Multi-parameter families of K3 surfaces and hypergeometric functions

Fields Institute
Workshop on Hodge Theory in String Theory
November 22, 2013

Andreas Malmendier, Colby College
(work in progress with C. Doran)

Examples of rigid rank-n local systems on $\mathbb{P}^1 \backslash \{0, 1, \infty\}$

- Euler integral transform:

$$ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1 \middle| t\right) \doteq \int_0^1 \frac{dx}{\sqrt{x(x-t)}} \, \frac{1}{\sqrt{1-x}} \,, $$

$$ {}_{n+1}F_n\left(\begin{array}{c} a_1, a_2, \dots, a_n, \frac{1}{2} \\ c_1, \dots \, c_{n-1}, 1 \end{array} \middle| t \right) \doteq \int_0^1 \frac{dx}{\sqrt{x(x-t)}} \; {}_nF_{n-1}\left(\begin{array}{c} a_1, a_2, \dots, a_n \\ c_1, \dots \, c_{n-1} \end{array} \middle| x \right) $$

- Families of twisted Legendre pencils:

$$ \begin{array}{ll} E & y_1^2 = (1-x_1)\, x_1\, (x_1 - t) \,, \\ K3 & y_2^2 = (1-x_1)\, x_1\, (x_1 - x_2)\, x_2 (x_2 - t) \,, \\ CY3 & y_3^2 = (1-x_1)\, x_1\, (x_1 - x_2)\, x_2 (x_2 - x_3)\, x_3\, (x_3 - t) \,. \end{array} $$

- Compute periods:

$$ \int_A \frac{dx_1}{y_1} = \int_0^1 \frac{dx_1}{\sqrt{x_1(x_1-t)}} \, \frac{1}{\sqrt{1-x_1}} \doteq {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1 \middle| t\right) \,, $$

$$ \iint_S \frac{dx_1 \wedge dx_2}{y_2} = \int_0^1 \frac{dx_2}{\sqrt{x_2(x_2-t)}} \int_0^1 \frac{dx_1}{y_1} \doteq {}_3F_2\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{array} \middle| t \right) \,, $$

$$ \iiint_C \frac{dx_1 \wedge dx_2 \wedge dx_3}{y_3} = \int_0^1 \frac{dx_3}{\sqrt{x_3(x_3-t)}} \iint_S \frac{dx_1 \wedge dx_2}{y_2} \doteq {}_4F_3\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{array} \middle| t \right) \,. $$

Examples of rigid rank-n local systems on $\mathbb{P}^1 \backslash \{0, 1, \infty\}$

- Euler integral transform:

$$
{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1 \,\middle|\, t\right) \doteq \int_0^1 \frac{dx}{\sqrt{x(x-t)}} \; {}_1F_0\left(\frac{1}{2}; \,\middle|\, t\right) ,
$$

$$
{}_{n+1}F_n\left(\begin{array}{c} a_1, a_2, \ldots, a_n, \frac{1}{2} \\ c_1, \ldots\; c_{n-1}, 1 \end{array} \,\middle|\, t\right) \doteq \int_0^1 \frac{dx}{\sqrt{x(x-t)}} \; {}_nF_{n-1}\left(\begin{array}{c} a_1, a_2, \ldots, a_n \\ c_1, \ldots\; c_{n-1} \end{array} \,\middle|\, x\right)
$$

- Families of twisted Legendre pencils:

$$
\begin{array}{ll}
E & y_1^2 = (1-x_1)\, x_1\, (x_1 - t) \,, \\
K3 & y_2^2 = (1-x_1)\, x_1\, (x_1 - x_2)\, x_2 (x_2 - t) \,, \\
CY3 & y_3^2 = (1-x_1)\, x_1\, (x_1 - x_2)\, x_2 (x_2 - x_3)\, x_3\, (x_3 - t) \,.
\end{array}
$$

