Optimal Covering of a Disk with Congruent Smaller Disks

Balázs Csikós

Eötvös Loránd University
Budapest

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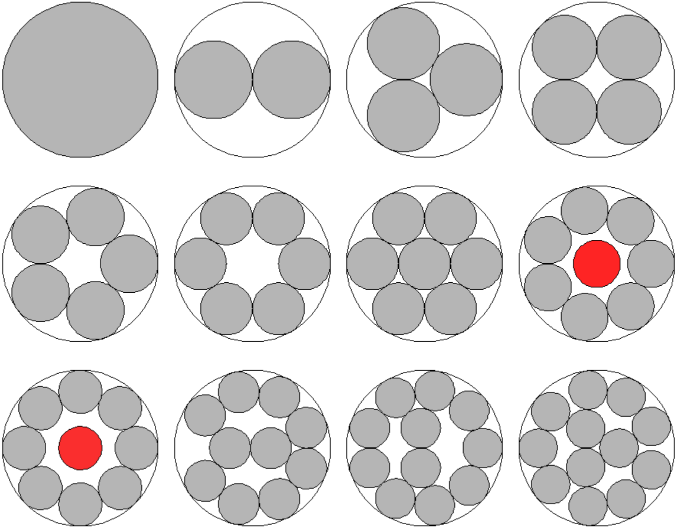
Problem: Find

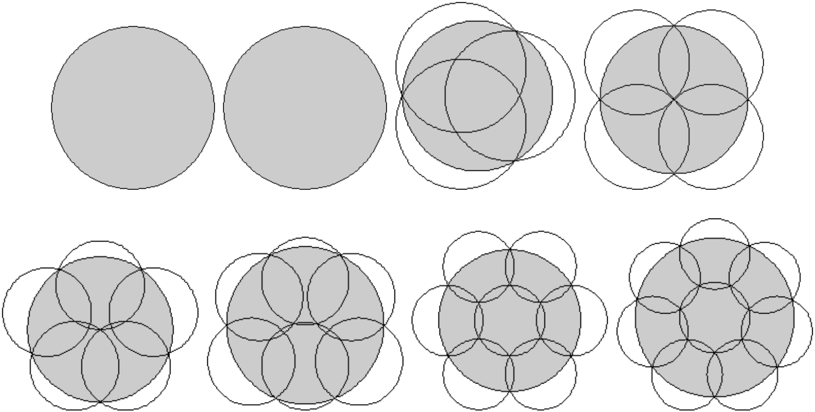
$r_p^n = \max\{r : n$ disks of radius r can be packed into a unit disk.$\}$



(1-4 trivial; 5-7 Graham (1968); 8-10 Pirl (1969); 11 Melissen (1994); 12 Fodor (2000); 13 Fodor (2003); 14 unsolved)

Covering of a Disk with Congruent Disks

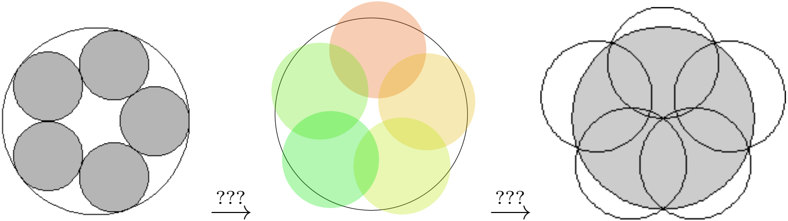
Problem: Find $r_c^n = \min\{r : n \text{ disks of radius } r \text{ can cover a unit disk.}\}$
Solutions for $n \leq 8$:



For $n = 5$ and 6 the optimal configuration has only a mirror symmetry.
(1-4, 7 trivial; 5-6 K. Bezdek (1979,1983); 8-9 G. Fejes Tóth (1999); 10 ?)

R. Connelly's Problem:

- Given n and $r_p^n \leq r \leq r_c^n$, find the configuration of n disks of radius r, that covers the most of the are of a unit disk.
- Understand how the rotational symmetry of the optimal configuration for $n = 5$, $r = r_p^5$ is lost as r grows continuously from r_p^5 to r_c^5.



