldsymbol{x}, t), \\
\delta\eta &: \ \frac{\partial\phi}{\partial t} + \frac{1}{2} |\boldsymbol{\nabla}\phi|^2 + g\eta = 0, \ \ y = \eta(\boldsymbol{x}, t).
\end{aligned}$$

- We recover the water wave problem by computing variations w.r.t. η and ϕ

Generalization of the Lagrangian density

D. Clamond & D. Dutykh, Phys. D (2012) [CD12]

- Introduce notation (traces):

$$\tilde{\phi} := \phi(\boldsymbol{x}, y = \eta(\boldsymbol{x}, t), t)\text{: quantity at the free surface}$$
$$\check{\phi} := \phi(\boldsymbol{x}, y = -d(\boldsymbol{x}, t), t)\text{: value at the bottom}$$

- Equivalent form of Luke's lagrangian:

$$\mathscr{L} = \tilde{\phi}\eta_t + \check{\phi}d_t - \frac{1}{2}g\eta^2 + \frac{1}{2}gd^2 - \int_{-d}^{\eta}\Big[\frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + \frac{1}{2}\phi_y^2\Big]\,\mathrm{d}y$$

- Explicitly introduce the velocity field: $\boldsymbol{u} = \boldsymbol{\nabla}\phi$, $v = \phi_y$

$$\mathscr{L} = \tilde{\phi}\eta_t + \check{\phi}d_t - \frac{1}{2}g\eta^2 - \int_{-d}^{\eta}\Big[\frac{1}{2}(\boldsymbol{u}^2 + v^2) + \boldsymbol{\mu}\cdot(\boldsymbol{\nabla}\phi - \boldsymbol{u}) + \nu(\phi_y - v)\Big]\,\mathrm{d}y$$

$\boldsymbol{\mu}, \nu$: Lagrange multipliers or pseudo-velocity field

Generalization of the Lagrangian density

D. Clamond & D. Dutykh, Phys. D (2012) [CD12]

- Relaxed variational principle:

$$\mathscr{L} = (\eta_t + \tilde{\boldsymbol{\mu}} \cdot \boldsymbol{\nabla}\eta - \tilde{\nu})\tilde{\phi} + (d_t + \check{\boldsymbol{\mu}} \cdot \boldsymbol{\nabla} d + \check{\nu})\check{\phi} - \frac{1}{2} g\eta^2 + \int_{-d}^{\eta} \Big[\boldsymbol{\mu} \cdot \boldsymbol{u} - \frac{1}{2}\boldsymbol{u}^2 + \nu v - \frac{1}{2} v^2 + (\boldsymbol{\nabla} \cdot \boldsymbol{\mu} + \nu_y)\phi \Big] \, \mathrm{d}y$$

- Classical formulation (for comparison):

$$\mathscr{L} = \tilde{\phi}\eta_t + \check{\phi} d_t - \frac{1}{2} g\eta^2 - \int_{-d}^{\eta} \Big[\frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + \frac{1}{2}\phi_y^2 \Big] \, \mathrm{d}y$$

Degrees of freedom: η, ϕ; $\boldsymbol{u}, v$; $\boldsymbol{\mu}, \nu$

- Ansatz:

$$\boldsymbol{u}(\boldsymbol{x}, y, t) \approx \bar{u}(\boldsymbol{x}, t), v(\boldsymbol{x}, y, t) \approx (y+d)(\eta+d)^{-1}\ \tilde{v}(\boldsymbol{x}, t)$$

$$\phi(\boldsymbol{x}, y, t) \approx \bar{\phi}(\boldsymbol{x}, t), \nu(\boldsymbol{x}, y, t) \approx (y+d)(\eta+d)^{-1}\ \tilde{\nu}(\boldsymbol{x}, t)$$