- Compute periods:

$$
\int_A \frac{dx_1}{y_1} = \int_0^1 \frac{dx_1}{\sqrt{x_1(x_1-t)}} \, \frac{1}{\sqrt{1-x_1}} \doteq {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1 \,\middle|\, t\right) ,
$$

$$
\iint_S \frac{dx_1 \wedge dx_2}{y_2} = \int_0^1 \frac{dx_2}{\sqrt{x_2(x_2-t)}} \int_0^1 \frac{dx_1}{y_1} \doteq {}_3F_2\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{array} \,\middle|\, t\right) ,
$$

$$
\iiint_C \frac{dx_1 \wedge dx_2 \wedge dx_3}{y_3} = \int_0^1 \frac{dx_3}{\sqrt{x_3(x_3-t)}} \iint_S \frac{dx_1 \wedge dx_2}{y_2} \doteq {}_4F_3\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{array} \,\middle|\, t\right) .
$$

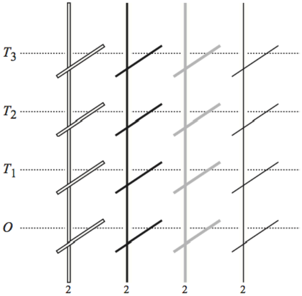
Kummer pencil in Het/F-theory duality

- Kummer $\mathrm{Kum}(E_1 \times E_2)$ of two elliptic curves

$$E_i: \quad y_i^2 = x_i(1-x_i)(x_i-\lambda_i), \quad \lambda_i = \lambda(\tau_i), \; i = 1, 2.$$

- (Isotrivial) elliptic fibration from projection to E_1, use $X = y_1^2 x_2$, $Y = y_1^2 (y_1 y_2)$ and $t = x_1$ on base:

$$Y^2 = X\,\left(y_1(t)^2 - X\right)\,\left(X - \lambda_2\, y_1(t)^2\right)\,, \qquad \begin{cases} 4\, I_0^*, x_1 = 0, 1, \lambda_1, \infty \\ j(\tau_F) = j(\tau_2) \end{cases}$$



Kummer pencil in Het/F-theory duality

- Kummer Kum$(E_1 \times E_2)$ of two elliptic curves

$$E_i: \quad y_i^2 = x_i(1-x_i)(x_i-\lambda_i), \quad \lambda_i = \lambda(\tau_i), \; i = 1, 2.$$

- (Isotrivial) elliptic fibration from projection to E_1, use $X = y_1^2 x_2$, $Y = y_1^2(y_1 y_2)$ and $t = x_1$ on base:

$$Y^2 = X\,(y_1^2 - X)\,(X - \lambda_2\, y_1^2)\;, \qquad \begin{cases} 4\, I_0^*, x_1 = 0, 1, \lambda_1 \\ j(\tau_F) = j(\tau_2) \end{cases}$$

- Periods of K3 surface (Picard rank 18):

$$\Omega_i = \oiint_{S_i} dt \wedge \frac{dX}{Y} = \oint_{A/B} \frac{dx_1}{y_1} \oint_{A/B} \frac{dx_2}{y_2} = \begin{cases} \omega_1\omega_2 \\ \tau_1\, \omega_1\omega_2 \\ \tau_2\, \omega_1\omega_2 \\ \tau_1\, \tau_2\, \omega_1\omega_2 \end{cases}$$

- Picard-Fuchs equations: rank-4 linear system

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1 \middle| \lambda_1\right) \boxtimes {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1 \middle| \lambda_2\right)$$

- Period Domain: $\left\{ [\Omega_1 : \cdots : \Omega_4] \in \mathbb{P}^3 \,\middle|\, \begin{array}{c} \Omega_1\Omega_4 - \Omega_2\Omega_3 = 0 \\ \mathrm{Re}(\Omega_1\bar{\Omega}_4 - \Omega_2\bar{\Omega}_3) > 0 \end{array} \right\}$

Embedding of gauge theory into F-theory

- Seiberg-Witten solution for $\mathcal{N}=2$ $SU(2)$ gauge theory with four quark flavors in $d=4$.
- Gauge coupling τ_G is encoded in rational elliptic fibration with section over u-plane, called **Seiberg-Witten curve**.
- Sen provided embedding of SW-curve into F-theory at limit point:

