Regular Covers and Monodromy Groups of Abstract Polytopes

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(from projects with L.B., M.M., D.O., E.S. and G.W.)

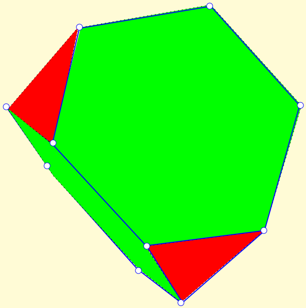
Fields Institute, November, 2013

(supported in part by NSERC)

Regularity is rare, despite its ubiquity

A d-polytope $\mathcal{P}$ is regular if $\mathrm{Aut}(\mathcal{P})$ is transitive on flags.
But most polytopes of rank $d \geq 3$ are not regular.

Eg. The truncated tetrahedron $\mathcal{Q}$, although quite symmetrical, has facets of two types (and 3 flag orbits under action of $\mathrm{Aut}(\mathcal{Q}) \simeq S_4$).



Now lift to covers ...

- Likewise, a map $\mathcal{Q}$ on a compact surface will not usually be regular.
- But it is 'well-known' that $\mathcal{Q}$ is covered by a regular map $\mathcal{P}$ (usually on some other surface).
- The regular cover $\mathcal{P}$ is unique (to isomorphism) if it covers $\mathcal{Q}$ minimally.
- The proof is straightforward and works for any abstract 3-polytope (e.g. if $\mathcal{Q}$ is a face-to-face tessellation of the plane). In fact,

  $\mathrm{Aut}(\mathcal{P}) \simeq \mathrm{Mon}(\mathcal{Q})$, the monodromy group of $\mathcal{Q}$.
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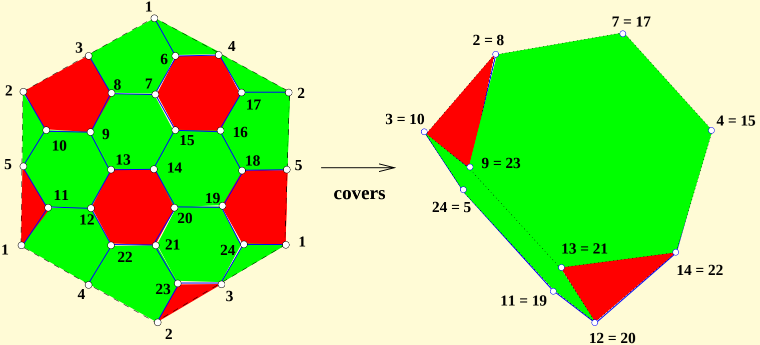
Example.

Hartley and Williams (2009) determined the minimal regular cover $\mathcal{P}$ for each classical (convex) Archimedean solid $\mathcal{Q}$ in $\mathbb{E}^3$.

Here the regular toroidal map $\mathcal{P} = \{6,3\}_{(2,2)}$ covers the truncated tetrahedron $\mathcal{Q}$.



Gripe

In the theory of covering spaces $f : C \to B$, the *monodromy group* is a representation of the fundamental group of the base B as a permutation group on a generic fibre $f^{-1}(x)$.

This is definitely not how we think of $\mathrm{Mon}(\mathcal{Q})$ in polytope theory!

The covering on the previous slide is $2:1$, except at four ramification points. There is no place for our monodromy group there.

But perhaps we can say, with futility, that the people working on covering spaces these last 200 years have misused the word!

Let's take stock:

- every polytope of small rank $d \leq 2$ is (combinatorially=abstractly) regular, hence equals its own minimal regular cover.
- every (abstract) 3-polytope $\mathcal{Q}$ has a unique minimal regular cover $\mathcal{P}$, and $\mathrm{Mon}(\mathcal{Q}) \simeq \mathrm{Aut}(\mathcal{P})$.
- So it's clear (in rank $d = 3$) that the cover $\mathcal{P}$ is finite if-f $\mathcal{Q}$ is finite.
- On the other hand, any polytope in any rank $d \geq 2$ is covered by the universal regular d-polytope $\mathcal{U} = \{\infty, \ldots, \infty\}$.
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Theorem (2013, to appear in J. Alg. Comb.)
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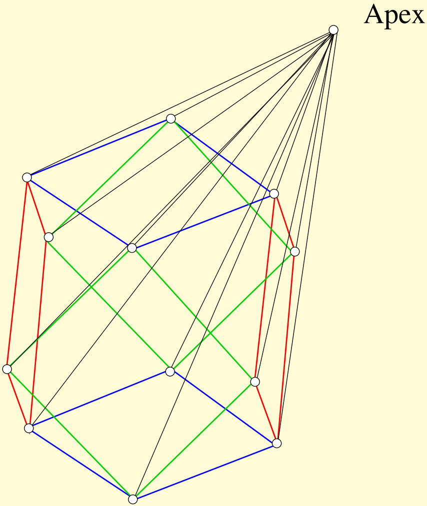
A 4-dimensional convex example