- Lagrangian density:

$$\mathscr{L} = \bar{\phi}\eta_t - \frac{1}{2}g\eta^2 + (\eta+d)\left[\bar{\mu}\cdot\bar{u} - \frac{1}{2}\bar{u}^2 + \frac{1}{3}\tilde{\nu}\tilde{v} - \frac{1}{6}\tilde{v}^2 - \bar{\mu}\cdot\boldsymbol{\nabla}\bar{\phi}\right]$$

- Nonlinear Shallow Water Equations:

$$h_t + \boldsymbol{\nabla}\cdot[h\bar{u}] = 0,$$

$$\bar{u}_t + (\bar{u}\cdot\boldsymbol{\nabla})\bar{u} + g\boldsymbol{\nabla}h = 0.$$

- Not so interesting...

CONSTRAINING WITH FREE SURFACE IMPERMEABILITY

- Constraint:

$$\tilde{\nu} = \eta_t + \bar{\mu} \cdot \nabla \eta$$

- Generalized Serre (Green–Naghdi) equations [Ser53]:

$$h_t + \nabla \cdot [h\bar{u}] = 0,$$

$$\bar{u}_t + \bar{u} \cdot \nabla \bar{u} + g\nabla h + \frac{1}{3} h^{-1} \nabla [h^2 \tilde{\gamma}] = (\bar{u} \cdot \nabla h) \nabla (h \nabla \cdot \bar{u}) - [\bar{u} \cdot \nabla (h \nabla \cdot \bar{u})] \nabla h$$

$$\tilde{\gamma} = \tilde{v}_t + \bar{u} \cdot \nabla \tilde{v} = h\big((\nabla \cdot \bar{u})^2 - \nabla \cdot \bar{u}_t - \bar{u} \cdot \nabla (\nabla \cdot \bar{u})\big)$$

CANNOT BE OBTAINED FROM LUKE'S LAGRANGIAN:

$\delta\bar{\mu}$: $\bar{u} = \nabla \bar{\phi} - \frac{1}{3} \tilde{v} \nabla \eta \neq \nabla \bar{\phi}$

- Ansatz and constraints ($v \neq \phi_y$):

$$\bar{\mu} = \bar{u}, \tilde{\nu} = \tilde{v}, \bar{u} = \nabla\bar{\phi}, \tilde{v} = -(\eta + d)\nabla^2\bar{\phi}$$

$$\mathscr{L} = \bar{\phi}\eta_t - \frac{1}{2}g\eta^2 - \frac{1}{2}(\eta + d)(\nabla\bar{\phi})^2 + \frac{1}{6}(\eta + d)^3(\nabla^2\bar{\phi})^2$$

- Generalized Kaup-Boussinesq equations:

$$\eta_t + \nabla \cdot \left[(\eta + d)\nabla\bar{\phi}\right] + \frac{1}{3}\nabla^2\left[(\eta + d)^3(\nabla^2\bar{\phi})\right] = 0,$$

$$\bar{\phi}_t + g\eta + \frac{1}{2}(\nabla\bar{\phi})^2 - \frac{1}{2}(\eta + d)^2(\nabla^2\bar{\phi})^2 = 0.$$

- Hamiltonian functional:

$$\mathscr{H} = \int_\Omega \left\{\frac{1}{2}g\eta^2 + \frac{1}{2}(\eta + d)(\nabla\bar{\phi})^2 - \frac{1}{6}(\eta + d)^3(\nabla^2\bar{\phi})^2\right\} \mathrm{d}\boldsymbol{x}$$

INCOMPRESSIBILITY AND PARTIAL POTENTIAL FLOW

- Ansatz and constraints ($v \neq \phi_y$):

$$\bar{\mu} = \bar{u}, \tilde{\nu} = \tilde{v}, \bar{u} = \nabla\bar{\phi}, \tilde{v} = -(\eta + d)\nabla^2\bar{\phi}$$

$$\mathscr{L} = \bar{\phi}\eta_t - \frac{1}{2}g\eta^2 - \frac{1}{2}(\eta + d)(\nabla\bar{\phi})^2 + \frac{1}{6}(\eta + d)^3(\nabla^2\bar{\phi})^2$$