	$\mathcal{N}=2$ **gauge theory**	$\to$	**F-theory**
total space:	rational elliptic surface	$\to$	elliptic K3 surface
fiberation:	$j(\tau_G)$ (isotrivial)	$=$	$j(\tau_F)$ (isotrivial)
sing. fibers:	$2I_0^*$ at $u=1,\infty$		$4\,I_0^*$ at $t=0,1,\lambda_1,\infty$
VHS:	elliptic curve period		K3 periods
local system:	rank=1	$\to$	rank=2
monodromy:	$(1,-1,-1)$		$(T^2,T^2,-T^2)$
periods:	$\frac{\omega_2}{\sqrt{1-u}} = {}_1F_0(\frac{1}{2};\vert u)\cdot\omega_2$		$\underbrace{\oint_A \frac{dt}{\sqrt{t(t-\lambda_1)}}\;{}_1F_0(\frac{1}{2};\vert u)}_{{}_2F_1(\frac{1}{2},\frac{1}{2};1\vert\,\lambda_1)}\cdot\omega_2$

Rational surfaces

- Rational elliptic surfaces **S** over $\mathbb{CP}^1$ with section:

$$\bar{\mathbf{S}}: \; y^2 = 4\,x^3 - g_2\,x - g_3\,, \qquad \begin{array}{l} g_2 \in H^0(\mathcal{O}(4)), \\ g_3 \in H^0(\mathcal{O}(6)), \end{array} \; [t:1] \in \mathbb{P}^1.$$

- Consider extremal rational elliptic srfc with $\mathrm{rk}(\mathrm{MW}) = 0$,
- classified by **Miranda, Persson ['86]**.

Examples:

- Legendre family over the t-line,
  $y^2 = x(x-1)(x-t)$
- Hesse pencil of cubics in $\mathbb{P}^2$,
  $x_1^3 + x_2^3 + x_3^3 - 3\,t\,x_1\,x_2\,x_3 = 0$

Extremal rational surfaces and their periods

- Rational elliptic surfaces **S**

$$\bar{\mathbf{S}}: \; y^2 = 4\,x^3 - g_2\,x - g_3\,, \qquad \begin{array}{l} g_2 \in H^0(\mathcal{O}(4)), \\ g_3 \in H^0(\mathcal{O}(6)), \end{array} \quad [t:1] \in \mathbb{P}^1.$$

- Extremal rational surfaces (up to $*$-transfer):

isotrivial		
I_0	I_0^*	I_0^*
I_0	IV	IV^*
I_0	III	III^*
I_0	II	II^*

gen. modular		
I_1	I_1	I_4^*
I_2	I_2	I_2^*
I_3	I_1	IV^*
I_2	I_1	III^*
I_1	I_1	II^*

modular			
I_4	I_2	I_2	I_4
I_2	I_1	I_1	I_8
I_3	I_3	I_3	I_3
I_9	I_1	I_1	I_1
I_5	I_1	I_1	I_5
I_6	I_1	I_2	I_3

Extremal rational surfaces and their periods

- Rational elliptic surfaces **S**

$$\bar{\mathbf{S}}: \; y^2 = 4\,x^3 - g_2\,x - g_3\,, \qquad \begin{array}{l} g_2 \in H^0(\mathcal{O}(4)), \\ g_3 \in H^0(\mathcal{O}(6)), \end{array} \quad [t:1] \in \mathbb{P}^1.$$

- Extremal rational surfaces (up to $*$-transfer):

isotrivial		
I_0	I_0^*	I_0^*
I_0	IV	IV^*
I_0	III	III^*
I_0	II	II^*

gen. modular		
I_1	I_1	I_4^*
I_2	I_2	I_2^*
I_3	I_1	IV^*
I_2	I_1	III^*
I_1	I_1	II^*

modular			
I_4	I_2	I_2	I_4
I_2	I_1	I_1	I_8
I_3	I_3	I_3	I_3
I_9	I_1	I_1	I_1
I_5	I_1	I_1	I_5
I_6	I_1	I_2	I_3

- Write down Picard-Fuchs first order linear system satisfied by periods of $\frac{dx}{y}$ and $\frac{x\,dx}{y}$ over cycles on the fibers:

$$\vec{u} = \left(\omega = \int_{A_t} \frac{dx}{y}, \;\; \eta = \int_{A_t} \frac{x\,dx}{y} \right)$$