Derivative of the Volume of Flowers

Definitions

- A lattice polynomial $f(x_1, \ldots, x_k)$ is a formal expression built from the variables $x_1, \ldots, x_n$ and the binary operation symbols $\wedge$ and $\vee$. Example: $x_1 \wedge (x_2 \vee x_3)$.
- A flower is a body obtained by evaluating a lattice polynomial $f(x_1, \ldots, x_k)$ on some balls $x_i = B_i$ with operations $\vee = \cup$, $\wedge = \cap$.
- The power of a point $\mathbf{p}$ with respect to a ball $B = B(\mathbf{c}, r)$ is $P_B(\mathbf{p}) = \|\mathbf{p} - \mathbf{c}\|^2 - r^2$.
- The (truncated) Dirichlet–Voronoi cell of the ball B_i in the flower $f(B_1, \ldots, B_k)$ is the set

  $$C_i = \{\mathbf{x} : f(P_{B_1}(\mathbf{x}), \ldots, f(P_{B_k}(\mathbf{x})) = f(P_{B_i}(\mathbf{x})\},$$

  where f is evaluated on the powers with operations $\vee = \min$, $\wedge = \max$.
- The wall W_{ij} between the Dirichlet–Voronoi cells C_i and C_j if $W_{ij} = C_i \cap C_j$.

Derivative of the Volume of Flowers

Theorem

Suppose that each variable x_i occurs in the lattice polynomial $f(x_1, \ldots, x_k)$ exactly once. Let ϵ_{ij} be 1 if the the shortest subexpression of f that contains both x_i and x_j is a $\vee$ of shorter lattice polynomials, -1 otherwise.
If the ball $B_i(t) = B(\mathbf{c}_i(t), r_i)$ move in a differentiable way, then the volume $V(t)$ of the flower $f(B_1(t), \ldots, B_k(t))$ is differentiable at each t at which the balls $B_i(t)$ are different and

$$V' = \sum_{1 \le i < j \le k} \epsilon_{ij} Vol_{n-1}(W_{ij}) d'_{ij},$$

where $d_{ij} = \|\mathbf{c}_i - \mathbf{c}_j\|$.

Observation (R. Connelly)

If a configuration of balls maximizes the volume of a flower, then the tensegrity obtained by connecting $\mathbf{c}_i$ and $\mathbf{c}_j$ by a cable if $\epsilon_{ij} = -1$ and a strut if $\epsilon_{ij} = 1$ is rigid. The volumes of the walls provide a self stress $\omega_{ij} = \epsilon_{ij} \frac{Vol_{n-1}(W_{ij})}{d_{ij}}$.

Formulae in the Euclidean Plane

- Orient the plane and all the circles in the positive direction.
- J – rotation by $+\frac{\pi}{2}$.
- For a collection of k disks $D_i = D(\mathbf{c}_i, r_i)$ with boundary circles $C_i = S(\mathbf{c}_i, r_i)$, denote by $\mathbf{p}_{ij}$ the point where C_i enters C_j, provided that this point exists.
- For a flower $f(D_1, \ldots, D_k)$, define the vertex set $\mathcal{V}$ as the set of crossings $\mathbf{p}_{ij}$, lying on the boundary of the flower.

Theorem

Assume that the disks $D_i = D(\mathbf{c}_i, r_i)$ move smoothly and the speed vectors of their centers are $\mathbf{v}_i$, $i = 1, \ldots, k$. If all the disks are different, then the derivative of the area V of the flower $f(D_1, \ldots, D_k)$ exists and can be expressed as follows

$$V' = \sum_{i=1}^{k} \left\langle \mathbf{v}_i, J\Big(\sum_{\mathbf{p}_{ij} \in \mathcal{V}} \epsilon_{ij} \mathbf{p}_{ij} - \sum_{\mathbf{p}_{ji} \in \mathcal{V}} \epsilon_{ij} \mathbf{p}_{ji} \Big) \right\rangle = \sum_{\mathbf{p}_{ij}} \epsilon_{ij} \langle \mathbf{v}_i - \mathbf{v}_j, J\mathbf{p}_{ij} \rangle.$$

- Given a lattice polynomial $f(x_1, \dots, x_k)$, an arrangement of disks $D_1, \dots, D_k$ is called critical configuration if the derivative of the area of $f(D_1, \dots, D_k)$ is 0 for any smooth variation of the disks.