- Generalized Kaup-Boussinesq equations:

$$\eta_t + \nabla \cdot \left[(\eta + d)\nabla\bar{\phi}\right] + \frac{1}{3}\nabla^2\left[(\eta + d)^3(\nabla^2\bar{\phi})\right] = 0,$$

$$\bar{\phi}_t + g\eta + \frac{1}{2}(\nabla\bar{\phi})^2 - \frac{1}{2}(\eta + d)^2(\nabla^2\bar{\phi})^2 = 0.$$

- Dispersion relation ($c^2 < 0$, $\kappa d > 1/\sqrt{3}$):

$$\eta = a\cos\kappa(x - ct), \quad c^2 = gd(1 - \frac{1}{3}(\kappa d)^2)$$

- Choice of the ansatz:

$$\{\phi; \boldsymbol{u}; v; \mu; \nu\} \approx \{\tilde{\phi}; \tilde{u}; \tilde{v}; \tilde{\boldsymbol{\mu}}; \tilde{\nu}\} e^{\kappa(y-\eta)}$$

$$2\kappa\mathscr{L} = 2\kappa\tilde{\phi}\eta_t - g\kappa\eta^2 + \frac{1}{2}\tilde{u}^2 + \frac{1}{2}\tilde{v}^2 - \tilde{u}\cdot(\boldsymbol{\nabla}\tilde{\phi} - \kappa\tilde{\phi}\boldsymbol{\nabla}\eta) - \kappa\tilde{v}\tilde{\phi}$$

- generalized Klein-Gordon equations:

$$\begin{aligned} \eta_t + \frac{1}{2}\kappa^{-1}\boldsymbol{\nabla}^2\tilde{\phi} - \frac{1}{2}\kappa\tilde{\phi} &= \frac{1}{2}\tilde{\phi}[\boldsymbol{\nabla}^2\eta + \kappa(\boldsymbol{\nabla}\eta)^2] \\ \tilde{\phi}_t + g\eta &= -\frac{1}{2}\boldsymbol{\nabla}\cdot\left[\tilde{\phi}\boldsymbol{\nabla}\tilde{\phi} - \kappa\tilde{\phi}^2\boldsymbol{\nabla}\eta\right] \end{aligned}$$

- Hamiltonian functional:

$$\mathcal{H} = \int_\Omega \left\{\frac{1}{2}g\eta^2 + \frac{1}{4}\kappa^{-1}[\boldsymbol{\nabla}\tilde{\phi} - \kappa\tilde{\phi}\boldsymbol{\nabla}\eta]^2 + \frac{1}{4}\kappa\tilde{\phi}^2\right\} \mathrm{d}\boldsymbol{x}$$

Deep water approximation

- Choice of the ansatz:

$$\{\phi; \boldsymbol{u}; v; \mu; \nu\} \approx \{\tilde{\phi}; \tilde{u}; \tilde{v}; \tilde{\boldsymbol{\mu}}; \tilde{\nu}\} e^{\kappa(y-\eta)}$$

$$2\kappa\mathscr{L} = 2\kappa\tilde{\phi}\eta_t - g\kappa\eta^2 + \frac{1}{2}\tilde{u}^2 + \frac{1}{2}\tilde{v}^2 - \tilde{u}\cdot(\boldsymbol{\nabla}\tilde{\phi} - \kappa\tilde{\phi}\boldsymbol{\nabla}\eta) - \kappa\tilde{v}\tilde{\phi}$$

- generalized Klein-Gordon equations:

$$\begin{aligned} \eta_t + \frac{1}{2}\kappa^{-1}\boldsymbol{\nabla}^2\tilde{\phi} - \frac{1}{2}\kappa\tilde{\phi} &= \frac{1}{2}\tilde{\phi}[\boldsymbol{\nabla}^2\eta + \kappa(\boldsymbol{\nabla}\eta)^2] \\ \tilde{\phi}_t + g\eta &= -\frac{1}{2}\boldsymbol{\nabla}\cdot\left[\tilde{\phi}\boldsymbol{\nabla}\tilde{\phi} - \kappa\tilde{\phi}^2\boldsymbol{\nabla}\eta\right] \end{aligned}$$