Extremal rational surfaces and their periods

- Rational elliptic surfaces **S**

$$\bar{\mathbf{S}}: \; y^2 = 4\,x^3 - g_2\,x - g_3\,, \qquad \begin{array}{l} g_2 \in H^0(\mathcal{O}(4)), \\ g_3 \in H^0(\mathcal{O}(6)), \end{array} \quad [t:1] \in \mathbb{P}^1.$$

- Extremal rational surfaces (up to $*$-transfer):

gen.	modular		μ
I_1	I_1	I_4^*	$1/2$
I_2	I_2	I_2^*	$1/2$
I_3	I_1	IV^*	$1/3$
I_2	I_1	III^*	$1/4$
I_1	I_1	II^*	$1/6$

modular				d	q
I_4	I_2	I_2	I_4	-1	0
I_2	I_1	I_1	I_8	-1	0
I_3	I_3	I_3	I_3	$\frac{1-i\sqrt{3}}{2}$	$\frac{3-i\sqrt{3}}{2}$
I_9	I_1	I_1	I_1	$\frac{1-i\sqrt{3}}{2}$	$\frac{3-i\sqrt{3}}{2}$
I_5	I_1	I_1	I_5	$\frac{8-5\varphi}{3+5\varphi}$	$\frac{816+165\varphi}{(3+5\varphi)^3}$
I_6	I_1	I_2	I_3	$\frac{1}{9}$	$\frac{1}{3}$

Solutions to Picard-Fuchs rank-2 linear system:

$$\omega = \,_2F_1(\mu, 1-\mu; 1|t) \qquad \omega = Hl(d, q; 1, 1, 1, 1|t)$$

One-parameter families of K3 surfaces

- *Construction 1*: quadratic twist with polynomial h

$$\begin{array}{rcl} \bar{\mathbf{X}}_1 = \bar{\mathbf{S}}_h : \ Y_1^2 & = & 4\,X_1^3 - h^2\,g_2\,X_1 - h^3\,g_3 \\ & \downarrow & \\ \bar{\mathbf{S}} : \ y^2 & = & 4\,x^3 - g_2\,x - g_3\,. \end{array}$$

- Quadratic twist adds 2 fibers of type I_0^*
- Parameter defines position of additional I_0^*, $h = t(t-A)$
- 1-parameter families of lattice-polarized K3 surfaces ($\rho = 19$)
- *Example:* $\mathrm{T}_{\mathbf{X}} = \langle 2 \rangle^{\oplus 2} \oplus \langle -2 \rangle$, $A \neq 0$:

E_{sing}	I_2	I_2	I_2^*
t	0	1	∞

S is rational

E_{sing}	I_2^*	I_2	I_2^*	I_0^*
t	0	1	∞	A

$\mathbf{X}_1$ is K3

One-parameter families of K3 surfaces

- *Construction 1*: quadratic twist with polynomial h

$$\begin{array}{rcl} \bar{\mathbf{X}}_1 = \bar{\mathbf{S}}_h :\ Y_1^2 & = & 4\,X_1^3 - h^2\,g_2\,X_1 - h^3\,g_3 \\ & \downarrow & \\ \bar{\mathbf{S}} :\ y^2 & = & 4\,x^3 - g_2\,x - g_3\,. \end{array}$$

- 2 I_0^*'s, $h = t(t-A)$, 2-form: $dt \wedge \frac{dX_1}{Y_1} = \frac{1}{\sqrt{h(t)}}\, dt \wedge \frac{dx}{y}$
- Represent K3-periods as **Euler transform** of a HGF
  $\Omega = \oiint_{S_{ij}} dt \wedge \frac{dX_1}{Y_1} = \int_{t_i^*}^{t_j^*} dt\, \frac{1}{\sqrt{h(t)}}\, \omega$
- They solve a 3rd oder ODE (=symmetric square of 2nd order).