Suppose $\mathcal{Q}$ is the pyramid over a cuboctahedral base.
Then from our theorem, $\mathcal{Q}$ has a regular cover $\mathcal{P}$ of type $\{12, 12, 12\}$ and with

$$2^{53} \cdot 3^{14} \cdot 5 \approx 2.15 \times 10^{23}$$

flags. (This isn't likely a minimal cover!)



Idea of proof.

- an induction based on rank of regular initial sections in $\mathcal{Q}$
- crucial case is when d-polytope $\mathcal{Q}$ has all facets isomorphic to some regular $(d-1)$-polytope $\mathcal{K}$
- in that case, extend $\mathcal{K}$ 'trivially' to a regular d-polytope $\bar{\mathcal{K}}$ of type $\{\mathcal{K}, 2\}$... Thanks ...
- next 'mix' to get

$$G = \mathrm{Mon}(\mathcal{Q}) \diamondsuit \mathrm{Aut}(\bar{\mathcal{K}})$$

- then $G = \mathrm{Aut}(\mathcal{P})$ for desired regular cover $\mathcal{P}$ of $\mathcal{Q}$ (quotient criterion).
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Time for some more pyramids?

Leah B., Mark M., Deborah O., Gordon W. and I have studied the monodromy group M_n of the ordinary pyramid $\mathcal{Q}_n$ over an n-gon (coming up in *Discrete Mathematics*).

The extreme cases M_2 and M_∞ are most interesting. In fact, M_∞ is isomorphic to one of the 4783 *space groups* acting on Euclidean 4-space. The '4' is because most 3-pyramids have 4 flag-orbits under automorphisms. Here is a

Problem of Sorts

What is special about a k-orbit d-polytope $\mathcal{Q}$ for which $\mathrm{Mon}(\mathcal{Q})$ has a normal subgroup $N \simeq \mathbb{Z}_b^k$? Maybe maximal among abelian subgroups? Here b should be 'meaningful'. For example, if $\mathcal{Q}$ were infinite, we might want $b = \infty$, or even N of finite index in $\mathrm{Mon}(\mathcal{Q})$.

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Many thanks to our organizers!

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What are abstract polytopes?

An **abstract *n*-polytope** $\mathcal{Q}$ is a poset having some of the key structural properties of the face lattice of a convex *n*-polytope, although $\mathcal{Q}$

- need not be a lattice
- need not be finite
- need not have a familiar geometric realization

The abstract 3-polytopes include all convex polyhedra, face-to-face tessellations and many less familiar structures. But

you can safely think of a finite 3-polytope as a *map on a compact surface*.

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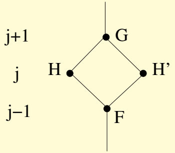
  whenever $F < G$ with rank$(F) = j-1$ and rank$(G) = j+1$ there exist exactly two j-faces H with $F < H < G$



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  via adjacency in the flag graph; this rules out, for example, the disjoint union of two polyhedra
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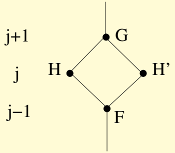

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The symmetry of $\mathcal{Q}$

is encoded in the group $\Gamma = \Gamma(\mathcal{Q})$ of all order-preserving bijections (= automorphisms) of $\mathcal{Q}$.

Each automorphism is det'd by its action on any one *flag* Φ; for a polyhedron, a flag

$$\Phi = \text{incident [vertex, edge, facet] triple}$$

Def. $\mathcal{Q}$ is *regular* if Γ is transitive on flags.

Examples:

- any polygon ($n = 2$) is (abstractly, i.e. combinatorially) regular
- the usual tiling of $\mathbb{E}^3$ by unit cubes is an infinite regular 4-polytope
- the Platonic solids ($n = 3$).