- Multi-symplectic structure:

$$\mathbb{M}\boldsymbol{z}_t + \mathbb{K}\boldsymbol{z}_x + \mathbb{L}\boldsymbol{z}_y = \boldsymbol{\nabla}_{\boldsymbol{z}}\mathcal{S}(\boldsymbol{z})$$

Comparison with exact Stokes wave

- Cubic Zakharov Equations (CZE):

$$\begin{aligned}
\eta_t - \mathfrak{d}\tilde{\phi} &= -\boldsymbol{\nabla}\cdot(\eta\boldsymbol{\nabla}\tilde{\phi}) - \mathfrak{d}(\eta\mathfrak{d}\tilde{\phi}) + \\
&\quad \tfrac{1}{2}\boldsymbol{\nabla}^2(\eta^2\mathfrak{d}\tilde{\phi}) + \mathfrak{d}(\eta\mathfrak{d}(\eta\mathfrak{d}\tilde{\phi})) + \tfrac{1}{2}\mathfrak{d}(\eta^2\boldsymbol{\nabla}^2\tilde{\phi}), \\
\tilde{\phi}_t + g\eta &= \tfrac{1}{2}(\mathfrak{d}\tilde{\phi})^2 - \tfrac{1}{2}(\boldsymbol{\nabla}\tilde{\phi})^2 - (\eta\mathfrak{d}\tilde{\phi})\boldsymbol{\nabla}^2\tilde{\phi} - (\mathfrak{d}\tilde{\phi})\mathfrak{d}(\eta\mathfrak{d}\tilde{\phi}).
\end{aligned}$$

- Phase speed :

$$\begin{aligned}
\text{Exact:}\quad & g^{-\frac{1}{2}}\kappa^{\frac{1}{2}}c = 1 + \tfrac{1}{2}\alpha^2 + \tfrac{1}{2}\alpha^4 + \tfrac{707}{384}\alpha^6 + \mathrm{O}(\alpha^8) \\
\text{CZE:}\quad & g^{-\frac{1}{2}}\kappa^{\frac{1}{2}}c = 1 + \tfrac{1}{2}\alpha^2 + \tfrac{\mathbf{41}}{\mathbf{64}}\alpha^4 + \tfrac{\mathbf{913}}{384}\alpha^6 + \mathrm{O}(\alpha^8) \\
\text{gKG:}\quad & g^{-\frac{1}{2}}\kappa^{\frac{1}{2}}c = 1 + \tfrac{1}{2}\alpha^2 + \tfrac{1}{2}\alpha^4 + \tfrac{\mathbf{899}}{384}\alpha^6 + \mathrm{O}(\alpha^8)
\end{aligned}$$

- n-th Fourier coefficient to the leading order: $\frac{n^{n-2}\alpha^n}{2^{n-1}(n-1)!}$ (the same in gKG & Stokes but not in CZE)

- Additional constraint:

$$\tilde{v} = \eta_t + \tilde{\boldsymbol{u}} \cdot \boldsymbol{\nabla}\eta$$

- Lagrangian density reads:

$$2\kappa\mathscr{L} = \tilde{\phi}\,(\kappa\,\eta_t + \boldsymbol{\nabla}\cdot\tilde{\boldsymbol{u}}) - g\kappa\,\eta^2 + \tfrac{1}{2}\tilde{\boldsymbol{u}}^2 + \tfrac{1}{2}(\eta_t + \tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\eta)^2$$