Solutions to the rank-3 integrable linear system of K3 periods:

$$\Omega = {}_3F_2\left(\left.\begin{array}{c} \mu, \frac{1}{2}, 1-\mu \\ 1, 1 \end{array} \right| A \right) \qquad \Omega = \left[\, Hl\left(d, \tfrac{q}{4}; \tfrac{1}{4}, \tfrac{3}{4}, 1, \tfrac{1}{2} \middle| A\right) \right]^2$$

One-parameter families of K3 surfaces

- *Construction 1*: quadratic twist with polynomial h

$$\begin{array}{rcl}\bar{\mathbf{X}}_1 = \bar{\mathbf{S}}_h :\ Y_1^2 & = & 4\,X_1^3 - h^2\,g_2\,X_1 - h^3\,g_3 \\ & \downarrow & \\ \bar{\mathbf{S}} :\ y^2 & = & 4\,x^3 - g_2\,x - g_3\,.\end{array}$$

- $2\ I_0^*$'s, $h = t(t-A)$, 2-form: $dt \wedge \frac{dX_1}{Y_1} = \frac{1}{\sqrt{h(t)}}\,dt \wedge \frac{dx}{y}$
- Represent K3-periods as **Euler transform** of a HGF $\Omega = \oiint_{S_{ij}} dt \wedge \frac{dX_1}{Y_1} = \int_{t_i^*}^{t_j^*} dt\,\frac{1}{\sqrt{h(t)}}\,\omega$
- They solve a 3rd oder ODE (=symmetric square of 2nd order).

Solutions to the rank-3 integrable linear system of K3 periods:

$$\Omega = \Big[\,{}_2F_1\Big(\tfrac{\mu}{2}, \tfrac{1-\mu}{2}; 1\Big|A\Big)\Big]^2 \qquad \Omega = \Big[\,Hl\,\Big(d, \tfrac{q}{4}; \tfrac{1}{4}, \tfrac{3}{4}, 1, \tfrac{1}{2}\Big|A\Big)\,\Big]^2$$

One-parameter families of K3 surfaces

Proposition (M.-Doran)

- *There is a fundamental set of solutions* $[\Omega_1 : \Omega_2 : \Omega_3]$ *such that*

μ	*quadric surface*	*series*
1/2	$\Omega_1^2 + \Omega_2^2 - \Omega_3^2$ $2\,\Omega_1^2 + 2\,\Omega_2^2 - 2\,\Omega_3^2$	${}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle\vert A\right) = \sum_{n=0}^{\infty} \frac{(2n)!^3}{n!^6} \frac{A^n}{2^{6n}}$
1/3	$4\,\Omega_1^2 + 3\,\Omega_2^2 - 3\,\Omega_3^2$	${}_3F_2\left(\begin{matrix} \frac{1}{3}, \frac{1}{2}, \frac{2}{3} \\ 1, 1 \end{matrix} \middle\vert A\right) = \sum_{n=0}^{\infty} \frac{(2n)!\,(3n)!}{n!^5} \frac{A^n}{2^{2n}3^{3n}}$
1/4	$4\,\Omega_1^2 + 2\,\Omega_2^2 - 2\,\Omega_3^2$	${}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{2}{4}, \frac{3}{4} \\ 1, 1 \end{matrix} \middle\vert A\right) = \sum_{n=0}^{\infty} \frac{(4n)!}{n!^4} \frac{A^n}{4^{4n}}$
1/6	$\Omega_1^2 + 4\,\Omega_2^2 - \Omega_3^2$	${}_3F_2\left(\begin{matrix} \frac{1}{6}, \frac{3}{6}, \frac{5}{6} \\ 1, 1 \end{matrix} \middle\vert A\right) = \sum_{n=0}^{\infty} \frac{(6n)!}{n!^3(3n)!} \frac{A^n}{2^{6n}3^{3n}}$

- *First 4 cases with 4 singularities are obtained as double covers.*
- *Cases 6 and 5 are related to Apery's recurrence for* $\zeta(3)$ *and* $\zeta(2)$*:*

$$\Omega = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \right) \frac{A^n}{4^n}, \quad \Omega = \left(\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} a^n \right)^2$$

One-parameter families of K3 surfaces

- *Construction 2*: base change by double cover

$$\begin{array}{ccc} \mathbf{X}_2 = \mathbf{S} \times_C B & \xrightarrow{\pi'} & B \\ \big\downarrow & & \big\downarrow {\scriptstyle t=f_A(s)} \\ \mathbf{S} & \xrightarrow{\pi} & C \end{array}$$