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The convex regular polyhedra (=Platonic solids) and the Kepler-Poinsot star-polyhedra $\mathcal{P}$

Local data for both polyhedron $\mathcal{P}$ and its group $\Gamma(\mathcal{P})$ reside in the Schläfli symbol or type $\{p, q\}$.

Platonic solids: $\{3, 3\}$ (tetrahedron), $\{3, 4\}$ (octahedron), $\{4, 3\}$ (cube), $\{3, 5\}$ (icosahedron),$\{5, 3\}$ (dodecahedron)

Kepler (ca. 1619) $\{\frac{5}{2}, 5\}$ (small stellated dodecahedron),
$\{\frac{5}{2}, 3\}$ (great stellated dodecahedron)

Poinsot (ca. 1809) $\{5, \frac{5}{2}\}$ (great dodecahedron),
$\{3, \frac{5}{2}\}$ (great isosahedron)

The classical convex regular polytopes, their Schläfli symbols and finite Coxeter groups with string diagrams

name	symbol	# facets	(Coxeter) group	order
$n = 4$:				
simplex	$\{3,3,3\}$	5	$A_4 \simeq S_5$	5!
cross-polytope	$\{3,3,4\}$	16	B_4	384
cube	$\{4,3,3\}$	8	B_4	384
24-cell	$\{3,4,3\}$	24	F_4	1152
600-cell	$\{3,3,5\}$	600	H_4	14400
120-cell	$\{5,3,3\}$	120	H_4	14400
$n > 4$:				
simplex	$\{3,3,\ldots,3\}$	$n+1$	$A_n \simeq S_{n+1}$	$(n+1)!$
cross-polytope	$\{3,\ldots,3,4\}$	2^n	B_n	$2^n \cdot n!$
cube	$\{4,3,\ldots,3\}$	$2n$	B_n	$2^n \cdot n!$

Regular polytopes and string C-groups

Schulte (1982) showed that the abstract regular n-polytopes $\mathcal{P}$ correspond exactly to the *string C-groups of rank n* (which we often study in their place).

The Correspondence Theorem.

Part 1. If $\mathcal{P}$ is a regular n-polytope, then $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ is a *string C-group*.

Part 2. Conversely, if $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ is a string C-group, then we can reconstruct an n-polytope $\mathcal{P}(\Gamma)$ (in a natural way as a coset geometry on Γ).

Furthermore, $\Gamma(\mathcal{P}(\Gamma)) \simeq \Gamma$ and $\mathcal{P}(\Gamma(\mathcal{P})) \simeq \mathcal{P}$.

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Recap: what is a string C-group?

Means: having fixed a base flag Φ in $\mathcal{P}$, for $0 \le j \le n-1$ there is a unique automorphism $\rho_j \in \Gamma(\mathcal{P})$ mapping Φ to the j-adjacent flag Φ^j. These involutions generate $\Gamma(\mathcal{P})$ and satisfy the relations implicit in some string (Coxeter) diagram, like

$$\bullet\overset{p_1}{\rule{2em}{0.4pt}}\bullet\overset{p_2}{\rule{2em}{0.4pt}}\bullet\rule{1em}{0.4pt}\;\ldots\;\rule{1em}{0.4pt}\bullet\overset{p_{n-1}}{\rule{2em}{0.4pt}}\bullet\,,$$

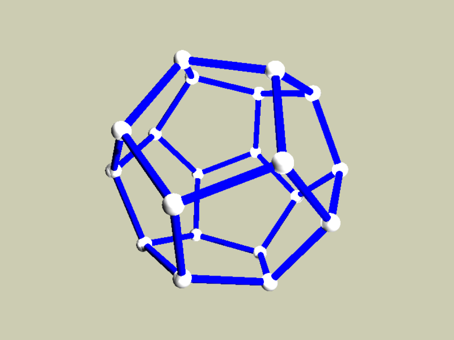
and perhaps other relations, so long as this *intersection condition* continues to hold:

$$\langle \rho_k : k \in I \rangle \cap \langle \rho_k : k \in J \rangle = \langle \rho_k : k \in I \cap J \rangle$$

(for all $I, J \subseteq \{0, \ldots, n-1\}$).
Notice that $\mathcal{P}$ then has Schläfli type $\{p_1, \ldots, p_{n-1}\}$.