$$\begin{aligned}
\delta\,\tilde{\boldsymbol{u}}: \quad & \boldsymbol{0} = \tilde{\boldsymbol{u}} + (\eta_t + \tilde{\boldsymbol{u}}\cdot\nabla\eta)\boldsymbol{\nabla}\eta - \boldsymbol{\nabla}\tilde{\phi},\\
\delta\,\tilde{\phi}: \quad & 0 = \kappa\eta_t + \boldsymbol{\nabla}\cdot\tilde{\boldsymbol{u}},\\
\delta\,\eta: \quad & 0 = 2g\kappa\eta + \kappa\tilde{\phi}_t + \eta_{tt} + (\tilde{u}\cdot\nabla\eta)_t + \nabla\cdot(\tilde{u}\eta_t) + \nabla\cdot[(\tilde{u}\cdot\nabla\eta)\tilde{u}]
\end{aligned}$$

- Cannot be derived from Luke's Lagrangian:

$$\boldsymbol{\nabla}\tilde{\phi} = \tilde{\boldsymbol{u}} + \tilde{v}\boldsymbol{\nabla}\eta$$

- Additional constraint:

$$\tilde{v} = \eta_t + \tilde{\boldsymbol{u}} \cdot \boldsymbol{\nabla} \eta$$

- Lagrangian density reads:

$$2\kappa \mathscr{L} = \tilde{\phi}\,(\kappa\,\eta_t + \boldsymbol{\nabla} \cdot \tilde{\boldsymbol{u}}) - g\kappa\,\eta^2 + \tfrac{1}{2}\tilde{\boldsymbol{u}}^2 + \tfrac{1}{2}(\eta_t + \tilde{\boldsymbol{u}} \cdot \boldsymbol{\nabla} \eta)^2$$

$$\begin{aligned}
\delta\,\tilde{\boldsymbol{u}}: \quad & \boldsymbol{0} = \tilde{\boldsymbol{u}} + (\eta_t + \tilde{\boldsymbol{u}} \cdot \boldsymbol{\nabla} \eta)\boldsymbol{\nabla} \eta - \boldsymbol{\nabla} \tilde{\phi}, \\
\delta\,\tilde{\phi}: \quad & 0 = \kappa\eta_t + \boldsymbol{\nabla} \cdot \tilde{\boldsymbol{u}}, \\
\delta\,\eta: \quad & 0 = 2g\kappa\eta + \kappa\tilde{\phi}_t + \eta_{tt} + (\tilde{u} \cdot \nabla\eta)_t + \nabla \cdot (\tilde{u}\eta_t) + \nabla \cdot [(\tilde{u} \cdot \nabla\eta)\tilde{u}]
\end{aligned}$$

- Incompressibility is satisfied identically:

$$0 = \kappa\eta_t + \boldsymbol{\nabla} \cdot \tilde{\boldsymbol{u}} \Longleftrightarrow \boldsymbol{\nabla} \cdot \boldsymbol{u} + v_y = 0$$

- Additional constraint:

$$\tilde{v} = \eta_t + \tilde{\boldsymbol{u}} \cdot \boldsymbol{\nabla}\eta$$

- Lagrangian density reads:

$$2\kappa\mathscr{L} = \tilde{\phi}\,(\kappa\,\eta_t + \boldsymbol{\nabla}\cdot\tilde{\boldsymbol{u}}) - g\kappa\,\eta^2 + \tfrac{1}{2}\tilde{\boldsymbol{u}}^2 + \tfrac{1}{2}(\eta_t + \tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\eta)^2$$

$$\begin{aligned}
\delta\,\tilde{\boldsymbol{u}}: &\quad \boldsymbol{0} = \tilde{\boldsymbol{u}} + (\eta_t + \tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\eta)\boldsymbol{\nabla}\eta - \boldsymbol{\nabla}\tilde{\phi},\\
\delta\,\tilde{\phi}: &\quad 0 = \kappa\eta_t + \boldsymbol{\nabla}\cdot\tilde{\boldsymbol{u}},\\
\delta\,\eta: &\quad 0 = 2g\kappa\eta + \kappa\tilde{\phi}_t + \eta_{tt} + (\tilde{u}\cdot\nabla\eta)_t + \nabla\cdot(\tilde{u}\eta_t) + \nabla\cdot[(\tilde{u}\cdot\nabla\eta)\tilde{u}]
\end{aligned}$$