- with $t = \frac{(s+A/4)^2}{s}$ we have $ds \wedge \frac{dX_2}{Y_2} = \frac{1}{\sqrt{t(t-A)}}\, dt \wedge \frac{dx}{y}$
- *Example* ($\mu = 1/6$): 1-param. family of K3 surfaces of Picard rank 19, $\mathrm{T}_{\mathbf{X}} = H \oplus \langle -2 \rangle$, $A \neq 0$:

E_{sing}	I_1	I_1	II^*
t	0	1	∞

S is rational

E_{sing}	I_2	$2\,I_1$	$2\,II^*$
s	$-A/4$	$f_A^{-1}(1)$	$0, \infty$

$\mathbf{X}_2$ is K3

- $\mathbf{X}_2$'s are one-parameter families with $n = 1, 2, 3, 4, 5, 6, 8, 9$ and $M_n = H \oplus E_8 \oplus E_8 \oplus \langle -2n \rangle$ lattice polarization.

One-parameter families of K3 surfaces

Proposition (M.-Doran)

- *The two constructions give rise to degree-two rational maps* $\mathbf{X}_2 \dashrightarrow \mathbf{X}_1$ *($\rho = 19$) that leave the holomorphic two-form invariant.*
- *The Picard-Fuchs equations of pairs $\{\mathbf{X}_2, \mathbf{X}_1\}$ coincide (rk=3).*

Remarks:

- The periods of the families with M_n lattice polarization for $n = 1, 2, 3, 4, 6$ agree with the results of **Lian, Yau ['96]**, **Dolgachev ['96]**, **Verrill, Yui['00]**, **Doran ['00]**, and **Beukers, Montanus, Peters, Stienstra ['84, '85, '86, '00]**.
- One can "undo" the Kummer construction and provide interpretation of K3 periods in terms of modular forms:

$$
{}_2F_1\left(\tfrac{\mu}{2}, \tfrac{1-\mu}{2}; 1 \middle| A\right) = {}_2F_1\left(\mu, 1-\mu; 1 \middle| a\right), \; A = 4a(1-a),
$$
$$
HI\left(d, \tfrac{q}{4}; \tfrac{1}{4}, \tfrac{3}{4}, 1, \tfrac{1}{2} \middle| A\right) \doteq HI\left(d, q; 1, 1, 1, 1 \middle| a\right), \; A = quartic(a).
$$

Two-parameter families of K3 surfaces

Remarks:

- Restrict (for simplicity) to case with 3-singular fibres.
- Repeat above constructions with **two** parameters (A, B).
- *Example* $(\mu = 1/6)$: $M = H \oplus E_8 \oplus E_8$-polarized case,

$$\underbrace{\begin{array}{c|ccc} E_{\text{sing}} & 2\,I_1 & 2\,I_1 & 2\,II^* \\ \hline s & t(s)=0 & t(s)=1 & 0,\ \infty \end{array}}_{\mathbf{X}_2 \text{ is } M\text{-polarized K3}} \dashrightarrow \underbrace{\begin{array}{c|ccc} E_{\text{sing}} & 2\,I_1 & II^* & 2\,I_0^* \\ \hline t & 0,1 & \infty & A,B \end{array}}_{\mathbf{X}_1 \text{ is } \mathrm{Kum}(E_1 \times E_2)}$$

- Other examples realize elliptic fibrations $\mathfrak{J}_3, \mathfrak{J}_4, \mathfrak{J}_6, \mathfrak{J}_7, \mathfrak{J}_{11}$ on $\mathrm{Kum}(E_1 \times E_2)$ from **Oguiso ['88]**.