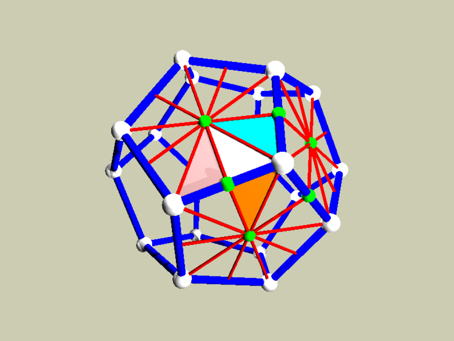
What is the monodromy group?

Look for example at the usual faithful realization of the regular dodecahedron $\mathcal{D}$



The flags of $\mathcal{D}$ correspond exactly to the triangles in a barycentric subdivision of the boundary. Here is part of that $\Rightarrow$

A base flag for $\mathcal{D}$, adjacent flags and generators



By transitivity, pick any base flag $= \Phi$ [white]

Then

0-adjacent flag $=: \ \Phi^0$ [pink]

1-adjacent flag $=: \ \Phi^1$ [cyan]

2-adjacent flag $=: \ \Phi^2$ [orange]

For $i = 0, 1, 2$, there is a unique automorphism

$$\rho_i : \Phi \mapsto \Phi^i \ .$$

Then $\Gamma(\mathcal{D}) = \langle \rho_0, \rho_1, \rho_2 \rangle$.

Can think reflections $\Rightarrow$

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Now DESTROY the polytope!

Consider any d-polytope $\mathcal{Q}$, not necessarily regular. For each flag Φ of $\mathcal{Q}$ and $i = 0, \ldots, d-1$, there is a unique *i-adjacent* flag Φ^i.

The mapping $s_i : \Phi \mapsto \Phi^i$ defines an involutory bijection s_i on the set $\mathcal{F}(\mathcal{Q})$ of all flags.

Defn. The *monodromy group* of $\mathcal{Q}$ is $\mathrm{Mon}(\mathcal{Q}) := \langle s_0, \ldots, s_{d-1} \rangle$.

(For maps, Steve Wilson [1994] calls this the "connection group".)

It is easy to check that $s_i^2 = 1$ and that $(s_i s_j)^2 = 1$, for $|j - i| > 1$, so $\mathrm{Mon}(\mathcal{Q})$ is an *sggi = string group generated by involutions*.
But for ranks $d \geq 4$, $\mathrm{Mon}(\mathcal{Q})$ can fail the intersection condition needed to to be a

string C-group = aut. group of regular d-poly.

Get back

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▸ Get back →

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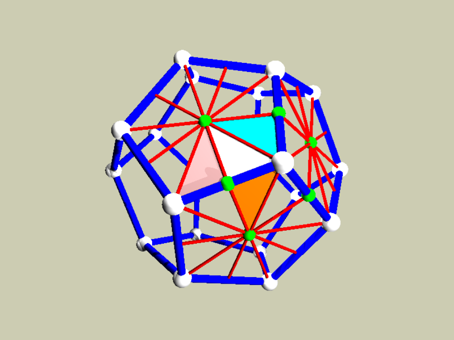
Example 1 - more on the regular dodecahedron $\mathcal{D}$



Note how seemingly destructive such flag swaps are.
(Think Rubik.)
Even so, here we do have

$$\mathrm{Mon}(\mathcal{D}) \simeq \Gamma(\mathcal{D}) \, .$$

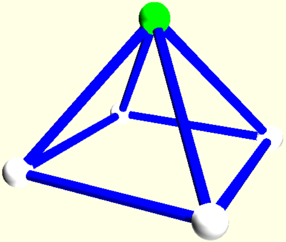
Theorem[ours in high rank]
For any abstract regular d-polytope $\mathcal{P}$,

$$\mathrm{Mon}(\mathcal{P}) \simeq \Gamma(\mathcal{P}) \, .$$

See *Mixing and Monodromy of Abstract Polytopes*, Monson, Pellicer and Williams, coming soon.

Example 2. The 4-gonal pyramid $\mathcal{E}$ is *not* regular

You can see that $\Gamma(\mathcal{E})$ has order 8. Guess the order of its monodromy group . . .



Example 2, continued

Here is a bit of the barycentric subdivison (left) with a few flags (right). Start flipping!