Corollary

A configuration of disks is critical for a given f, if and only if

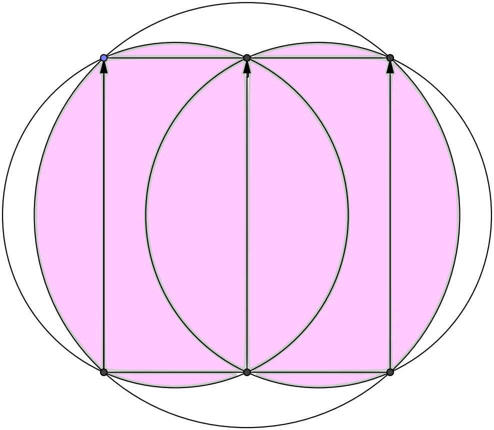
$$\forall i: \qquad \sum_{\mathbf{p}_{ij} \in \mathcal{V}} \epsilon_{ij} \mathbf{p}_{ij} - \sum_{\mathbf{p}_{ji} \in \mathcal{V}} \epsilon_{ij} \mathbf{p}_{ji} = \mathbf{0}.$$

Theorem

If for a given critical configuration, there are no tangent pair of circles, the contact point of which is on the boundary of the flower, then the area of the flower is twice differentiable at this configuration and its Hessian can be computed by the formula

$$Hess(\mathbf{V}, \mathbf{V}) = \sum_{\mathbf{p}_{ij} \in \mathcal{V}} \frac{\epsilon_{ij}}{r_i r_j \sin \theta_{ij}} (\mathbf{v}_i - \mathbf{v}_j)^T (\mathbf{c}_i - \mathbf{p}_{ij})(\mathbf{c}_j - \mathbf{p}_{ij})^T (\mathbf{v}_i - \mathbf{v}_j),$$

Covering the Most with 2 Congruent Disks



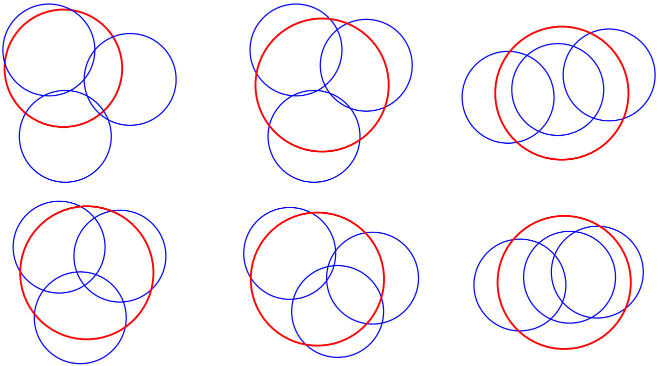
Optimal Coverings by 3 Disks

joint work with B. Szalkai

First Step. Classify all combinatorially different arrangements that can possibly give critical configurations.

- Find criteria that rule out geometrically not realizable combinatorial structures.
- Find criteria that are necessarily satisfied by optimal configurations.
- Develop a software that lists all the remaining cases.

This produces a list of combinatorial configuration types to be dealt with.



Optimal Coverings by 3 Disks

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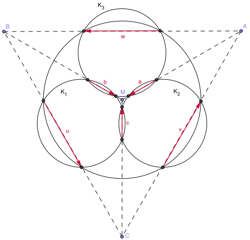
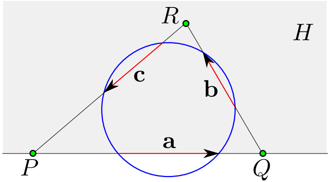
Second Step. Find all critical configurations belonging to one of the listed combinatorial types. The hardest configuration is





Lemma

Given a circle C, an intersecting straight line e and two points $P, Q \in e$ on different sides of C, and one of the half planes H bounded by e, there is a unique point $R \in H$ such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$.

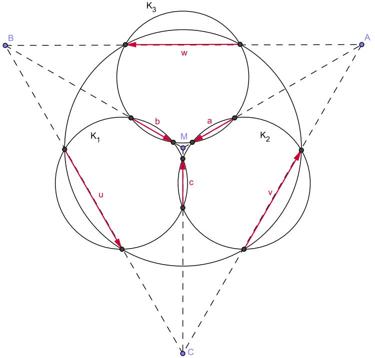
Optimal Coverings by 3 Disks

joint work with B. Szalkai

The proof of the Lemma requires to show that two algebraic curves of degree four have exactly one intersection point in H.