Two-parameter families of K3 surfaces

Set $h(t) = (t - A)(t - B)$ in $\mathbf{X}_1$ and $t = f_{A,B}(s)$ in $\mathbf{X}_2$ $(A \neq B)$ s.t.

$$dt \wedge \frac{dX_1}{Y_1} = ds \wedge \frac{dX_2}{Y_2} = \frac{1}{\sqrt{h(t)}}\, dt \wedge \frac{dx}{y}$$

Proposition (M.-Doran)

- *The two constructions give rise to degree-two rational maps* $\mathbf{X}_2 \dashrightarrow \mathbf{X}_1$ $(\rho = 18)$ *that leave the holomorphic two-form invariant.*
- *The Picard-Fuchs equations for pairs* $\{\mathbf{X}_2, \mathbf{X}_1\}$ *coincide.*
- *K3-periods solve an integrable rank-4 linear system in* ∂_A, ∂_B.
- *Fundamental solutions* $[\Omega_1 : \Omega_2 : \Omega_3, : \Omega_4]$ *form a quadric in* $\mathbb{P}^3$.

Two-parameter families of K3 surfaces

Remarks:

HG local system $\xrightarrow{\text{E.T.}}$ $\mathcal{A}$-HG local system

$$\omega(t) = {}_2F_1(\alpha, \beta; \gamma | t) \quad \xrightarrow{\text{E.T.}} \quad \Omega(A,B) = \int_{t_i^*}^{t_j^*} \frac{dt}{\sqrt{(t-A)(t-B)}}\, {}_2F_1(t)$$

GKZ HG system (rk=2) $\longrightarrow$ **GKZ HG system** (rk=4)

$$\mathbf{c} = (\gamma-\beta, \alpha, \beta) \qquad \mathbf{C} = (\gamma-\beta, 1/2, \alpha, 1/2, 1+\alpha-\beta)$$

$$\mathfrak{l} = [1,-1,-1,1] \quad \longrightarrow \quad \mathfrak{L} = \begin{bmatrix} 1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\Box_t\, \omega(t) = 0 \qquad \vec{\Box}_{A,B}\, \Omega(A,B) = 0$$

- Condition for non-resonance: $\alpha, \beta, \gamma-\alpha, \gamma-\beta \notin \mathbb{Z}$
- Condition for quadrically related solutions: $\gamma = 1, \alpha+\beta = 1$
- $\because\ \alpha = \mu \in (0,1),\ \beta = 1-\mu,\ \gamma = 1$

$$\Box_t^{\mu} \quad \xrightarrow{E.T.} \quad \vec{\Box}_{A,B} = \Box_X^{\mu/2} \boxtimes \Box_Y^{\mu/2}$$

with $XY = (A-B)^2$ and $(1-X)(1-Y) = (1-A-B)^2$.

Two-parameter families of K3 surfaces

Remarks:

HG local system	$\xrightarrow{\text{E.T.}}$	$\mathcal{A}$-HG local system
$\omega(t) = {}_2F_1(\alpha,\beta;\gamma\mid t)$		$\Omega(A,B) = \int_{t_i^*}^{t_j^*} \frac{dt}{\sqrt{(t-A)(t-B)}}\,{}_2F_1(t)$
GKZ HG system (rk=2)	$\longrightarrow$	**GKZ HG system** (rk=4)
$\mathbf{c} = (\gamma-\beta,\alpha,\beta)$ $\mathfrak{l} = [1,-1,-1,1]$ $\Box_t\,\omega(t) = 0$		$\mathbf{C} = (\gamma-\beta, 1/2, \alpha, 1/2, 1+\alpha-\beta)$ $\mathfrak{L} = \begin{bmatrix} 1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 1 \end{bmatrix}$ $\vec{\Box}_{A,B}\,\Omega(A,B) = 0$

- Condition for non-resonance: $\alpha,\beta,\gamma-\alpha,\gamma-\beta \notin \mathbb{Z}$
- Condition for quadrically related solutions: $\gamma = 1, \alpha+\beta = 1$
- $\because\ \alpha = \mu \in (0,1),\ \beta = 1-\mu,\ \gamma = 1$

$$\Box_t^{\mu} \xrightarrow{E.T.} \vec{\Box}_{A,B} = \Box_X^{\mu/2} \boxtimes \Box_Y^{\mu/2}$$

- **Beukers ['13]** determined full monodromy group for this system in terms of **C**.