- Exact dispersion relation if $k = \kappa$:

$$\eta = a\cos k(x_1 - ct), \quad c^2 = 2\,g\,\kappa\,(k^2 + \kappa^2)^{-1}$$

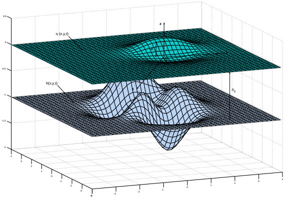
Arbitrary depth case

No available small parameters...

- Finite depth ansatz:

$$\phi \approx \frac{\cosh \kappa Y}{\cosh \kappa h}\tilde{\phi}(\boldsymbol{x}, t), \qquad \boldsymbol{u} \approx \frac{\cosh \kappa Y}{\cosh \kappa h}\tilde{\boldsymbol{u}}(\boldsymbol{x}, t),$$

$$\boldsymbol{\mu} \approx \frac{\cosh \kappa Y}{\cosh \kappa h}\tilde{\boldsymbol{\mu}}(\boldsymbol{x}, t), \qquad \nu \approx \frac{\sinh \kappa Y}{\sinh \kappa h}\tilde{\nu}(\boldsymbol{x}, t).$$



- Finite depth Lagrangian:

$$\mathscr{L} = [\,\eta_t + \tilde{\boldsymbol{\mu}}\cdot\boldsymbol{\nabla}\eta\,]\,\tilde{\phi} - \tfrac{1}{2}\,g\,\eta^2 + [\,\tilde{\nu}\,\tilde{v} - \tfrac{1}{2}\,\tilde{v}^2\,]\,\frac{\sinh(2\kappa h) - 2\kappa h}{2\kappa\cosh(2\kappa h) - 2\kappa}$$
$$+\,[\tilde{\boldsymbol{\mu}}\cdot\tilde{\boldsymbol{u}} - \tfrac{1}{2}\tilde{\boldsymbol{u}}^2 + \tilde{\phi}\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{\mu}} - \kappa\tanh(\kappa h)\tilde{\phi}\,\tilde{\boldsymbol{\mu}}\cdot\boldsymbol{\nabla}\eta]\,\frac{\sinh(2\kappa h) + 2\kappa h}{2\kappa\cosh(2\kappa h) + 2\kappa}$$
$$+\,\tfrac{1}{2}\,\tilde{\phi}\,\tilde{\nu}\left[\frac{2\kappa h}{\sinh(2\kappa h)} - 1\right].$$

Conclusions & Perspectives

Conclusions:

- A relaxed variational principle was presented
- Practical usage of this principle was illustrated
- All models automatically possess the Lagrangian structure
  - In most cases the Hamiltonian as well!

Perspectives:

- Further validation of derived models is needed
- Deeper study of their properties
- Development of variational discretizations

Thank you for your attention!

http://www.denys-dutykh.com/

References I

D. Clamond and D. Dutykh.
Practical use of variational principles for modeling water waves.
Physica D: Nonlinear Phenomena, 241(1):25–36, 2012.

W. Craig and C. Sulem.
Numerical simulation of gravity waves.
J. Comput. Phys., 108:73–83, 1993.

J. C. Luke.
A variational principle for a fluid with a free surface.
J. Fluid Mech., 27:375–397, 1967.

F. Serre.
Contribution à l'étude des écoulements permanents et variables dans les canaux.
La Houille blanche, 8:374–388, 1953.

References II

V. E. Zakharov.
Stability of periodic waves of finite amplitude on the surface of a deep fluid.
J. Appl. Mech. Tech. Phys., 9:190–194, 1968.