Theorem

For any $r_p^3 \le r \le r_c^3$, the opimal covering of a unit disk with three congruent disks of radius r is given by the unique rotationally symmetric critical configuration of the following combinatorial type

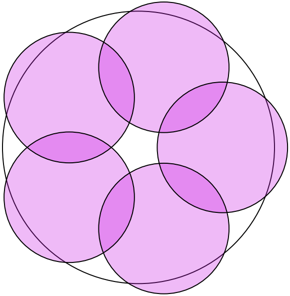


Optimal Coverings by 5 Disks

Computer search for optimal covering was done by Tarnai–Gáspár–Hincz, using a mechanical model.

They observed that

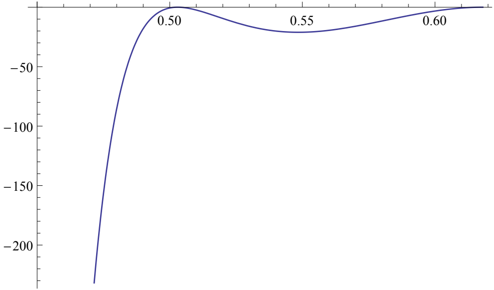
- there is a unique rotationally symmetric critical configuration;
- for small values of $r \geq r_p^5$, the unique rotationally symmetric critical configuration is the best;
- as r increases above a certain value r_0, two critical configurations grow out of the rotationally symmetric one, an egg shaped and a pumpkin shaped.

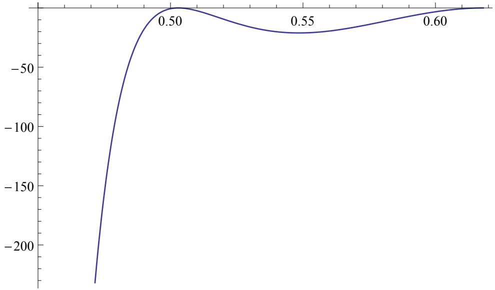


Question

How can we compute r_0? In which directions should we deform the rotationally symmetric critical configuration at r_0 to push toward the pumpkin or egg shaped critical configurations?

- The Hessian can be computed using the general formulae presented above.
- Fixing the big circle, the Hessian is defined on a 10-dimensional Euclidean space.
- Infinitesimal rotations about the origin are in the kernel.
- We consider the Hessian on the orthogonal complement.
- Here is a numeric plot of the graph of its determinant:





- H is negative definite at the beginning, has two positive eigenvalues in the valley.

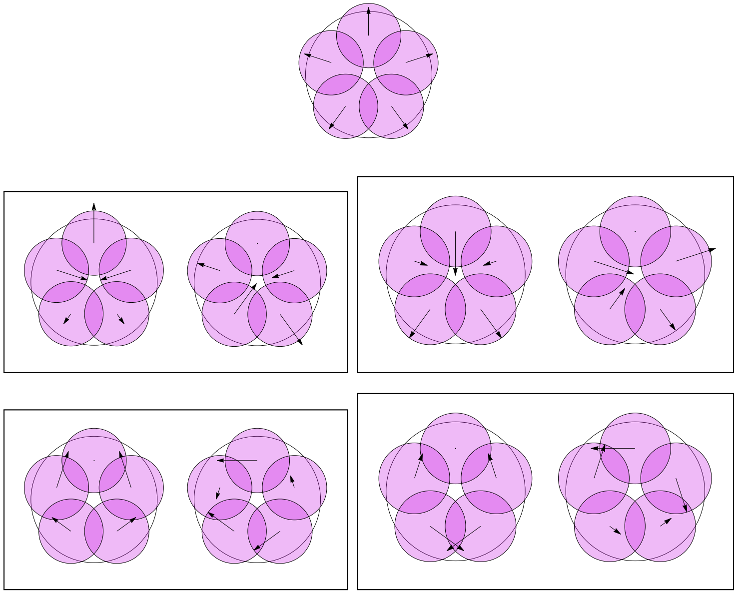
Question

Is there a double root? or two roots very close to one another?

Theorem

The dihedral group D_5 is a symmetry group of the configuration. Thus, it acts on the tangents space, on which the Hesse form is defined. The Hesse form is invariant under the action of D_5. Thus, the irreducible factors of this representation are invariant under the Hesse map. As a result, we obtain an orthonormal basis diagonalizing the Hesse form. The Hesse map has consant eigenvalue on irreducible factors.

Eigenvectors of the Hesse map



Thank you for your attention!!!