Three-parameter families of K3 surfaces

- Use $h(t) = (t-A)(t-B)(t-C)$ to obtain 3-parameter families $\mathbf{X}_1 = \mathbf{S}_h$ of K3 surfaces by quadratic twist.
- Picard-Fuchs equations in A, B, C form a GKZ system:
  - resonant for $\mu = \frac{1}{2}$, rk=5
  - non-resonant for $\mu \neq \frac{1}{2}$, rk=6
- There is one family where $\mathbf{X}_1$ is a 3-parameter family of K3 surfaces with lattice polarization of Picard rank 17 : $\mu = \frac{1}{2}$.
- Families of curves from SW-theory:

$$\begin{array}{ccccc} I_0^* + I_0^* & \xleftarrow{\text{mass}=0} & 3I_2 + I_0^* & \xrightarrow{RG} & 2\,I_2 + I_2^* \\ \text{extremal} & & \text{rk(MW)} = 1 & & \text{extremal} \end{array}$$

- K3 surfaces from twisting:
  Sen limit: 2-par. twist$(I_0^* + I_0^*)$
  Deformation: 2-par. twist$(3I_2 + I_0^*)$ = 3-par. twist$(2I_2 + I_2^*)$

Kummer surfaces from $SU(2)$-Seiberg-Witten curve

Proposition (M.-Doran)

- *The family* $\mathbf{X}_1 = \mathbf{S}_h \to \mathbb{P}^1$ $(\mu = 1/2)$ *is a family of Jacobian K3 surfaces of Picard rank 17.*
- *There is a family* $\mathbf{X}_2 \to \mathbb{P}^1$ *obtained from the covering map* $t = (C\,s^2 - B)/(s^2 - 1)$.
- $\mathbf{X}_2 = \mathrm{Kum}(\mathbf{A})$ *where*

ρ	*parameter*	$\mathbf{A}$	*equation*	*moduli space*
17	A, B, C	$\mathrm{Jac}\, C^{(2)}$	$y^2 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$	$\Gamma_2(2)\backslash\mathbb{H}_2$
18	$A, B, C=\infty$	$E_1 \times E_2$	$y_i^2 = (2x_i - 1)\left[(4x_i+1)^2 + 9r_i\right]$	$\Gamma\backslash\mathbb{H}\times\mathbb{H}$
19	$A, B=0, C=\infty$	$E_1 \times E_1$	$y_1^2 = (2x_1 - 1)\left[(4x_1+1)^2 + 9r_1\right]$	$\Gamma_0(2)\backslash\mathbb{H}$

Mayr, Stieberger ['95], Kokorelis ['99]: moduli space of genus-two curves with level-two structure = moduli space of $\mathcal{N} = 2$ heterotic string theories compactified on $K3 \times T^2$ with one Wilson line.

One-parameter families of Calabi-Yau 3folds

- *Construction 1*: quadratic twist with polynomial h

$$\begin{array}{rcl} \bar{\mathbf{X}}_1 = \bar{\mathbf{S}}_h : \; Y_1^2 & = & 4\,X_1^3 - h^2\,g_2\,X_1 - h^3\,g_3 \\ & \downarrow & \\ \bar{\mathbf{S}} : \; y^2 & = & 4\,x^3 - g_2\,x - g_3\,. \end{array}$$

- $h = u_1\,u_2^2\,(u_1 - A\,u_2)\,t_1\,(t_1 - u_1\,t_2)$ in $[t_1 : t_2]$ and $[u_1 : u_2]$
- *Construction 2*: birational family of 3-folds fibered by M_n-polarized K3 surfaces ($\mu = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6} \leftrightarrow n = 4, 3, 2, 1$)
- Represent CY-periods as iterated **Euler transform** :
$\Omega = \iiint_C du \wedge dt \wedge \frac{dX_1}{Y_1} = \int_{u_i^*}^{u_j^*} \frac{du}{\sqrt{u(u-A)}} \int_{t_k^*}^{t_l^*} \frac{dt}{\sqrt{t(t-u)}}\,\omega(t)$

Solutions to the rank-4 integrable linear system of CY periods:

$${}_4F_3\left(\begin{array}{c} \mu, \frac{1}{2}, \frac{1}{2}, 1-\mu \\ 1, 1, 1 \end{array} \middle| A\right)$$

$\mu = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ *give cases* $(m, a) = (16, 8), (12, 7), (8, 6), (4, 5)$